Iteration and Spider Diagrams

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In this session we have been looking mainly at functions. But what is a function? It is simply a rule or set of rules that take one number and map it onto another.

For example, you could have as your rule “If the given number is greater than zero, then the function's output is 1, and if the given number is less than or equal to zero, then the function's output is –1”.

Often, however, we define a formula to calculate the output of a function from the input. The notation used is “f(x)” meaning the function “f” takes one input, which is represented in the formula on the right hand side of the “=” by x. Functions can be called by any letter, but by tradition they are usually f, g, h, etc.

<u>Examples</u>
$f(x) = 2x - 0.5$
$f(2) = 2 \times 2 – 0.5 = 3.5$
$f(0.4) = 2 \times 0.4 – 0.5 = 0.3$

$g(x) = 0.5x + 0.2$
$g(2) = 0.5 \times 2 + 0.2 = 1.2$
$g(0.4) = 0.5 \times 0.4 + 0.2 = 0.4$

Notice that g(0.4) = 0.4, ie. when 0.4 is input into the function “g”, 0.4 is the resulting output. This is an example of a **fixed point** of g.

Is 0.4 unique as a fixed point of g? Do any fixed points exist for f?

The answers to these questions can be found easily using a simple fact:
Any fixed points of f satisfy the equation f(x) = x

This is obvious if you think about it, since all it says is that when this particular value of x is input into the equation, the same value is output - this is the definition of a fixed point.

Given this fact and the original formula for f(x), you can now find any fixed points.

For example, with $f(x) = 2x – 0.5$
Fixed points satisfy: $x = 2x – 0.5$
$0 = x – 0.5$ [taking x off both sides]
$x = 0.5$ [adding 0.5 to both sides]

See if you can do the same for g to check that 0.4 is a fixed point.

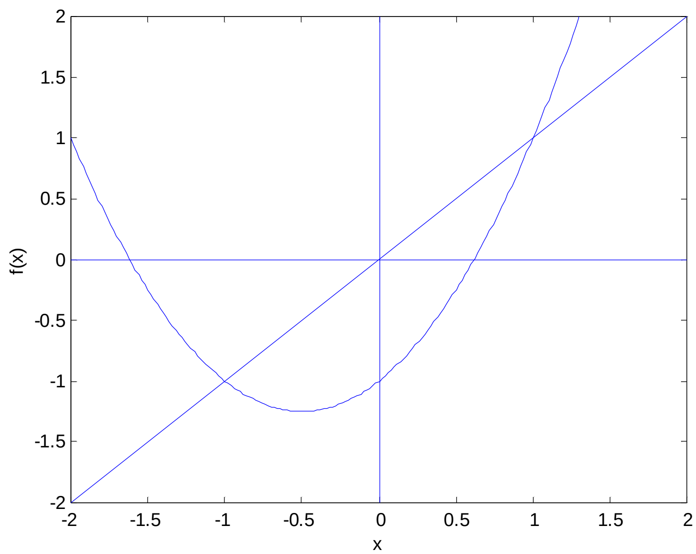
Those of you who have met quadratic equations may like to try finding the fixed points of $h(x) = x^2 + 2x - 2$

Hint: There are two.

Can fixed points be represented on a graph in any way?

Yes, they can. In fact, we use a similar idea to that which we used to find the fixed points, ie. the fact that f(x) = x at fixed points. If we superimpose the graph of f(x) = x on top of the graph of the original f(x) then fixed points occur at the intersection(s) of the graphs.

For example, for $i(x) = x^2 + x - 1$:



So clearly i has two fixed points, located at 1 and -1.

We're now going to look at the function $k(x) = 2 - x^2$

$$k\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)=\frac{1}{2}-\frac{1}{2}\sqrt{5}$$

$$k\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)=\frac{1}{2}+\frac{1}{2}\sqrt{5}$$

Notice how when $\frac{1}{2}+\frac{1}{2}\sqrt{5}$ is input into the function, $\frac{1}{2}-\frac{1}{2}\sqrt{5}$ is output. However, when $\frac{1}{2}-\frac{1}{2}\sqrt{5}$ is input into the function, $\frac{1}{2}+\frac{1}{2}\sqrt{5}$ is output.

This is an interesting phenomenon, and in general whenever f(a) = b and f(b) = a, a and b are called **period 2 points** of the function, and together form a **period 2 orbit**.

It is also possible to have period 3 points, and higher.

For example, if $f(a) = b$
$f(b) = c$
$f(c) = d$
and $f(d) = a$

then a, b, c & d are period 4 points of the function f and together form a period 4 orbit. Any period n point (for some whole number n) is called a **periodic point**.

It is possible, but slightly complicated, to find periodic points algebraically. The method is similar to that used for fixed points but instead of having $f(x) = x$, you have:
$f(\ f(x)\ ) = x$ for period 2 points
$f(\ f(\ f(x)\ )\ ) = x$ for period 3 points
etc.

To find the period 2 points of $k(x) = 2 - x^2$, then, we'd use the following method:

$k(\ k(x)\ ) = x$ (remember k(x) is the name of the function, *not* k times x)
$k(\ 2 - x^2\ ) = x$ (substituting the formula for the inner k(x))
$2 - [2 - x^2]^2 = x$ (substituting the formula now for $k(\ 2 - x^2\ )$
$2 - [4 - 4x^2 + x^4] = x$ (multiplying out the brackets)
$x^4 - 4x^2 + x + 2 = 0$ (rearranging)

Now this final equation is tricky to solve, but employing a few tricks it can be rewritten as:

$$(x - 1)(x + 2)(x^2 - x - 1) = 0$$

So the four period two points in this case are 1, -2 and the solutions to the quadratic $x^2 - x - 1 = 0$, which are $\frac{1}{2} + \frac{1}{2}\sqrt{5}$ and $\frac{1}{2} - \frac{1}{2}\sqrt{5}$.

Try putting x = 1 and x = -2 through the function k. What happens?

Can periodic points be represented on the graph at all?

The answer is yes, they can, but they're not as easy to spot as fixed points. The technique we use to find them is the **spider diagram**.

Perhaps I should mention at this point that the process of putting some value into a function, taking the output and putting this back in again (often repeating this process many times) is called **iteration**.

The first step in constructing a spider diagram of a function is to draw the graph of the function, and then superimpose the graph of $f(x) = x$ on top. This is often referred to as simply the “diagonal”.

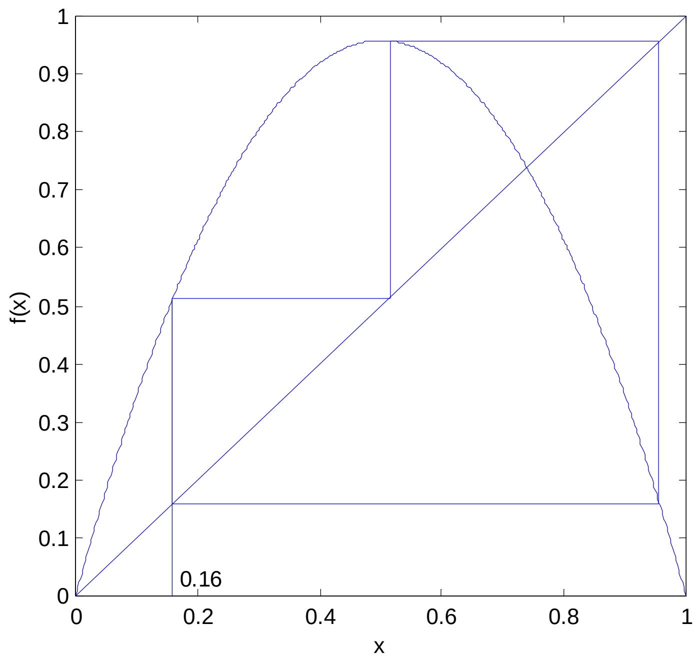
Next, you find the point you want to start with on the x-axis. Draw a vertical line up from this point to the graph of your function.

This represents putting the first value into the function (let's call this a), and the height of the line shows the value of the function's output (let's call this b). To put b back into the function draw a horizontal line across to the diagonal. Where this line hits the diagonal, you are level with the value of b on the x-axis so can draw another vertical line up (or down) to the graph.

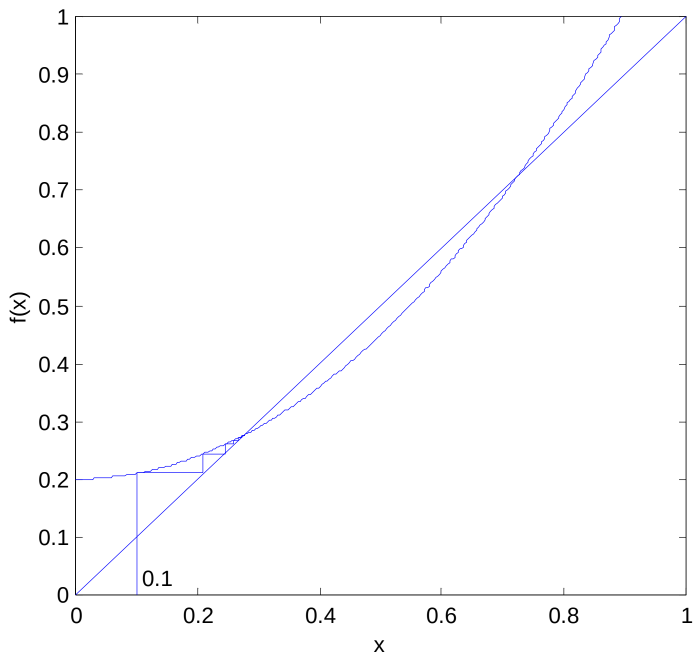
By repeating these two steps, you graphically iterate the function. If you end up at a point you've already visited, then you've found a periodic point.

Here are the spider diagrams of the eight exercises you were set during the session - compare them with your answers to see how close you were!

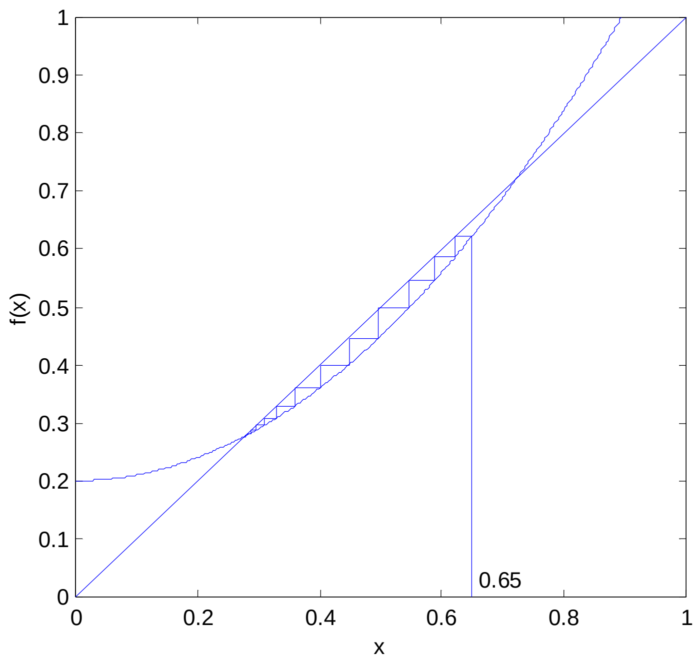
1. $f(x) = 3.829x(1 - x)$
Starting value of $x = 0.16$
This is a period 3 orbit.



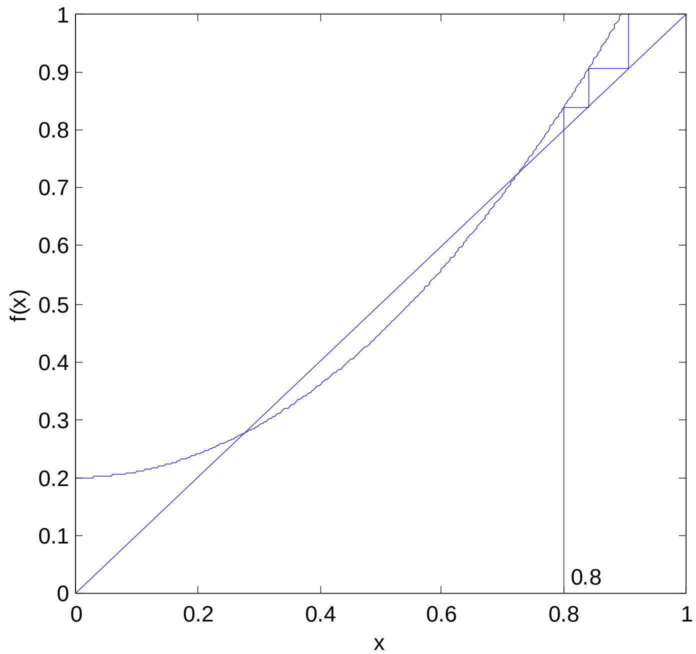
2. $f(x) = x^2 + 0.2$
Starting value of x = 0.1
This shows that the more iterations you perform, the closer you get to a fixed point.



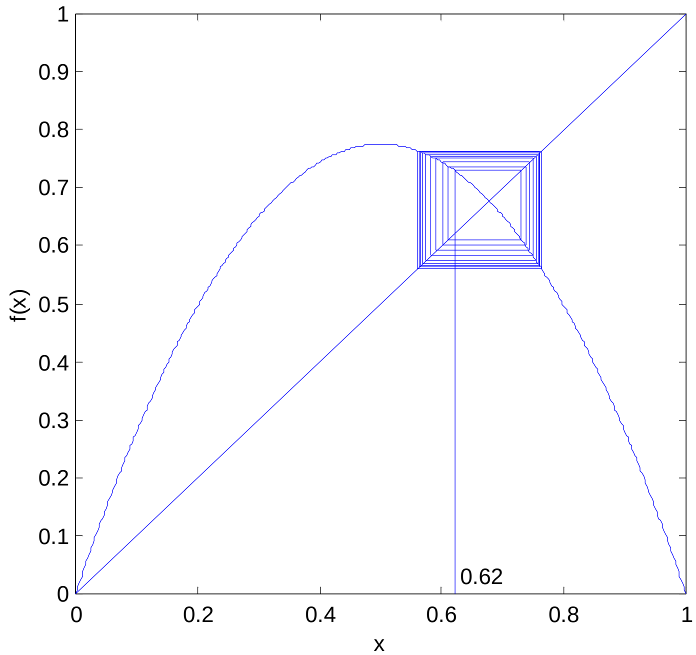
3. $f(x) = x^2 + 0.2$
Starting value of x = 0.65
This shows that starting above the fixed point you still get "sucked in" towards it.



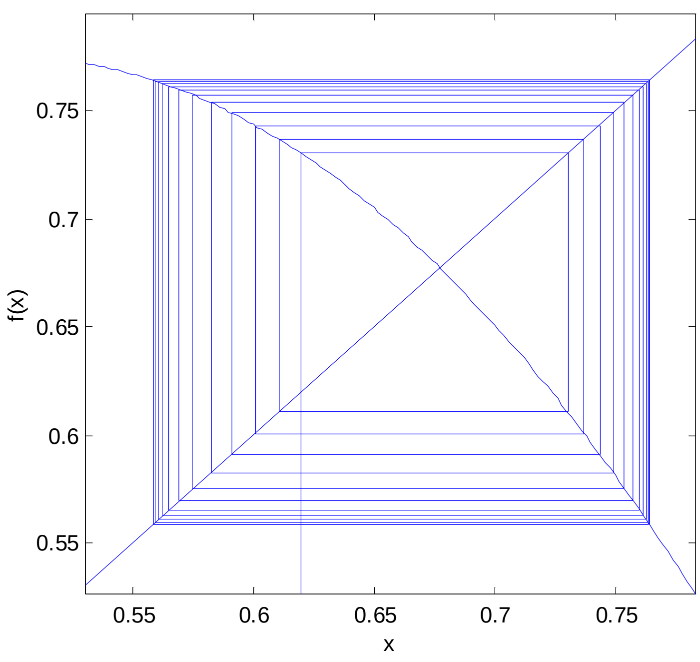
4. $f(x) = x^2 + 0.2$
Starting value of $x = 0.8$
Now we're starting above the second fixed point, but don't seem to be getting sucked towards it at all. In fact, you go off the edge of the graph and the numbers keep getting larger and larger forever.



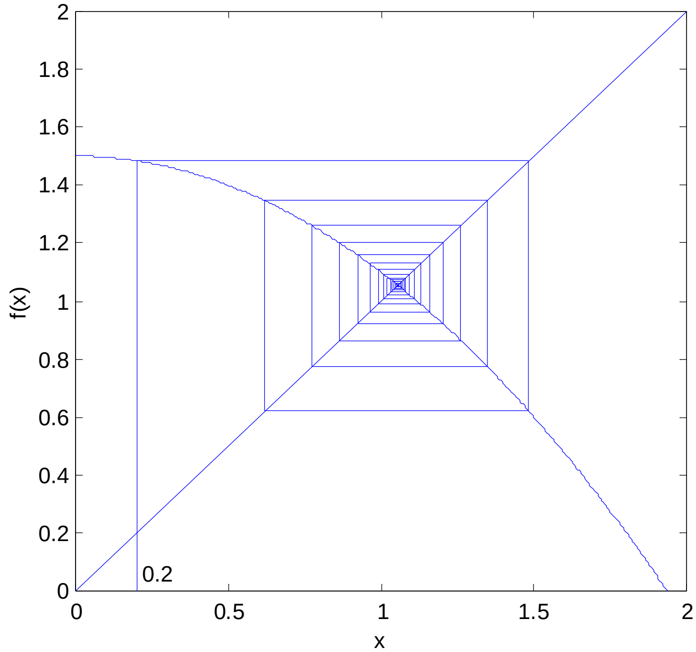
5. $f(x) = 3.1x(1 - x)$
Starting value of $x = 0.62$
This time we start near a period 2 orbit, and get "sucked into" the orbit.



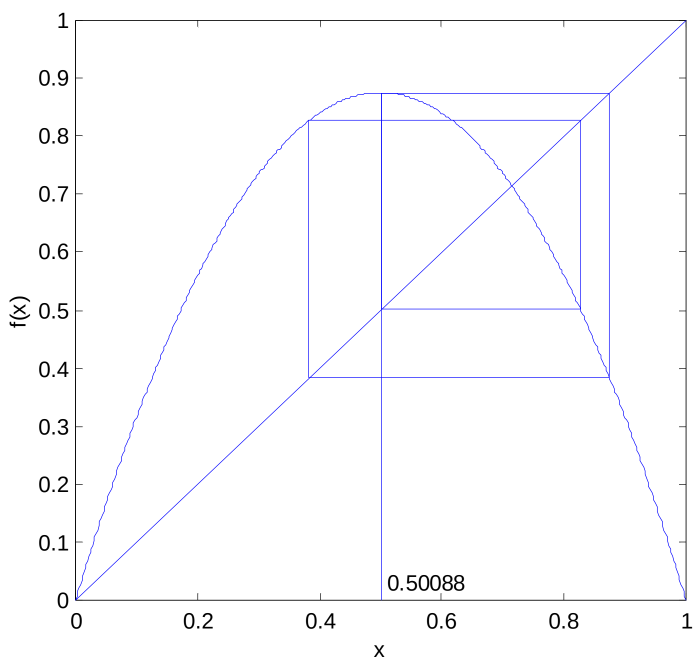
An enlargement of the interesting area is as follows:



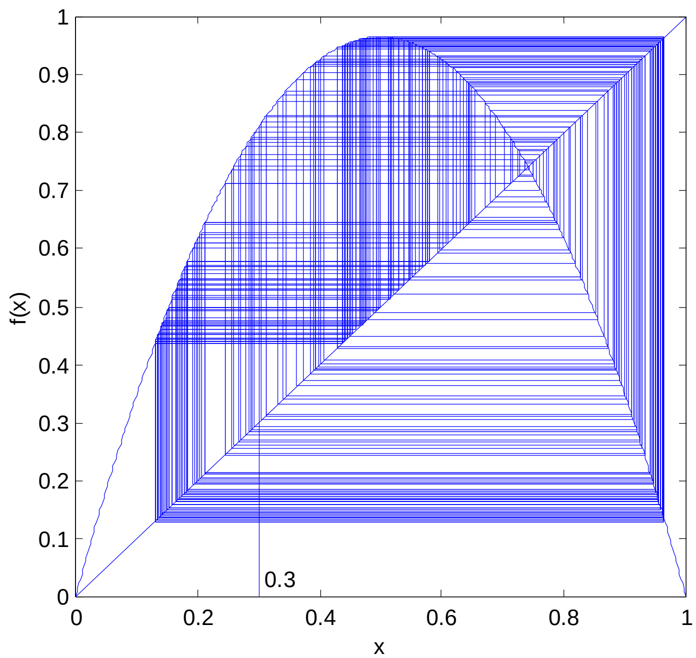
6. $f(x) = 1.5 - 0.4x^2$
Starting value of x = 0.2
Here we start near a fixed point and get gradually closer to it, but we jump either side of it on each iteration - generating a cobweb-like pattern.



7. $f(x) = 3.5x(1 - x)$
Starting value of x = 0.50088
This is a period 4 orbit.



8. $f(x) = 3.86x(1 - x)$
Starting value of $x = 0.3$
There is no discernable pattern here. This is called chaos (it's easy to see why!).



It is clear from the above examples that there are two different types of fixed points - those that "suck in" nearby points on a spider diagram and those that repel them. Fixed points that "suck in" nearby points are called **stable fixed points** and those that repel nearby points are called **unstable fixed points**.
In fact, there are stable and unstable periodic orbits as well (example 5 shows a stable period 2 orbit).

If you look closely at the diagrams, you may notice that the stable fixed points all occur where the graph's slope is quite gentle, and the unstable fixed points occur where the graph's slope is more steep. In fact, fixed points are stable if the slope of the graph where they occur is less than 1 (or, if the slope is negative, greater than -1). They are unstable if the graph's slope is greater than one (for negative slopes, less than -1). If the slope is 1 or -1, we can't be sure exactly what will happen (the point could attract from one side and repel the other, for example).

You may have noticed that many of the above functions are of the form $rx(1 - x)$ where r is a fixed value. These are all various forms of the function called the **logistic map**, which has the formula $f(x) = rx(1 - x)$ for values of r between 0 and 4. It is a very interesting function which has applications in modelling populations amongst other things.

The final picture I've got for you is rather different to the earlier ones. It is a graph showing the logistic map, but rather than having x along the horizontal axis, it has r (from 3 to 4). On the vertical axis, f(x) is plotted as usual.

Any thin vertical slice taken through the graph below shows the result of iterating some point many times without displaying it (to allow it to be sucked in to any stable orbits) and then displaying the next few hundred iterations. This has the effect of isolating the stable orbits for that particular value of r. For example, a slice taken at r = 3.2 would show that there are 2 stable periodic points (ie. one period 2 orbit). At about r = 3.45, however, there suddenly start being four stable periodic points and after r = 3.6 the picture starts to become really interesting. An almost black slice indicates chaos. See if you can find where the above examples (1, 5, 7 and 8) would fit in on this picture.