Liverpool University Maths Club, 28 October 2000

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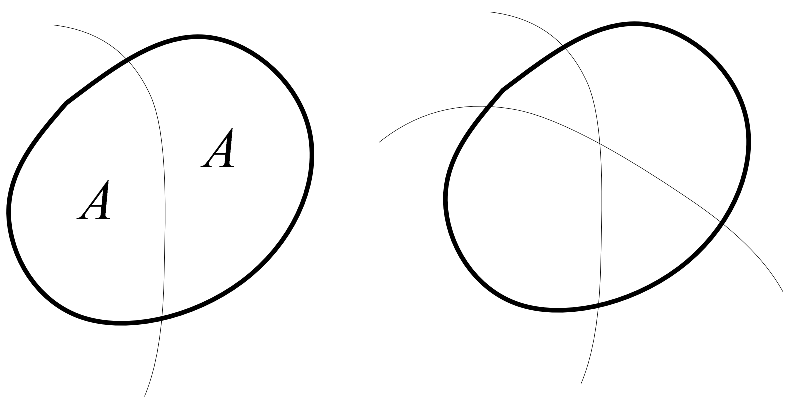
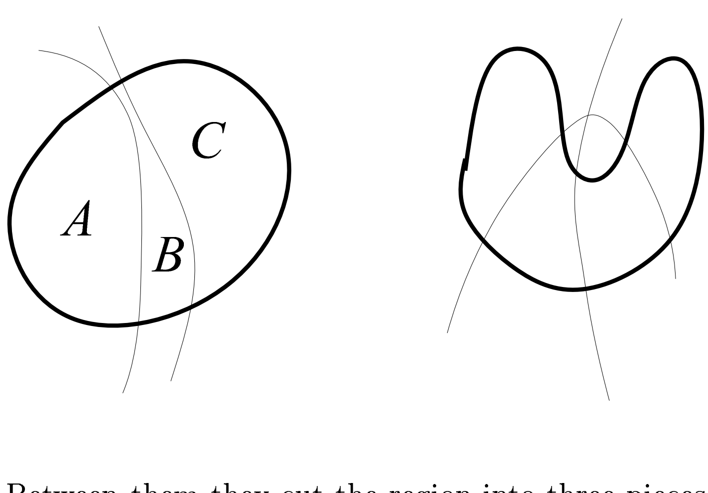
The first thing we looked at was the idea of cutting the area of a region in half by means of a curve cutting across the region:



Look at the left-hand figure, where the curve cuts the region (inside the heavy curve) in half, both parts having area equal to A. The right-hand figure shows two curves, each cutting the region in half. Notice that they *meet inside the area*. We should think about this: must it happen that two curves, both cutting the region in half, must meet inside?

Look at the left hand figure below: Suppose two curves both cut the region in half but



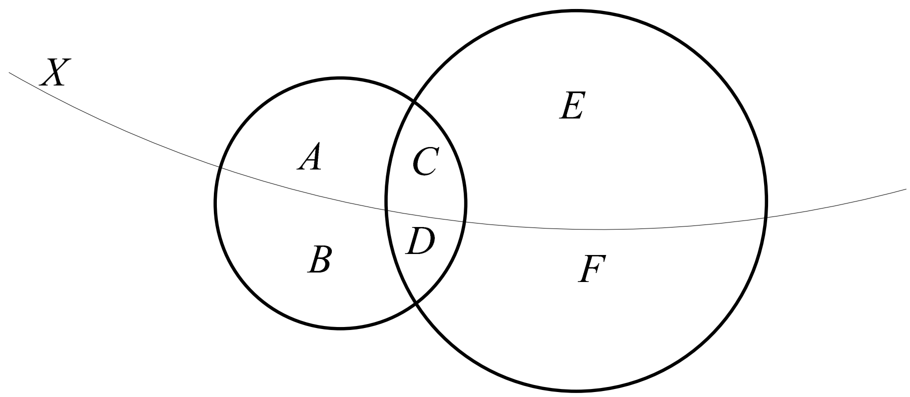
don't meet inside. Between them they cut the region into three pieces, marked A, B and C. Then because the curve which is further left cuts the region in half we have $A = B+C$. Because the curve which is further right cuts the region in half we have $A+B = C$. Now look at these two equations and show that in fact B must be 0. That means that actually the two curves were the same, since there is no area between them!

So it's not possible to have two different curves cutting the region in half without intersecting somewhere inside............or is it? Look at the right-hand figure above. Maybe these both cut the region in half but intersect *outside* the region?

The region on the left (inside the heavily drawn curve) is called *convex*: it doesn't have any inlets or bays. The right-hand region is not convex.[1]

What our argument shows is that if two curves both cut a convex region in half, then they meet inside the region.

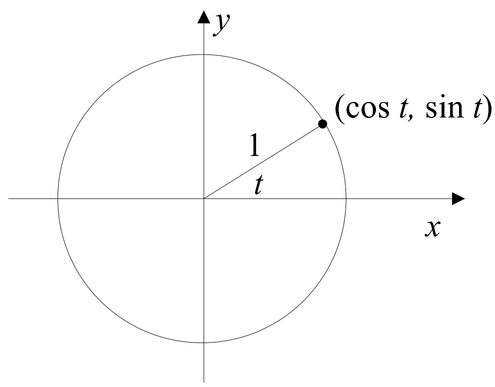
Now take two circles (heavy lines) and ask the question whether it is possible to draw another circle X which cuts each the three regions shown in half, that is $A = B$, $C =$



D, $E = F$. Each of the regions is convex (no inlets or bays, or look at the footnote for a proper definition), so if we can find something else, Y say, cutting all three regions in half it must be that X and Y meet inside all three regions. However it is easy to see that we can take Y to be the straight line through the centres of the two original circles! For sure this cuts the three regions in half. But Y and X cannot meet inside all three regions since that would give a circle and a line which met in three points.

So we've shown that there is no circle X cutting all three regions in half.

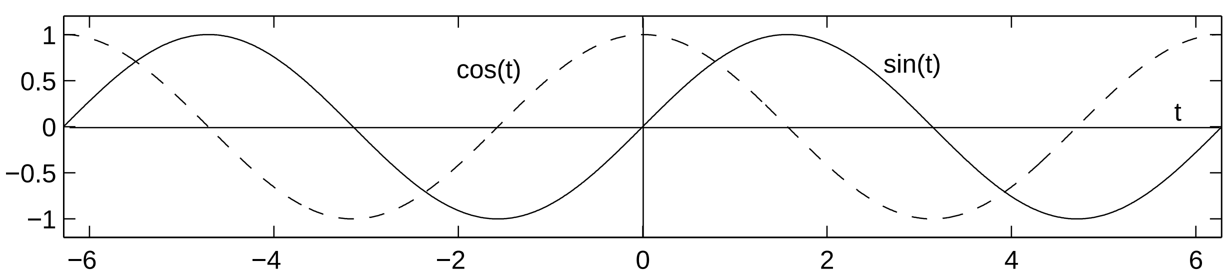
[1]Here's the official definition of convex: draw the straight line between any two points inside a convex region. Then the *whole straight line* lies inside the region. Does that make sense for the two regions in the bottom figure on the previous page? (Convex on the left and not convex on the right.)



The point on the circle with radius 1 centred at the origin, where the angle round from the x-axis is t, as in the figure above, has coordinates $x = \cos t$ and $y = \sin t$. In full sin is 'sine' and cos is 'cosine' but we always write sin and cos. Using Pythagoras' theorem,

$$\cos^2 t + \sin^2 t = 1, \tag{1}$$

where we write $\cos^2 t$ as shorthand for $(\cos t)^2$ and similarly for sin. This is a very important equation indeed. The graph of the sine function is shown below as a solid line and the



graph of the cosine function as a dashed line. The horizontal axis is in *radians*, where π radians equals 180°. Remember π is about 3.14. The sine curve crosses the horizontal axis ($\sin t = 0$) at $t = -2\pi, -\pi, 0, \pi, 2\pi$; in degrees these angles are $-360, -180, 0, 180, 360$. The cosine curve crosses the horizontal axis ($\cos t = 0$) at $t = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$. In degrees these are $-270, -90, 90, 270$.

Here is a table of some angles and their radian equivalents, also a few values of the sine and cosine.

radians	0	$\frac{\pi}{10}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{\pi}{5}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
degrees	0	18°	20°	30°	36°	45°	60°	90°	120°	135°	150°	180°
cos	1			$\frac{\sqrt{3}}{2}$		$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
sin	0			$\frac{1}{2}$		$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Sine and cosine are linked by another crucial formula:

$$\sin\left(\frac{\pi}{2} - t\right) = \cos t; \qquad \cos\left(\frac{\pi}{2} - t\right) = \sin t. \tag{2}$$

Now let's try some problems:

1. Let's work out the graph of the function

$$y = \frac{1 - \sin^4 t - \cos^4 t}{1 - \sin^6 t - \cos^6 t}. \tag{3}$$

You can try it on your calculator if it draws graphs. It might have trouble!

But we can simplify the expression enormously. Remember the expansions which come from Pascal's triangle:

$$(a+b)^2 = a^2 + 2ab + b^2; \qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Let's put $a = \cos^2 t,\ b = \sin^2 t$. Then using the $(a+b)^2$ formula and equation (1)

$$1 = (\sin^2 t + \cos^2 t)^2 = \sin^4 t + 2\sin^2 t\cos^2 t + \cos^4 t.$$

Rearranging, it follows that $1 - \sin^4 t - \cos^4 t = 2\sin^2 t\cos^2 t$.

Maybe you can work out that using the $(a+b)^3$ formula and equation (1) you end up with

$$1 - \sin^6 t - \cos^6 t = 3\sin^2 t\cos^4 t + 3\sin^4 t\cos^2 t = 3\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t) = 3\sin^2 t\cos^2 t$$

using equation (1) again.

So our horrible function (3) has been reduced to

$$y = \frac{2\sin^2 t\cos^2 t}{3\sin^2 t\cos^2 t} = \frac{2}{3}.$$

So in fact y is always the same value! The graph is a horizontal line at height $\frac{2}{3}$. (But note that $\sin t$ and $\cos t$ have been cancelled. This means that really y isn't defined when $\sin t = 0$ or $\cos t = 0$.)

2. Usually we have to rely on a calculator to give us a numerical value of sine or cosine but sometimes it's possible to work out the sine or cosine of an angle in a nice way. Here's an example. We'll have to quote a few more formulae which you can take on trust for now; if you haven't met them before you'll find out why they are true later on.

$$\begin{aligned} \sin(2t) &= 2\sin t\cos t && (4)\\ \cos(2t) &= 2\cos^2 t - 1 = 1 - 2\sin^2 t && (5)\\ \sin(3t) &= 3\sin t - 4\sin^3 t && (6)\\ \cos(3t) &= 4\cos^3 t - 3\cos t && (7) \end{aligned}$$

Now let's work out $\sin\frac{\pi}{10}$, that is the sine of 18°. Let's write $t = \frac{\pi}{10}$. Notice that $2t + 3t = \frac{\pi}{2}$ so that by equation (2) we have $\sin(3t) = \cos(2t)$. Now we apply equations (6) and (5). To make the equation even shorter let's write s for $\sin t$. Then we get

$$3s - 4s^3 = 1 - 2s^2, \quad \text{that is} \quad 4s^3 - 2s^2 - 3s + 1 = 0.$$

Maybe you can factorize the last equation; it comes to

$$(s-1)(4s^2 + 2s - 1) = 0.$$

Well, the solution $s = 1$ isn't right. (Why? and Why does it come up as $s = \sin t$ where $\sin(3t) = \cos(2t)$? Both these questions are worth thinking about.) Solving the quadratic by the Famous Formula we get

$$s = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

(this uses $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$). One of these solutions is negative and *that* can't be right (ask yourself the same questions as before!), so the correct answer is

$$\sin\left(\frac{\pi}{10}\right) = \frac{-1 + \sqrt{5}}{2}.$$

This is a rare example of a sine which comes out nicely, other than those in the table above.

3. As an exercise, use this and equation (5) to show that

$$\cos\left(\frac{\pi}{5}\right) = \frac{1 + \sqrt{5}}{4}.$$

4. Since $\frac{2\pi}{5} + \frac{\pi}{10} = \frac{\pi}{2}$ you'll easily deduce from equation (2) that

$$\cos\left(\frac{2\pi}{5}\right) = \frac{-1 + \sqrt{5}}{4}.$$

5. From a couple of results above you'll quickly find that

$$\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{\pi}{5}\right) = \frac{1}{4},$$

which is also quite nice. (There are lots of other ways of proving this result if you know more trigonometrical formulae.)

6. The following formulae are not so nice (can you find any better ones?); they follow from some results already proved and the very basic equation (1).

$$\sin\left(\frac{\pi}{5}\right) = \frac{1}{4}\sqrt{2}\sqrt{5 - \sqrt{5}}, \qquad \cos\left(\frac{\pi}{10}\right) = \frac{1}{4}\sqrt{2}\sqrt{5 + \sqrt{5}}.$$

7. Using $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ (one of the relatively 'easy' results in the table, which follows by examining a right-angled triangle with the other angles both equal to $\frac{\pi}{4} = 45°$), and equation (5) you'll get the first result below. Maybe you can get the second one too!

$$\cos\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}, \qquad \sin\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}.$$

8. By taking angles away from $\frac{\pi}{2}$ and using equation (2) you'll be able to find $\sin\frac{3\pi}{8}$, $\cos\frac{3\pi}{8}$.

9. Another two basic facts about sine and cosine which follow from the fact that the point at an angle t on the unit circle (see one of the diagrams above) has coordinates $(\cos t,\ \sin t)$:

$$\cos(-t) = \cos t; \qquad \sin(-t) = -\sin t. \tag{8}$$

To get a negative angle, just go clockwise from the x-axis rather than anticlockwise.

10. Here are a couple more formulae which follow from the numbered equations above. See if you can follow the reasoning!

$$\sin\left(\frac{\pi}{2}+t\right) = \sin\left(\frac{\pi}{2}-(-t)\right) = \cos(-t) = \cos(t).$$

$$\cos(\pi - t) = \cos\left(\frac{\pi}{2} - \left(t - \frac{\pi}{2}\right)\right) = \sin\left(t - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - t\right) = -\cos t.$$

Both of these also follow easily from formulae for $\cos(a+b)$, $\sin(a+b)$, if you know those.

11. Finally, an application of the formulae just above, together with equation (2). We'll give just the bones of the calculation and leave it to you to say where each step comes from. To save some writing, let $t = \frac{\pi}{8}$ here. We don't need the explicit values of $\cos t$ and $\sin t$ in Number 7 above, but as a separate exercise you might try using those instead! We'll write down a complicated expression X and then simplify it as follows. Remember $t = \frac{\pi}{8}$.

$$\begin{aligned} X &= \sin^4 t + \cos^4(3t) + \sin^4(5t) + \cos^4(7t) \\ &= \sin^4 t + \cos^4(\tfrac{1}{2}\pi - t) + \sin^4(\tfrac{1}{2}\pi + t) + \cos^4(\pi - t) \\ &= 2\sin^4 t + 2\cos^4 t. \end{aligned}$$

Now we can use an old trick, rather like Number 1 above. Namely $a^2 + b^2 = (a+b)^2 - 2ab$. Applied to $a = \sin^2 t$ and $b = \cos^2 t$ this makes

$$X = 2(\sin^2 t + \cos^2 t)^2 - 4\sin^2 t\cos^2 t.$$

Now use equations (1) and (4), and also one of the sine values from the table (remembering $t = \frac{\pi}{8}$). See if you can arrive at the final answer: $X = \frac{3}{2}$.