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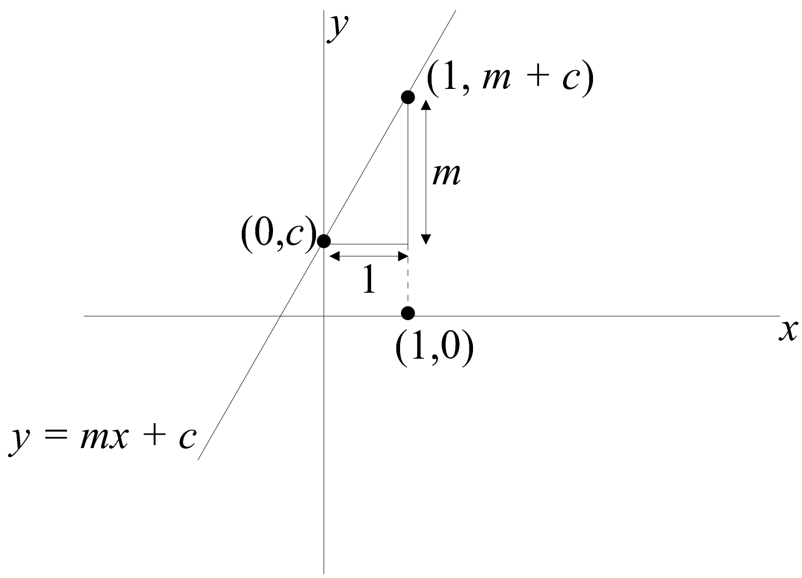
You are familiar with the idea that there is a straight line through any two points in the plane. Very often we use the formula

$$y = mx + c \tag{1}$$

for a straight line, where m is the 'slope' and c is the 'intercept'. See the figure. For every



1 unit move to the right a point on the line moves up by m units. If $m < 0$ then the point moves down instead of up. Here is an example.

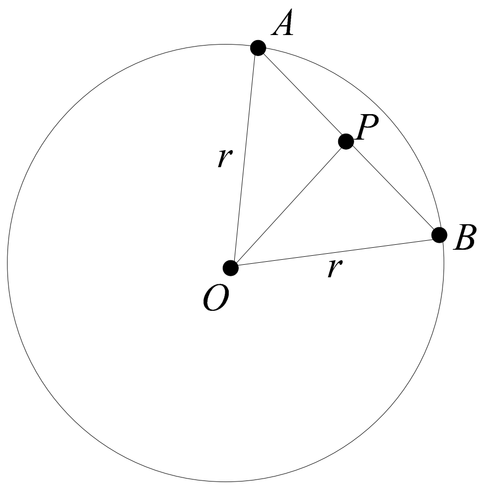
Example 1 Find the equation of the line through $(2, 3)$ and $(1, 5)$.
Substitute $x = 2, y = 3$ into $y = mx + c$. This gives $3 = 2m + c$.
Substitute $x = 1, y = 5$ into $y = mx + c$. This gives $5 = m + c$.
Subtracting the two equations gives $-2 = m$, and substituting gives $c = 5 - m = 7$. So the equation of the line is $y = -2x + 7$.

Notice that something goes wrong if we try to find the equation of the line through $(2, 3)$ and $(2, 5)$ by this method. The equations become $3 = 2m + c$ and $5 = 2m + c$ and these contradict one another: subtracting gives $-2 = 0$ which is nonsensical. This is because the line in this case is 'vertical', in fact has equation $x = 2$. Lines of the form '$x =$ constant' are the only exceptions to equations of the form $y = mx + c$ for lines. It's as well to be aware of this.

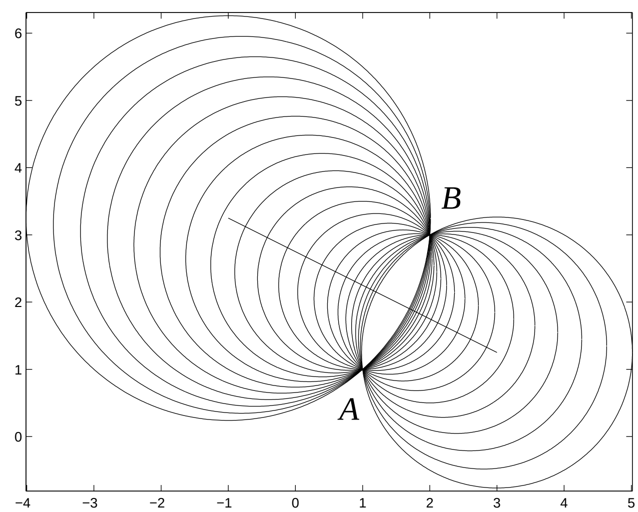
A completely general equation for a straight line is $ax + by = c$ though this has two disadvantages: (i) we don't allow $a = b = 0$; (ii) we can multiply the equation through by any nonzero number k, giving $kax + kby = kc$, and it's the same line as before. For example, $x - 3y = 4$ and $2x - 6y = 8$ and $-3x + 9y = -12$ are all the same line. [There is a way round these problems which is called using the 'projective plane' but we won't go into that today!]

Question We've seen that 'vertical' lines are the exception to being of the form $y = mx + c$. Which lines *cannot* be represented as $ax + by = 1$? Which lines *cannot* be represented as $x + by = c$?

We'll go on now to talk about *circles*. First of all, circles through two given points. The left-hand figure shows a circle through A and B, centre O. The conclusion is the



OP is perpendicular to *AB*. When a circle centre *O* passes through *A* and *B* the triangles *OAP* and *OBP* are congruent, so *P* is the midpoint of *AB*. So *OP* is the perpendicular bisector of *AB*.



important fact that the centre O lies on the perpendicular bisector of AB, that is, on the line perpendicular to AB and passing through the midpoint of AB. In the right-hand figure, we take $A = (1, 1)$ and $B = (2, 3)$. The centres of the circles through A and B all lie along the perpendicular bisector of AB, and this is drawn in the figure. To find the equation of this perpendicular bisector we need to use the fact that,

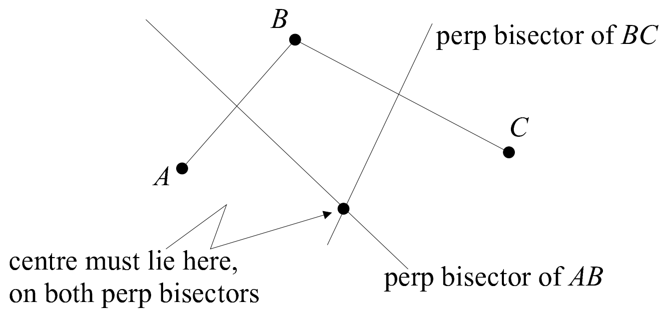
> *if (x, y) is the centre of any circle through A and B, then (x, y) is the same distance from A and from B.*

Turning this into algebra we get

$$(x-1)^2+(y-1)^2 = (x-2)^2+(y-3)^2, \quad \text{or} \quad x^2-2x+1+y^2-2y+1 = x^2-4x+4+y^2-6y+9,$$

which after cancelling (note the x^2 and y^2 both go!) gives $2x + 4y = 11$. So this is the line containing all the centres of the circles in the figure.

What about a circle through *three* given points? The figure suggests one way to find the centre of such a circle.



Example 2 Let $A = (-2, -1), B = (0, 1)$ and $C = (3, 3)$. The perpendicular bisectors of AB and BC are

$$(x+2)^2 + (y+1)^2 = x^2 + (y-1)^2, \quad \text{that is} \quad 4x + 4y = -4, \quad \text{and}$$

$$x^2 + (y-1)^2 = (x-3)^2 + (y-3)^2, \quad \text{that is} \quad 6x + 4y = 17.$$

Solving these two equations we get the centre as $(\frac{21}{2}, -\frac{23}{2})$. The radius is the distance from this point to say A, which works out as $\sqrt{1066}/2$ or about 16.3.

The equation of the circle centre (a, b) and radius r is

$$(x-a)^2 + (y-b)^2 = r^2. \tag{2}$$

The equation of the circle through A, B and C then works out as $x^2 + y^2 - 21x + 23y = 24$.

Here is another way of working out the equation of a circle through three points. Let's rewrite (2) as

$$x^2 + y^2 - 2ax - 2by + c = 0, \quad \text{where} \quad c = a^2 + b^2 - r^2. \tag{3}$$

Then let's use the same method as we used for lines to determine a, b and c. So we substitute $x = -2, y = -1$ in (3), because the circle passes through A; then we substitute $x = 0, y = 1$ and finally $x = 3, y = 3$. These give the three equations

$$\begin{aligned} 4a + 2b + c &= -5 \\ 2b - c &= 1 \\ 6a + 6b - c &= 18 \end{aligned}$$

These are 'simultaneous equations' for a, b and c, and solving them (try it!) gives the same answer $a = \frac{21}{2}, b = -\frac{23}{2}, c = -24$ as before. The centre of the circle is of course (a, b).

Here are some **Examples** for you to try for yourself.

3 $A = (0, 0), B = (0, 2), C = (2, 0)$. This example you might be able to do just by drawing a picture to find the centre $(1, 1)$ and radius $\sqrt{2}$. The equation is $x^2 + y^2 - 2x - 2y = 0$.

4 $A = (0, 6), B = (12, 0), C = (-12, -6)$. The values of a, b, c work out as 1, –7, –120. (So what is the radius?)

5 Try $A = (0,1), B = (1,2), C = (2,3)$. Something strange happens here. Can you explain it?

6 Is it possible for a circle to have $a = 1, b = 1, c = 3$? (Try working out r.)

7 Can you devise a test, given *four* points, to decide whether they lie on the same circle?

Finally let us look at circles through three points which are *close together on the same given curve*, say a parabola $y = x^2$. For a start let's take three points $A = (0,0), B = (s, s^2)$ and $C = (-s, s^2)$ which all lie on $y = x^2$. When s is small they are all close together. Proceeding as before we take (3) as the equation of the circle through the three points, and substitute the coordinates of the points A, B and C is one by one. This gives the three equations

$$\begin{aligned} c &= 0 \\ 2as + 2bs^2 - c &= s^2 + s^4 \\ -2as + 2bs^2 - c &= s^2 + s^4 \end{aligned}$$

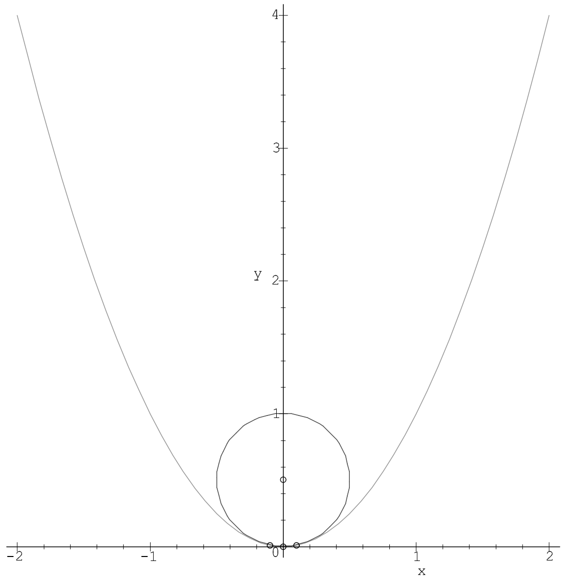
These are quite easy to solve, giving $a = 0, b = \frac{1}{2} + \frac{1}{2}s^2, c = 0$. So the radius of the circle is in this case just $r = \frac{1}{2} + \frac{1}{2}s^2$ (distance of (a,b) from one of the three points, $(0,0)$). It is particularly interesting to notice that when s is very small this radius is very close to $\frac{1}{2}$. As s becomes very small (as 's tends to zero' is the official phrase) the circle is said to approach the *circle of curvature* of the parabola $y = x^2$ at the origin. It is the *best fitting circle* there. Its radius $\frac{1}{2}$ is called the *radius of curvature* of the parabola at the origin. See the upper left figure on the next page where $s = 0.1$.

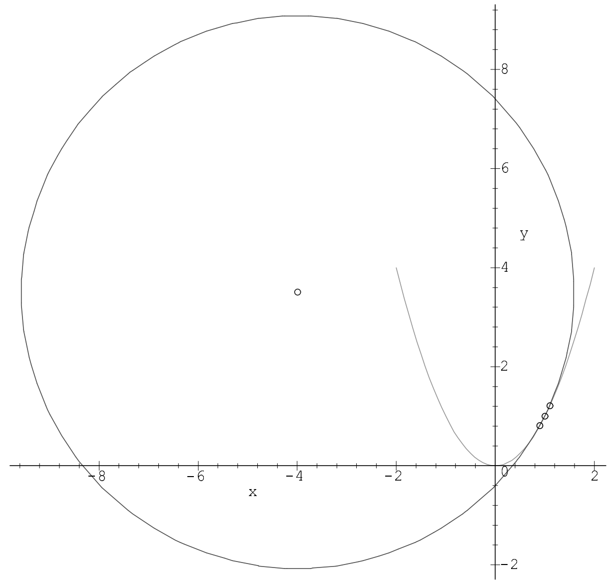
If we do the same calculations for three points $(1,1), (1+s, (1+s)^2), (1-s, (1-s)^2)$ which are all close to $(1,1)$ things are a bit more complicated but work out to $a = -4 + s^2, b = \frac{7}{2} + \frac{1}{2}s^2, c = -3 + 3s^2$ and the radius comes to $\frac{1}{2}\sqrt{125 - 30s^2 + 5s^4}$. When s is very small this is close to $\frac{1}{2}\sqrt{125} = \frac{5}{2}\sqrt{5}$ or about 5.59. This is the radius of curvature of the parabola at $(1,1)$. See the upper right figure on the next page where $s = 0.1$.

The circle of curvature is 'one step beyond' the tangent line to the parabola. The tangent line is obtained by taking the straight line through two very close points of the parabola, and then letting the distance between the points 'tend to zero'. The slope of the tangent is the 'limit' of the slopes of these lines. In the same way the radius of curvature is the 'limit' of the radii of the circles through three very close points on the parabola. By the way the *curvature* of the parabola at a point P is $1/r$ where r is the radius of curvature. So for P at the origin the curvature is 2, and for P at (1,1) the curvature is $2/(5\sqrt{5})$. The larger the curvature, the more curved the parabola is!

The same ideas apply to any curve, not just to the parabola: we can find the best fitting circle at any point P. Its radius r is the radius of curvature and $1/r$ is the curvature.

The animations which follow are designed to show the circle of curvature at various points of a curve C. Remember the circle of curvature at P is the circle through three points all extremely close to P, for example P itself and a very nearby point on each side of P. It is the circle which 'best approximates' the curve C at P. The centres of all these circles of curvature, as P is taken at different points along the curve, trace out something called the





evolute of the curve. Typically this evolute has a number of sharp points called *cusps*. In fact there is a beautiful theorem called the 'Four Vertex Theorem' which says that there are always at least four such cusps provided the curve C closes up and does not cross through itself. In one of the animations C is an ellipse, and then the evolute has exactly four cusps. In the other animation the curve C is more complicated and the evolute has more cusps. As you watch the circle's centre move round the evolute, you'll be struck with this fact: as the centre passes through a cusp the circle either stops getting bigger and starts to get smaller again, or stops getting smaller and starts to get bigger again. These are 'turning points' or 'extreme points' of the radius of the circle.

Example 8 Going back to the parabola $y = x^2$ let's take the point (t, t^2) and the two points $(t + s, (t + s)^2)$ and $(t - s, (t - s)^2)$ on either side. Let the equation of the circle through these three points be, as before, $x^2 + y^2 - 2ax - 2by + c = 0$. Using the same method of substitution as before, and after a rather marathon session of solving for a, b, c, maybe you can arrive at

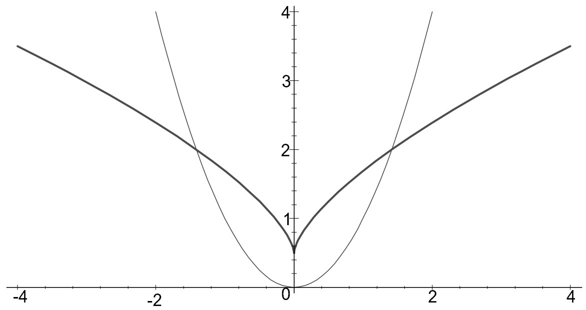
$$a = -4t^3 + ts^2, \quad b = \tfrac{1}{2} + 3t^2 + \tfrac{1}{2}s^2, \quad c = -3t^4 + 3t^2s^2.$$

At any rate you might like to check that, with these values of a, b, c, the circle does pass through the three points above! The interesting thing is to let s become very small. Then a, b, c are very nearly $-4t^3, \frac{1}{2} + 3t^2, -3t^4$ respectively. We can deduce two things from this:
(i) the centre of curvature (a, b) of the parabola at (t, t^2) is $(-4t^3, \frac{1}{2} + 3t^2)$;
(ii) the radius of curvature r is given by

$$r^2 = a^2 + b^2 - c = 16t^6 + (\tfrac{1}{4} + 3t^2 + 9t^4) + 3t^4 = \tfrac{1}{4}(1 + 12t^2 + 48t^4 + 64t^6) = \tfrac{1}{4}(1 + 4t^2)^3.$$

Note the beautiful final form of the answer!

As the point $P = (t, t^2)$ sweeps along the parabola the centre of curvature $x = -4t^3, y = \frac{1}{2} + 3t^2$ sweeps along the evolute. Note that $x^2 = 16t^6, (y - \frac{1}{2})^3 = 27t^6$ so the *equation of the evolute* is $27x^2 = 16(y - \frac{1}{2})^3$. It's pretty amazing to be able to deduce this just from equations of circles (and lots of algebra!). This curve is shown in the figure. It contains the centres of all the circles of curvature of the parabola.



Note The best way to find the radius of curvature is not by the method used here but by using calculus. You'll meet this incredibly powerful and beautiful subject later if you haven't met it already. If you know about calculus then you can understand the amazing formula for the radius of curvature of a curve $y = f(x)$: it is

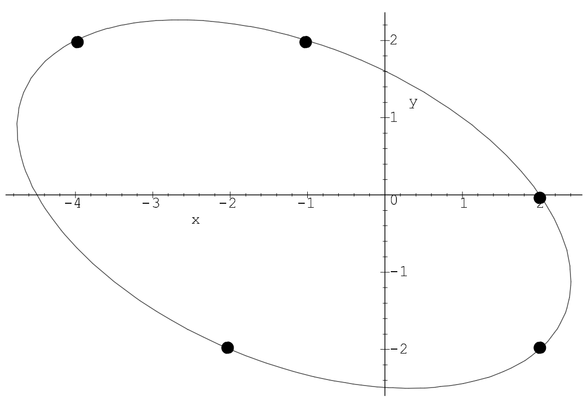
$$\frac{(1 + f'(x)^2)^{3/2}}{f''(x)},$$

where f', f'' are the first and second derivatives of f with respect to x. Notice that this is 'infinite' if $f''(x) = 0$. In this case the circle through three nearby points gets more and more like a straight line as the points get closer to the place where $f''(x) = 0$. The formula displayed here is the one which is used in the animations to draw the circles of curvature.

Of course we don't need to stop at circles. For example we can look at 'conics' which are curves with a more general equation of the form

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

We can be given now *five* points and expect to find a conic through all of them. Two examples are illustrated in the figure on the next page. When doing the calculations it is convenient to choose say $a = 1$ and solve for the other five coefficients b, h, g, f, c.



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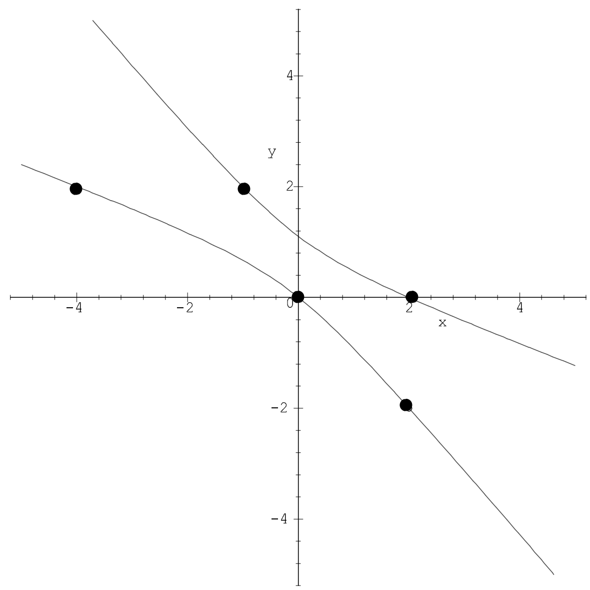

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