Lines and Circles. 27 January, 2001

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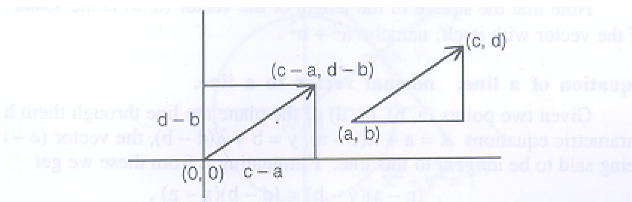
The Euclidean plane; Pythagoras' theorem.

The *plane* consists of pairs of real numbers (x, y), the points of the plane, the *distance* of a point from the origin $(0, 0)$ being the square root of $x^2 + y^2$.

Vectors and scalars. Scalar multiples of a vector.

The *vector* from a point (a, b) to a point (c, d) is the pair $(c - a, d - c)$.



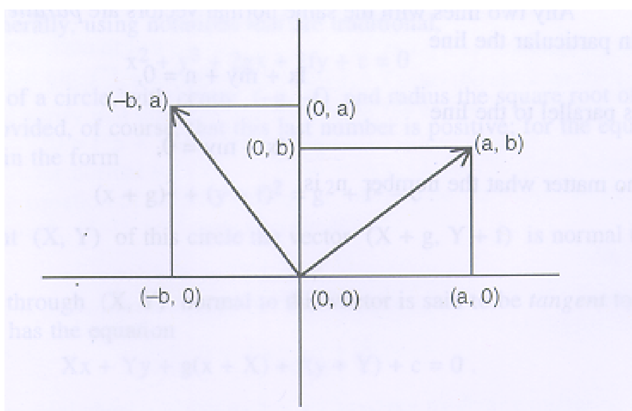
So vectors are represented by points.

In this context real numbers are often called *scalars*. Given a vector (a, b) and a scalar λ we may form the *scalar multiple* $(\lambda a, \lambda b)$.

Parametric equation of a line through the origin.

The scalar multiples of a vector (a, b) are *collinear* with the origin, that is the points representing them all lie on the straight line through the origin $(0, 0)$ and the point (a, b). The *parametric equations* of the line are $x = \lambda a, y = \lambda b$.

Normal vectors.



Two vectors are said to be *normal* (or *orthogonal*) to one another if the lines joining their representative points to the origin are at right angles to one another. For example the vectors (a, b) and $(-b, a)$ are normal to one another.

Proposition 1 *Two vectors* (a, b) *and* (c, d) *are normal to one another if and only if* $ac + bd = 0$.

Scalar product.

The scalar $ac + bd$ is said to be the *scalar product* of the vectors (a, b) and (c, d). One writes $(a, b) \cdot (c, d) = ac + bd$.

Note that the square of the length of the vector (a, b) is the scalar product of the vector with itself, namely $a^2 + b^2$.

Equation of a line; normal vector to a line.

Given two points $(a, b), (c, d)$ of the plane the line through them has the parametric equations $x = a + \lambda(c - a), y = b + \lambda(d - b)$, the vector $(c - a, d - b)$ being said to be *tangent* to this line. Eliminating λ from these we get

$$(c - a)(y - b) = (d - b)(x - a),$$

or equivalently

$$-(d - b)x + (c - a)y + ad - bc = 0.$$

Note that the coefficients of x and y form a vector $(-(d - b), c - a)$ normal to the line.

Proposition 2 *In general an equation of the form*

$$lx + my + n = 0,$$

where l, m *and* n *are real numbers, represents a line of the plane, with normal vector* (l, m).

Parallel lines.

Any two lines with the same normal vectors are parallel to one another. In particular the line

$$lx + my + n = 0,$$

is parallel to the line

$$lx + my = 0,$$

no matter what the number n is.

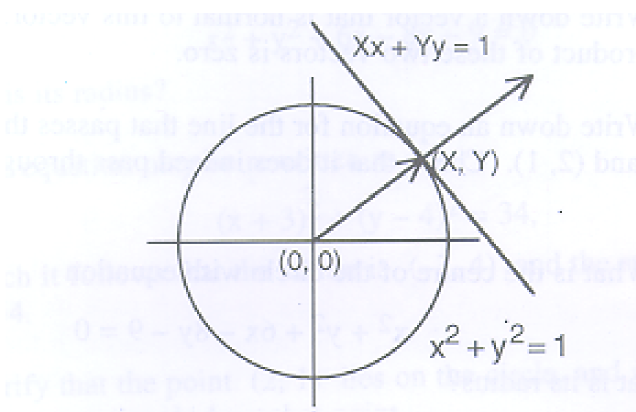
The unit circle, its normal vectors and tangent lines.

The circle, centre the origin $(0, 0)$ and radius 1 has, by Pythagoras' theorem, the equation

$$x^2 + y^2 = 1,$$

the vector (X, Y) being normal to the circle at a point (X, Y) of it.

This circle is known as the *unit circle*.



The line through a point (X, Y) of the circle normal to the vector (X, Y) is said to be *tangent* to the circle there. It has the equation

$$Xx + Yy = 1,$$

since $X^2 + Y^2 = 1$.

Circles in general, their normal vectors and tangent lines.

The circle, centre (a, b) and radius r has, by Pythagoras' theorem, the equation

$$(x - a)^2 + (y - b)^2 = r^2,$$

the vector $(X - a, Y - b)$ being *normal* to the circle at the point (X, Y) of it.

More generally, using notations that are traditional,

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is the equation of a circle, with centre $(-g, -f)$ and radius the square root of $g^2 + f^2 - c$, provided, of course, that this last number is positive; for the equation can be written in the form

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c.$$

At a point (X, Y) of this circle the vector $(X + g, Y + f)$ is normal to the circle.

The line through (X, Y) normal to this vector is said to be *tangent* to the circle there. It has the equation

$$Xx + Yy + g(x + X) + f(y + Y) + c = 0.$$

Inversion in a circle.

Inversion in the unit circle sends any point (x, y) of the plane, other than the origin, to the point $(x', y') = (x^2 + y^2)^{-1}(x, y)$, the positive scalar multiple of (x, y), at a distance $(x^2 + y^2)^{-\frac{1}{2}}$ from the origin. Points inside the circle go outside, points outside come inside, while the circle itself remains fixed.

Now, for any fixed numbers a, c, f and g,

$$a(x^2 + y^2) + 2gx + 2fy + c = 0$$

represents a circle, provided $a \neq 0$ and $g^2 + f^2 - ac > 0$, but represents a line, if $a = 0$, with g and f not both zero. Either passes through the origin if $c = 0$. Dividing this by $x^2 + y^2$, this implies that its inverse has equation

$$a + 2gx' + 2fy' + c(x'^2 + y'^2) = 0.$$

So the inverse of a circle is a circle, unless the original circle passes through the *centre of inversion*, the origin, in which case its inverse is a line. Likewise the inverse of a line is a circle that passes through the origin, unless the line passes through the origin, in which case its inverse is the same line.

Next, it is straightforward to verify that at a point (X, Y) of

$$a(x^2 + y^2) + 2gx + 2fy + c = 0,$$

whether a circle or a line, the vector $(aX + g, aY + f)$ is normal to it.

Now suppose that the circles or lines

$$a'(x^2 + y^2) + 2g'x + 2f'y + c' = 0 \text{ and } a"(x^2 + y^2) + 2g"x + 2f"y + c" = 0$$

intersect at a point (X, Y), and are normal to each other there, that is the vectors $(X + g', Y + f')$ and $(X + g", Y + f")$ are normal to each other. Then, as in Exercise 3, below,

$$g'g" + f'f" - \frac{1}{2}(a"c' + a'c") = 0.$$

It follows that the inverses intersect normally at the inverse of the point (X, Y).

Circles or lines that touch at a point (X, Y) invert to circles or lines that touch at the inverse of the point (X, Y). For they share a line that passes through (X, Y) normally to both, and this line inverts to a circle or line normal to both at the inverse of (X, Y).

Inversion in any circle is defined in the obvious way, the points inside the circle going outside, and the points outside going inside, as before. Preservation of normality remains true.

Theorem 1 *Suppose that three circles touch each other, two by two. Then a unique circle passes through the three points of contact, being normal to each of them.*

To prove this, invert with respect to a circle whose centre is one of the three points of contact.

Exercises on Lines and Circles

1(a) What is the vector with initial point $(-3, 4)$ and final point $(2, 1)$, and what is its length?

1(b) Write down a vector that is normal to this vector. Verify that the scalar product of these two vectors is zero.

1(c) Write down an equation for the line that passes through the points $(-3, 4)$ and $(2, 1)$. Check that it does indeed pass sthrough both these points.

2(a) What is the centre of the circle with equation

$$x^2 + y^2 + 6x - 8y - 9 = 0,$$

and what is its radius?

2(b) Verify that the point $(2, 1)$ lies on the circle, and write down a normal vector to the circle at that point.

2(c) Write down an equation for the tangent line to the circle at the point $(2, 1)$.

2(d) Write down an equation for the normal line to the circle at the point $(2, 1)$, and verify that this line passes through the centre of the circle.

3 Suppose that circles

$$a'(x^2 + y^2) + 2g'x + 2f'y + c' = 0 \text{ and } a"(x^2 + y^2) + 2g"x + 2f"y + c" = 0$$

intersect at a point (X, Y), and are normal to each other there; that is the vectors $(X + g', Y + f')$ and $(X + g", Y + f")$ are normal to each other. Verify that in that case

$$g'g" + f'f" - \frac{1}{2}(c' + c") = 0.$$

Solution to Exercise 2

2(a) What is the centre of the circle with equation

$$x^2 + y^2 + 6x - 8y - 9 = 0,$$

and what is its radius?

The equation may be rewritten as

$$(x + 3)^2 + (y - 4)^2 = 34,$$

from which it follows that the centre is $(-3, 4)$ and the radius the square root of 34.

2(b) Verify that the point $(2, 1)$ lies on the circle, and write down a normal vector to the circle at that point.

The point lies on the circle since $4 + 1 + 12 - 8 - 9 = 0$ or, alternatively, since $(2 + 3)2 + (1 - 4)2 = 25 + 9 = 34$.

A normal vector at a point (X, Y) of the circle is $(X + 3, Y - 4)$. With $X = 2, Y = 1$ this becomes the vector $(5, -3)$.

2(c) Write down an equation for the tangent line to the circle at the point $(2, 1)$.

Since $(5, -3)$ is a normal vector, the left-hand side of the equation of the tangent can be taken to be $5x - 3y$. Since it passes through $(2, 1)$ we have as its equation

$$5x - 3y = 7.$$

2(d) Write down an equation for the normal line to the circle at the point $(2, 1)$, and verify that this line passes through the centre of the circle.

A vector normal to $(5, -3)$ is the vector $(3, 5)$. So the left-hand side of the equation of the normal may be taken to be $3x + 5y$. Since it passes through $(2, 1)$ we have as its equation

$$3x + 5y = 11.$$

Finally, this line passes through the centre of the circle $(-3, 4)$, since

$$3.(-3) + 5.4 = -9 + 20 = 11.$$

This final check assures us that we have not made any mistakes!