Liverpool University Maths Club, 25 November 2000

Peter Giblin

First of all, something about graphs. The *circle* of radius 1 centred at the origin has equation $x^2 + y^2 = 1$. Pythagoras' theorem tells us that the distance from the origin is $\sqrt{x^2 + y^2} = 1$ and squaring gives the equation. See Figure 1.

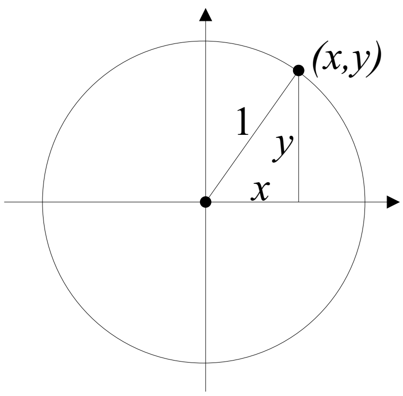

Figure 1: A circle, radius 1, centre $(0, 0)$.

Suppose that in the equation, we replace x by $2x$, giving $(2x)^2+y^2 = 1$, or $4x^2+y^2 = 1$. This has the effect of squashing the circle in the x-direction, by a factor of 2. See Figure 2, top right. This curve is called an *ellipse*. Similarly replacing y by $2y$ to give $x^2 + 4y^2 = 1$ squashes by a factor of 2 in the y-direction, as in Figure 2, bottom left. As an exercise,

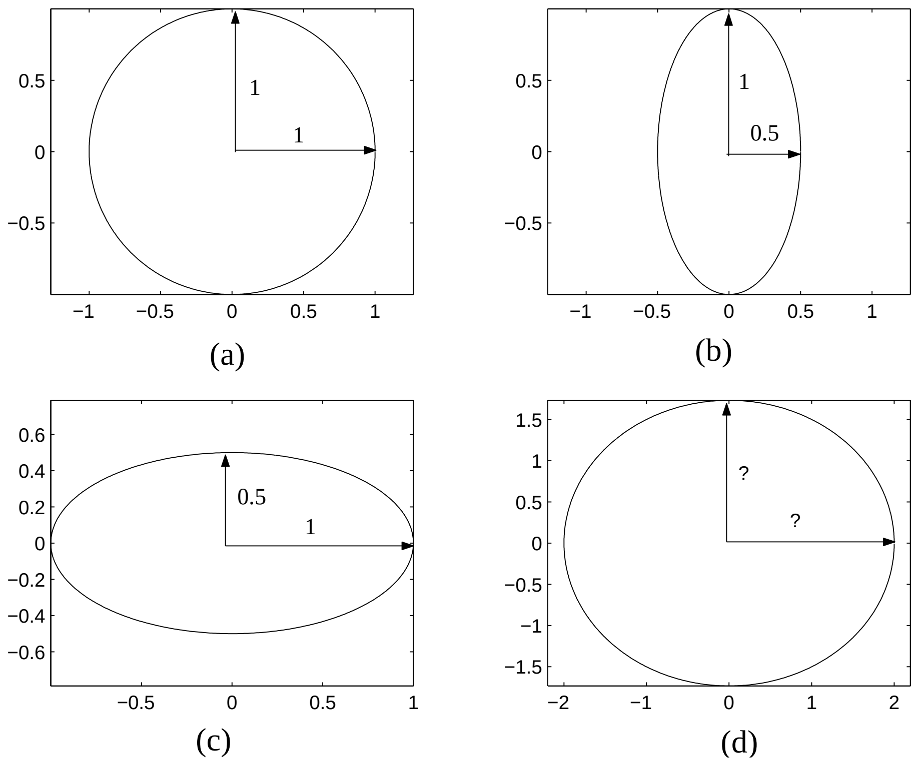

Figure 2: Ellipses. (a) $x^2+y^2 = 1$; a circle. (b) $4x^2+y^2 = 1$, where the circle is squashed in the x-direction; (c) $x^2+4y^2 = 1$, where the circle is squashed in the y-direction; (d) $3x^2+4y^2 = 12$, where the circle is...what?

Figure 2, bottom right, shows the curve $3x^2 + 4y^2 = 12$. Note the 12 on the right-hand side! Here's a good method of finding how much the circle is squashed in each direction.
Put $y = 0$ in the equation. Then $3x^2 = 12$, so $x =$
Put $x = 0$ in the equation. Then $4y^2 = 12$, so $y =$
What do these tell you about the graph in Figure 2(d)?

In the rest of this session we'll see some ellipses and circles arising in different ways.

First, back to the definition of a circle: all the points are the same distance from the centre. Here's a slightly different idea. Suppose we take two points, say $A = (-1, 0)$ and $B = (1, 0)$, and look for all points (x, y) such that the *sum* of the distances from the A and B is always the same, say 4. See Figure 3, left. For a start, let's look at the point P which lies on the x-axis with the distances from A and B adding to 4. Where is this?

Answer:

What about the point Q which lies on the y-axis? Here, QA and QB are equal and they add to 4, so both of them equal 2. Now using Pythagoras' theorem, what are the coordinates of Q?

Answer:

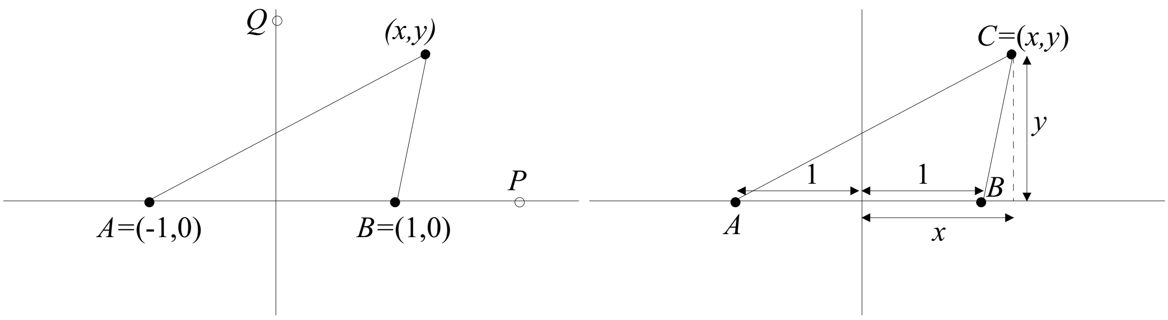

Figure 3: Look for points (x, y) with the distances from A and B adding to 4.

It's actually quite a hard calculation to find the equation of the curve traced out by (x, y), but here goes. It's really all Pythagoras' theorem again. Look at Figure 3, right. Using Pythagoras' theorem,

$$CA + CB = \sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2} = 4.$$

Now to sort this out it's *much* better to take one of those horrible square roots on to the other side of the equation:

$$\sqrt{(x+1)^2 + y^2} = 4 - \sqrt{(x-1)^2 + y^2}$$

and now square both sides. Remember that $(p + q)^2 = p^2 + 2pq + q^2$.

Eventually you should arrive at

$$16 - 4x = 8\sqrt{(x-1)^2 + y^2}, \quad \text{or} \quad 4 - x = 2\sqrt{(x-1)^2 + y^2}.$$

Now square again!!

You should arrive, after some more work, at

$$3x^2 + 4y^2 = 12,$$

which we've seen before somewhere. See Figure 2, bottom right.

So this is an ellipse. It's always true, in fact, that when C moves so that the sum of the distances to two points is given, C moves on an ellipse. A and B can be any points here, they don't have to be $(-1, 0)$ and $(1, 0)$. But they must be given in advance.

When $CA + CB$ is always the same, then C moves on an ellipse.

The points A and B are called the *foci* of the ellipse. Each one of them is called a *focus*. (Like radius→radii, so focus→foci.)

Now let's apply this idea to a very interesting situation. We shall look at circles which are simultaneously tangent to (= touching) two other circles. See Figure 4 for an explanation of why, when two circles touch, their centres and the point of contact are in a straight line.

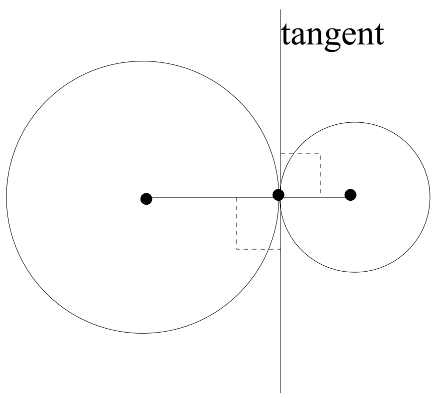

Figure 4: The tangent is at right-angles to each radius, so the two radii are in a straight line.

Now let's look at two circles, as in Figure 5, and find some other circles which are tangent to *both* of them. Can you see that $CA = a - r$ and $CB = b + r$? We then get the

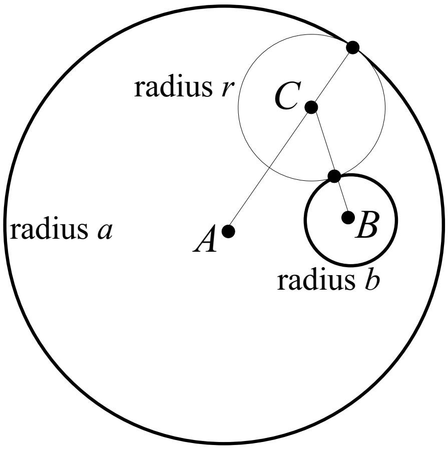

Figure 5: We are given two heavily drawn circles, of radius a and b. We now fit in another circle which is tangent to both of them. Say it has radius r.

amazing result

$$CA + CB = a + b,$$

which is a constant number, not dependent on the particular circle we chose tangent to both of the given circles! Now look at Figure 6.

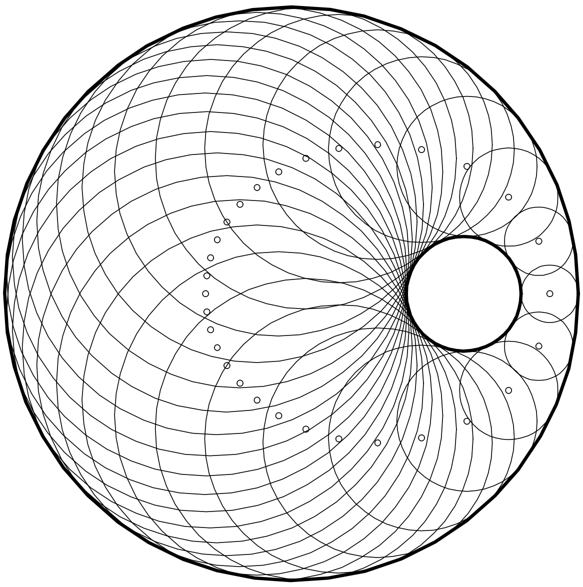

Figure 6: Taking lots of circles which are all tangent to two given (heavily drawn) circles, their centres move on....................

Some Challenges

1. There are some other circles which are tangent to two given circles.............see Figure 7.

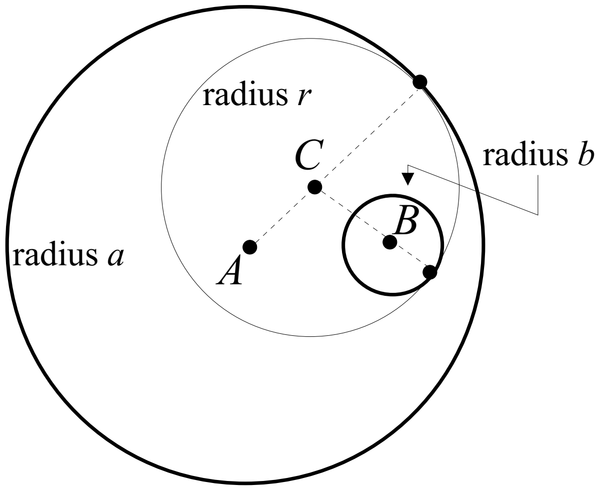

Figure 7: Two heavily drawn circles as before, centre A and radius a, and centre B and radius b. But this time we take circles tangent to both of them and *containing* the smaller circle. Is $CA + CB$ still the same no matter which of these circles we take? If so then the centres line on...................

2. Does it matter if the two given circles meet? What if one is completely *outside* the other? This last question might lead you to ask another question: suppose (x, y) moves so that the *difference* of its distances from A and B is always the same. What kind of a curve does it move on? You could take $A = (-1, 0)$, $B = (1, 0)$ as we did before, and the constant difference of distances equal to 1: $CA - CB = 1$. [Another question: why can't you take it equal to 4 this time? Could CA ever be as much as $4 + CB$?] After some rearranging and squaring as before, you should eventually find the equation $12x^2 - 4y^2 = 3$. [But this actually also includes the points where $CB - CA = 1$. Never mind, it's a simple enough equation.] This is what is called a *hyperbola*. Take a look at Figure 8.

3. This is for people who know something about trigonometry. It's an interesting exercise to find the radius r in Figure 5, given a and b and also the angle $CAB = \theta$, say. Here is the answer. You might enjoy proving the formula. This formula was used in plotting the circles in Figure 6. We'll write d for the length of AB: the distance between the centres of the circles.

$$r = \frac{a^2 - b^2 + d^2 - 2ad\cos\theta}{2(a + b - d\cos\theta)}.$$

Of course you can ask the same question in the situation of the previous challenge, that is when the circles are external to each other. You find

$$r = \frac{a^2 + d^2 - b^2 - 2ad\cos\theta}{2(b - a + d\cos\theta)}.$$

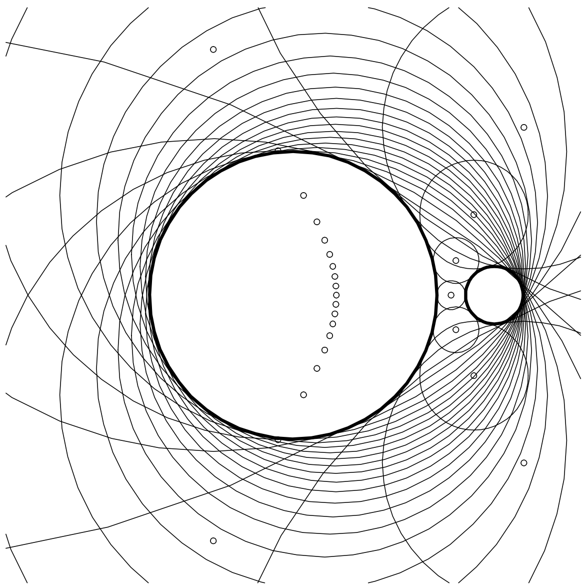

Figure 8: Two circles which are outside each other. Lots of circles have been drawn which are tangent to both of them. Because these circles get big, some of them only appear as parts of circles. The centres are again marked and this time they trace a...................