Tangles, Fractions and Factors

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1 How to play the game of Tangles

The game of Tangles is played with two pieces of string of contrasting colours. The game starts at position 0, which is illustrated in Fig. 1. Each subsequent

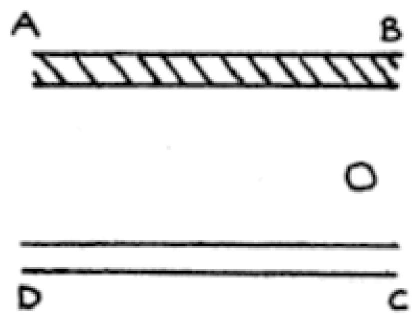

Figure 1: The start: position 0.

position of the game will also be labelled by a fraction (with one exception—see the penultimate paragraph of this section). We will demonstrate the game with four people, A, B, C and D, each holding one end of string. They start in the positions shown in Fig. 1. There are two allowed moves in the game:

- **Twist 'em up** For this move, the two people on the right swap places and, as they swap over, the person who was at top right lifts her string over the string of the person who was at bottom right. For example, when we start from 0 (Fig. 1), B and C swap places, and B lifts her string over C's. Whenever we **Twist 'em up**, we add one to the position. We write this as

$$\textbf{Twist 'em up}: \ t \mapsto t+1.$$

So, since we started from position 0, we are now at position 1 (Fig. 2). Let's **Twist 'em up** again. So C and B swap places again, and C lifts

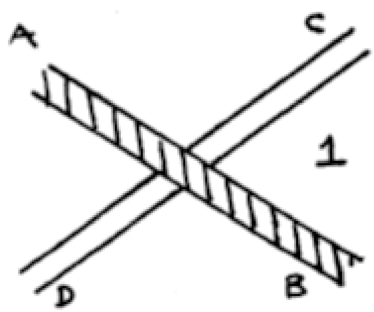

Figure 2: Position 1.

his string over B's. And we add one to the position again, so we are now at position 2 (Fig. 3). And let's **Twist 'em up** one more time, to

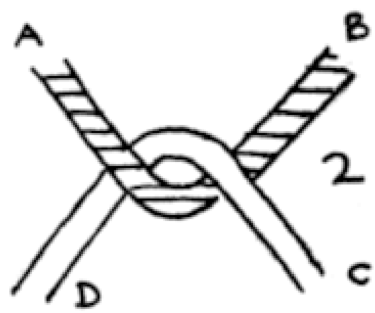

Figure 3: Position 2

get to position 3 (Fig. 4).

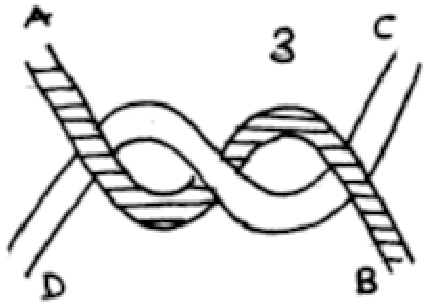

Figure 4: Position 3.

- **Turn 'em round** This is the other move of the game. For this move, the person at top left moves to top right, the person at top right moves to bottom right, the person at bottom right moves to bottom left, and the person at bottom left moves to top left. So, to apply **Turn 'em round** to position 3 (Fig. 4), A moves to where C was standing, C moves to where B was standing, B moves to where D was standing, and D moves to where A was standing. Whenever we apply **Turn 'em round** to position t, we arrive at position $-1/t$. We write this as

$$\textbf{Turn 'em round}: \ t \mapsto -1/t.$$

So, since we were at position 3, we are now at $-1/3$ (Fig. 5).

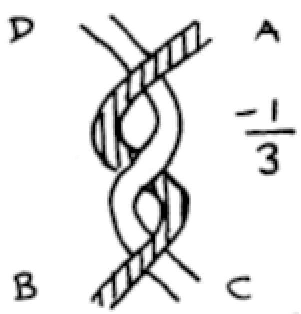

Figure 5: Position $-1/3$.

We won't **Turn 'em round** twice in a row (what would happen if we did?). Instead let's **Twist 'em up** (A and C swap places, A lifts her string over C's). As before, we add one to the position. We were at $-1/3$, so we are now at $2/3$. Something interesting happens here. The string can be made to lie in one of at least four different forms (Fig. 6). Notice that only two of

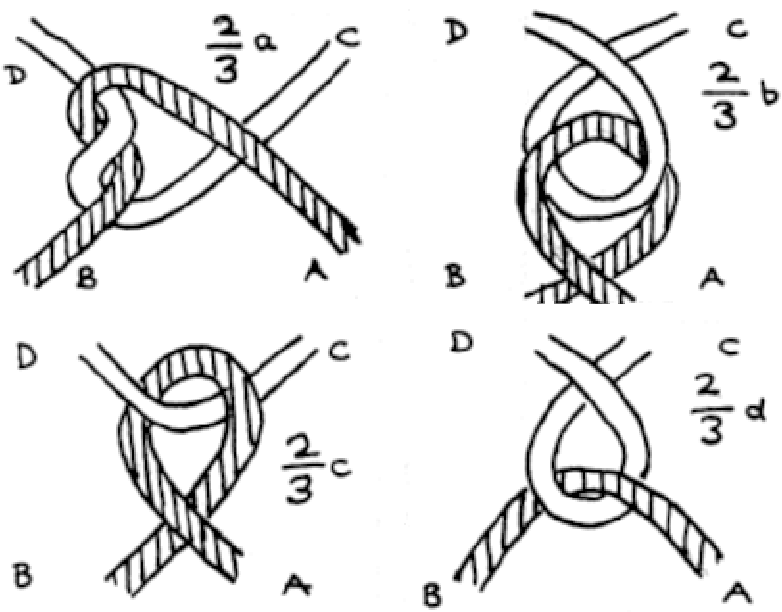

Figure 6: Four different forms for position 2/3.

them (a and b) are symmetric, in the sense that both strings make the same shape. Let's **Twist 'em up** again, to get from $2/3$ to $5/3$. Again, there are several different forms, but we only illustrate one (Fig. 7).

Now that we have seen how the game works, here is a challenge. Can you get back from $5/3$ to 0? Remember that you are only allowed to use the two moves **Twist 'em up** and **Turn 'em round**. Twisting or Turning backwards is not allowed, and you can only **Twist 'em up** on the right hand side. It might help later on if you keep a record of the moves you make, and the sequence of fractions you pass through on the way.

Try following these tangle trails. Again, keeping a record may help later.

a) $0 \mapsto 2 \mapsto 3/2 \mapsto 4/3 \mapsto 5/4 \mapsto 6/5 \mapsto \cdots$

b) $0 \mapsto 1 \mapsto 3/2 \mapsto 8/5 \mapsto 21/13 \mapsto 55/34 \mapsto \cdots$

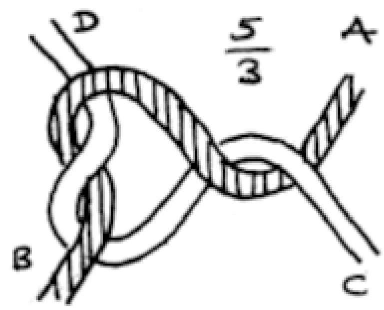

Figure 7: One of the forms for position 5/3.

The sequence of integers f_n defined by $f_0 = f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ is called the Fibonacci sequence.

Here are some more exercises. Again, it may help to keep a record. As extra exercises, try getting back to zero.

c) $0 \mapsto 11/2$, $0 \mapsto 11/3$, $0 \mapsto 11/4$, $0 \mapsto 11/6$.

d) $0 \mapsto 2/5$, $0 \mapsto 3/5$, $0 \mapsto 5/7$, $0 \mapsto 5/8$, $0 \mapsto 4/9$, $0 \mapsto -3$.

e) $0 \mapsto 1/2$, $0 \mapsto 1/3$, $0 \mapsto 1/4$, $0 \mapsto 1/5$, $0 \mapsto 1/6, \ldots$. Is there a pattern here? Can you guess how to do $0 \mapsto 1/7$, $0 \mapsto 1/8$, $0 \mapsto 1/9, \ldots$?

f) $0 \mapsto 11/5$, $0 \mapsto 11/7$, $0 \mapsto 11/8$, $0 \mapsto 11/10$.

g) $0 \mapsto 11/9$.

One form for the first tangle in each section is illustrated in Fig. 8. As for 2/3, there may be several different forms for each one. It seems that there is always a symmetric form, in which both strings make the same shape.

Here are some other things you might like to try. Start from any tangle, not necessarily zero. What happens when you **Twist 'em up**, **Turn 'em round**, **Twist 'em up**, **Turn 'em round**, **Twist 'em up**, **Turn 'em round**. Does it matter which tangle you start from? Now start from 0. What happens if you **Turn 'em round** straight away? Can you describe this position with a fraction? Now, from the new position, **Twist 'em up**. What position are you at now?

The game of Tangles is the most beautiful piece of mathematics I know. I find the symmetric forms particularly beautiful. And it amazes me that

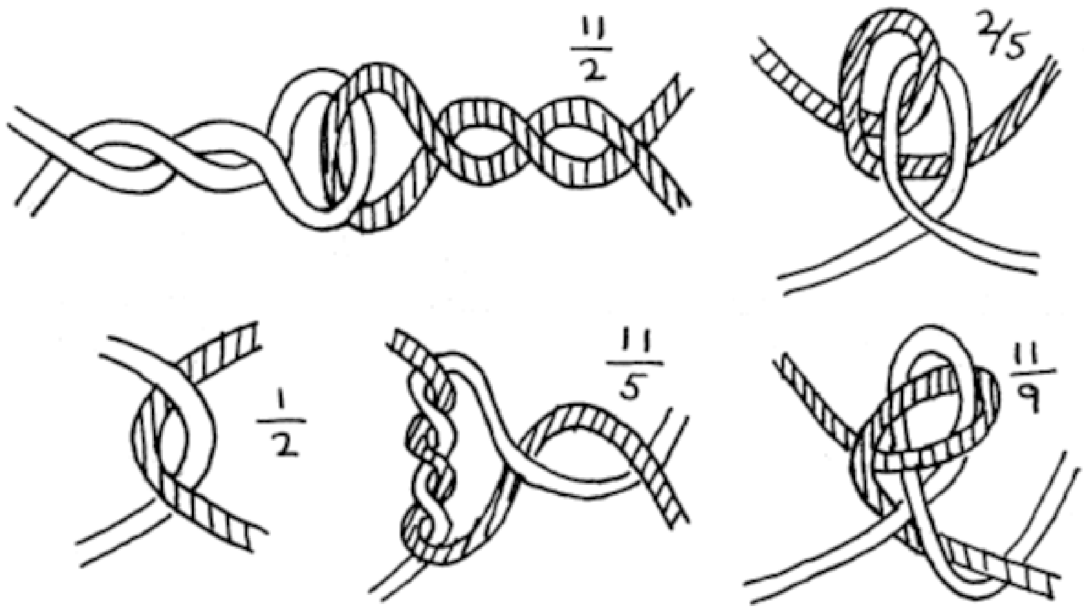

Figure 8: Some of the tangles for the exercises.

something so complicated (a tangle of string) can correspond to something so simple (a fraction) in such a simple way.

2 Highest Common Factor

What is a highest common factor? We will explain one word at a time. Let M and N be positive whole numbers. We say that M is a *factor* of N if M divides exactly into N, with no remainder. For example, the factors of 14 are 1, 2, 7, 14, and the factors of 34 are 1, 2, 17, 34. The *common factors* of M and N are the numbers which are factors of both. For example, the common factors of 14 and 34 are 1 and 2. The *highest common factor* of M and N is the highest of the common factors of M and N. The highest common factor of 14 and 34 is 2. We can write the last sentence in symbols as $\mathrm{hcf}(14, 34) = 2$. In principle, we could find the highest common factor of any two numbers in the same way, by finding the factors of each number, then finding the common factors, and hence finding the highest common factor. But in practice, it can be very hard to find the factors of large numbers. Fortunately there is a method, called the Euclidean algorithm, for reducing the size of the numbers under consideration. We will actually use an adaptation of the Euclidean algorithm which will find highest common factors, and find a route to any Tangle. The method is best illustrated with an example.

Let us calculate the highest common factor of 14 and 34 by the new method. The trick is to find a number n which is smaller than both 14 and 34, but such that $\text{hcf}(14, 34) = \text{hcf}(n, 14)$. How do we find this number n? Refer to the number line in Fig. 9. We make a row of blocks of the smaller

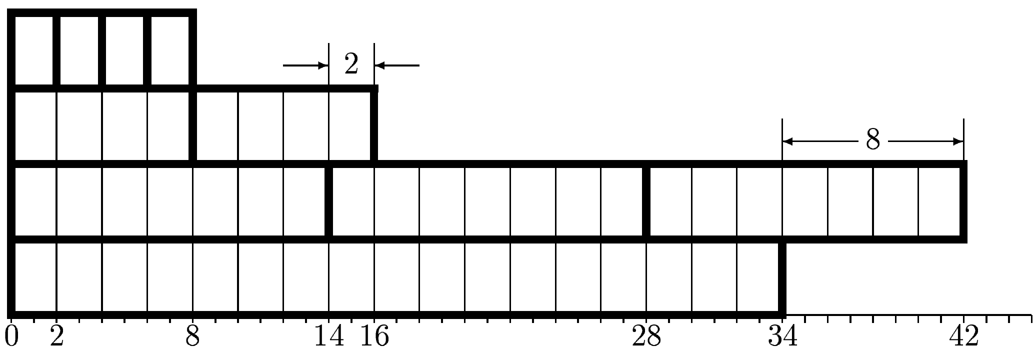

Figure 9: A number line to illustrate the method of finding the hcf.

number, 14, until it equals or exceeds the larger number, 34. We see that we need a total of three blocks of 14, of total length $3 \times 14 = 42$, in order to exceed 34. Now, we can see from Fig. 9 that the difference in length between the two rows will be made up from the same highest common factor as the two original numbers, 14 and 34. So we can choose the number n to be the difference in length between the two rows, that is $n = 42 - 34 = 8$, and we have $\text{hcf}(8, 14) = \text{hcf}(14, 34)$. It is important to keep a record by writing down the equation

$$34 = 3 \times 14 - 8.$$

So now we have simplified our problem by replacing the large number 34 by the smaller number 8. But we can simplify further by applying the same method once again. The smaller number is now 8. We need to build this up until it exceeds or equals 14, so we need one more block of 8 to make a row of length 16. The difference in length is $16 - 14 = 2$, so we have $\text{hcf}(14, 34) = \text{hcf}(8, 14) = \text{hcf}(2, 8)$. Again, it is important to keep a record:

$$14 = 2 \times 8 - 2.$$

If we try to simplify again we find that we can make 8 exactly out of blocks of 2:

$$8 = 4 \times 2.$$

This is the sign that we have found the answer: the highest common is then the last number n that we found, i.e. $\mathrm{hcf}(14, 34) = 2$. Encouragingly, this agrees with the answer we found by factoring 14 and 34.

Here is one more example. Let us calculate $\mathrm{hcf}(91, 343)$. The sequence of equations is

$$\begin{aligned} 343 &= 4 \times 91 - 21, \\ 91 &= 5 \times 21 - 14, \\ 21 &= 2 \times 14 - 7, \\ 14 &= 2 \times 7, \end{aligned}$$

so the answer is $\mathrm{hcf}(91, 343) = 7$. As further exercises, try showing that $\mathrm{hcf}(738, 921) = 3$, $\mathrm{hcf}(496, 899) = 31$ and $\mathrm{hcf}(644, 821) = 1$. Please do feel free to use a calculator for the arithmetic (I used a computer). See how much easier this is than factoring.

What has all this got to do with Tangles? We will show how to use our new method for highest common factors to find the Tangle 14/9. First we will find $\mathrm{hcf}(9, 14)$:

$$\begin{aligned} 14 &= 2 \times 9 - 4, \\ 9 &= 3 \times 4 - 3, \\ 4 &= 2 \times 3 - 2, \\ 3 &= 2 \times 2 - 1, \\ 2 &= 2 \times 1, \end{aligned}$$

so the answer is $\mathrm{hcf}(9, 14) = 1$. Now we can simply read off the route! Look at the sequence of multipliers on the right hand side: working upwards, they are 2, 2, 2, 3, 2. Therefore we can get from 0 to 14/9 by 2×**Twist 'em up** (to position 2), **Turn 'em round** (−1/2), 2×**Twist 'em up** (3/2), **Turn 'em round** (−2/3), 2×**Twist 'em up** (4/3), **Turn 'em round** (−3/4), 3×**Twist 'em up** (9/4), **Turn 'em round** (−4/9) and 2×**Twist 'em up** (14/9). Why does this work?

One last note: when finding a route to a Tangle like 5/9 in which the numerator is less than the denominator, start the hcf calculation by writing

$5 = 9 - 4$, i.e. using the difference in length between the larger number and the smaller number. Then write $9 = 3 \times 4 - 3$, i.e. replace the smaller number with the difference. Then continue as normal.

3 History

Tangles were discovered in around 1955 by Prof. John H. Conway while he was still at school here in Liverpool. For more on Tangles and Knots, [1, §2.3] is very highly recommended. You could also try [2]. Other work of Prof. Conway can be found in [3, 4]. Some Web references are: [5, 6, 7]

The method for highest common factors described in §2 is adapted from the Euclidean algorithm, which dates from around 300BC. Most books on 'Number Theory' will cover the Euclidean algorithm; see for example [8, §1.6 and §1.8] or [9, §1.2]. Continued fractions, which demonstrate the relation between fractions and the Euclidean algorithm, are also covered in most books on Number Theory; see for example [8, §4.1]. How does our adaptation of the Euclidean algorithm affect continued fractions?

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