Noncommuting limits and effective properties for oblique propagation of electromagnetic waves through an array of aligned fibres

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We consider formulations for the Helmholtz operator for conically propagating modes on an array of high contrast ferromagnetic cylindrical inclusions in the limit when their wavelength outside the inclusions tends to infinity. If one considers a trajectory in the coordinates (frequency, conical parameter), then the effective phase refractive index will depend on this trajectory in the vicinity of the origin.

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I. INTRODUCTION

The task of finding the effective properties for a medium in the context of electrostatics is an old one, and it is a task which has been worked on extensively from the writings of Lorenz 1 and Lorentz 2 until the present day. However, when the fields exhibit some dependence on time one moves into the area of electromagnetism, and this is an area where the designation of effective properties is perhaps less well studied. 3 Interesting mathematical studies are related to effective properties of inhomogeneous media containing highly conducting cylinders (see e.g. Refs. 4 and 11), and in particular to the influence of the skin effect on the effective moduli of high-contrast composites. The idea that an array of metallic photonic crystals may be a technological base for newly discovered left-handed materials (LHM) appears as a result of analytical, numerical and experimental studies 4-7 which have found the electromagnetic waves in these high-contrast microstructured fibres propagating above a very low cutoff frequency.

Many researchers find that they can confidently apply the traditional static formulas in this dynamic regime. This technique is generally known as the quasistatic limit and relies, often tacitly, on a set of assumptions about the fields in the material; namely, that their wavelengths are sufficiently long and that their frequencies are sufficiently low compared to the size of an elementary cell within a composite structure. Difficulties can be demonstrated with the supposition of the quasistatic limit in the case when the material possesses regions with a high dielectric constant, since the wavelength in these regions becomes finite whilst remaining infinite in other parts of the medium. On the other hand, in a highly conducting body the amplitude of the electric field is suppressed and the magnetic field remains almost constant, and so the temptation exists to simply replace these regions with materials which obey the requisite Dirichlet or Neumann conditions. If this is done then the boundary layer which exists near the high-contrast interface is neglected, and the effective properties of the material change markedly depending on whether the long-wavelength limit or the high-conductivity limit is taken first. 8,9 It appears then that the resulting effective properties of the body depend on the trajectory of a path in an abstract space involving wavelength and conductivity. 10

Homogenization problems of electromagnetism in high contrast microstructured media were analyzed by Felbacq and Bouchitte 11 and Felbacq 12,13 In particular, they studied effects of noncommuting limits for long-wave asymptotic approximations in transverse polarization of electromagnetic signals.

In this paper we demonstrate that there are further limitations on taking the quasistatic limit. We broaden previous analysis 10 to examine the important case of oblique propagation of electromagnetic waves through an array of aligned fibres, and also to include ferro-magnetic materials. We show that there is indeed a clear relationship between the static effective properties of the material and their dynamic equivalents, and that this again depends on the order in which various limits are taken. By this we hope to clarify the situation for those who wish to apply static effective medium theory to their dynamic models. The duality relations for conical incidence were studied in Ref. 14 using a Fourier expansion method in the low-frequency limit.

II. MOTIVATIONS

It is known from the existing literature 8,10 that effects of noncommuting limits in models of composite structures are associated with asymptotic theory known as the theory of homogenization (see Ref. 22 and Appendix). The idea of homogenization is very simple: one would like to replace multiscaled composite structures by homogeneous ones with averaged “effective” properties. Sometimes this idea works, sometimes it does not; it is important from the point of view of practical applications to identify the cases where homogenization fails and hence replacement of a composite by a uniformly homogeneous material is not possible. These special cases are also referred to as “multiple porosity problems” where the leading term in the asymptotic approximation of the field still depends on the small parameter characterizing the scaling within the structure of the composite. A good illustrative example was thoroughly described in Ref. 10, which was associated with the analysis of a quasistatic limit...
for fields within a composite containing high-contrast inclusions.

The second important issue linked with the effect of non-commuting limit refers to local analysis near the boundaries and interfaces. In particular, in Ref. 10 the authors identified the presence of a boundary layer and hence of a singular perturbation in the mathematical model. Although a singular perturbation analysis may look rather formal, the boundary layers have very clear physical interpretation and they represent high-gradient physical fields near the interfaces and/or exterior contours. The detailed study of a boundary layer itself is a separate and important problem. However, it is even more important to be able to identify physical configurations and corresponding mathematical models where boundary layers occur. The motivation for the analysis given in this paper came from our work on modeling of photonic crystal fibres.15–18 The practical importance of homogenization models and effective phase refractive index for photonic crystal fibres structures is discussed in Ref. 19. Anyone who has done a thorough analysis of physical fields within photonic crystal fibres observes numerical instability for the case of oblique propagation of electromagnetic fields. We feel that this modeling difficulty. We classify it as an effect of non-commuting limits and the formal details are given in the text below.

III. FORMULATION OF THE SPECTRAL PROBLEM

We examine the case of time-harmonic electromagnetic waves moving through an array of identical fibres which are infinitely extended in the $z$ direction and are spaced periodically in the $(x,y)$ plane. Specifically, we consider modes which propagate obliquely through the array (in this case the fields have nonzero axial components), and so have the form

$$E(x) = E(x,y)e^{i\beta z}, \quad (1)$$

$$H(x) = H(x,y)e^{i\beta z}. \quad (2)$$

We allow the fibres to have arbitrary cross section and a permittivity $\epsilon$ and permeability $\mu$ which are different to the surrounding matrix material, which we assume to be a vacuum. Outside the inclusions the electric and magnetic fields satisfy the vector Helmholtz equations

$$(\nabla^2 + \omega^2/\epsilon^2)E = 0, \quad (3)$$

$$(\nabla^2 + \omega^2/\epsilon^2)H = 0, \quad (4)$$

where $\omega$ is the radian frequency. Within the inclusions the fields satisfy the equations

$$(\nabla^2 + n^2\omega^2/c^2)E = 0, \quad (5)$$

$$(\nabla^2 + n^2\omega^2/c^2)H = 0, \quad (6)$$

where $n = \sqrt{\epsilon\mu}$ is the refractive index of the inclusions, defined relative to the surrounding material.

Because the material is periodic in each of the three coordinate directions, all fields must satisfy the Bloch-Floquet condition

$$E(x+d) = E(x)e^{ik_{\text{bloch}} \cdot d}, \quad (7)$$

$$H(x+d) = H(x)e^{ik_{\text{bloch}} \cdot d}, \quad (8)$$

$$d = (pd_1,qd_2,rd_3), \quad (9)$$

where $p,q,r$ are integers and $d_1,d_2,d_3$ are the periods of the array in the three directions. The vector $k_{\text{bloch}}$ is known as the Bloch vector and is the equivalent of the quasimomentum in solid-state physics.

The relationship between the Bloch vector $k_{\text{bloch}}$ and the radian frequency $\omega$ defines the effective phase refractive index of the array (here we use the standard definition; see, for example, Refs. 8–10), through the relation

$$N_{\text{eff}} = \lim_{\omega,k_{\text{bloch}} \to 0} \frac{\partial \omega}{\partial k_{\text{bloch}}}, \quad (10)$$

from which it is clear that the effective refractive index will depend on the orientation of $k_{\text{bloch}}$. The quantity $N_{\text{eff}}$ is inherently dynamic, and could be ascertained for a material by measuring the phase shift of a beam of light passing through the material in different directions. We henceforth write

$$k_{\text{bloch}} = (0,k_1,\beta), \quad (11)$$

and thereby assume, without loss of generality, that the wave propagates in the $(y,z)$ plane.

There is an apparent link with the two-scale homogenization problem where instead of doubly periodic array one would consider a finite size body containing a microstructure. Within such a microstructure we can select an elementary cell of small diameter and construct an asymptotic approximation of physical fields in terms of slow and fast variables. An outline of such an asymptotic analysis of the oblique incidence problem is given in the Appendix.

Because the rods are infinitely extended in the $z$ direction, the $x$ and $y$ components of the $E$ and $H$ fields can be reconstructed from their $z$ components, via the equations

$$E_z = \frac{1}{n^2\omega^2 - \beta^2}(i\beta \nabla_z E_z - i\omega \mu e_z \times \nabla_z H_z), \quad (12)$$

$$H_z = \frac{1}{n^2\omega^2 - \beta^2}(i\beta \nabla_z H_z + i\omega \epsilon e_z \times \nabla_z E_z). \quad (13)$$

It should be noted that in the subsequent analysis we retain the $z$ dependence of the fields.

The boundary conditions which apply on the surface of each inclusion come from the continuity of the tangential components of the $E$ and $H$ fields. If we write the tangent vector at any given point on the inclusion surface as $t$ and the normal vector as $n$ then the quantities which must be continuous across the inclusion boundary are
It is shown in Ref. 10 that boundary conditions will be obtained in the limits

\[ \begin{align*}
\mathbf{n} \times \mathbf{E} &= \mathbf{E}_z, \\
\mathbf{n} \times \mathbf{H} &= \mathbf{H}_z^i,
\end{align*} \]

and

\[ \begin{align*}
\frac{i \beta}{\omega^2 - \beta^2} \frac{\partial E_z}{\partial t} - \frac{i \omega}{\omega^2 - \beta^2} \frac{\partial H_z}{\partial n} &= \frac{i \beta}{n^2 \omega^2 - \beta^2} \frac{\partial E_z^i}{\partial t} - \frac{i \omega}{n^2 \omega^2 - \beta^2} \frac{\partial H_z^i}{\partial n}, \\
\frac{i \beta}{\omega^2 - \beta^2} \frac{\partial H_z}{\partial t} + \frac{i \omega}{\omega^2 - \beta^2} \frac{\partial E_z}{\partial n} &= \frac{i \beta}{n^2 \omega^2 - \beta^2} \frac{\partial H_z^i}{\partial t} + \frac{i \omega}{n^2 \omega^2 - \beta^2} \frac{\partial E_z^i}{\partial n}.
\end{align*} \]

From the above equations it can be seen that different boundary conditions will be obtained in the limits \( e, \mu \rightarrow \infty \), and \( \beta, \omega \rightarrow 0 \), depending on which limit is taken first. The task is now to see what effect this has on the calculation of the material’s effective refractive index.

**IV. OBLIQUE PROPAGATION OF SMALL ANGLE**

We consider the case when \( \beta_{\text{max}}, \mu, \varepsilon \ll 1 \), and the dielectric and magnetic constants \( \varepsilon \) and \( \mu \) associated with the inclusions are large. We also assume that these constants are related by \( \mu / \varepsilon \ll 1 \). Both equations, for \( E_z^i \) and \( H_z^i \), are singularly perturbed [see Eqs. (5) and (6)], as the product \( \varepsilon \mu \gg 1 \). If the boundary layer near the interface \( C \) (Fig. 1) is neglected then

\[ E_z^i = E_z |_C = 0. \]

It is shown in Ref. 10 that

\[ -\omega^2 = \int_{C_{\Omega}} \frac{\partial E_z}{\partial n} ds + O(k_{\text{block}}), \]

where \( \mathbf{n} \) is the inward normal to the boundary \( C \) of the cylindrical fibre, and \( \Omega \) the cubic elementary cell (Fig. 1).

Hence, when \( k_{\text{block}} \) tends to 0, \( \omega \) takes a finite positive value and the effective refractive index \( N_{\text{eff}} \) cannot be defined when a “charge neutrality” condition within the elementary cell is not satisfied. This phenomenon is known in the literature as the effect of noncommuting limits and it was studied in Ref. 10. The effective refractive index becomes continuous if the boundary layer near the interface is properly taken into account.

Assuming that \( H_z \) is represented in the form

\[ H_z = \sum_{k=0}^{\infty} \beta^k h_z^k(\varepsilon, \mu, \omega), \]

we neglect all terms of order \( O(\beta^3) \). In this case, the transmission condition (18) implies

\[ \left. \frac{\partial H_z}{\partial n} \right|_C = 0, \]

to second order in \( \beta \). We note that Eq. (23) holds both for \( H_z \) and \( \mathcal{H}_z \) due to the definition (2) and the fact that \( \mathbf{n} \) is perpendicular to the \( z \) axis. The Helmholtz equation for \( H_z \) takes the form

\[ (\nabla^2 + \omega^2) H_z = 0, \]

where \( \nabla^2 \) stands for the Laplacian in the \((x, y)\) plane. Also, \( \beta \) is explicitly involved in the system of the Bloch-Floquet conditions on the outer boundary of \( \Omega \):

\[ H_z(x, y + d, z) = H_z(x, y, z) e^{i kd}, \]

\[ H_z(x, y, z + d) = H_z(x, y, z) e^{i \beta d}. \]

We remark that the different situation when \( \varepsilon / \mu \ll 1 \) would imply that \( H_z \) would be replaced by \( E_z \) in the above equations (23), (24), and (25).

We now introduce a new problem for the field \( \mathcal{H}_z \), which we will describe as being a perturbation on the conjugate of \( H_z \). \( \mathcal{H}_z \) satisfies the Helmholtz equation

\[ (\nabla^2 + \omega^2) \mathcal{H}_z = 0, \]
with a Bloch-Floquet condition which is conjugate to the one in Eq. (25):
\[
\vec{H}_\perp(x,y+d,z) = \vec{H}_\perp(x,y,z) e^{-i\vec{k}d},
\]
\[
\vec{H}_\perp(x,y,z+d) = \vec{H}_\perp(x,y,z) e^{-i\vec{\beta}d}.
\]  
In Eqs. (27) we have perturbed the components of the Bloch vector, so that
\[
\vec{k} = k + \delta k, \quad \vec{\beta} = \beta + \delta \beta.
\]  
This in turn will affect the allowed value of the frequency of the vibration, so that
\[
\tilde{\omega} = \omega + \delta \omega.
\]  
We also specify that \( \vec{H}_\perp \) satisfies the same boundary conditions as \( \vec{H}_\perp \), namely the Neumann condition (23).

We now apply Green’s theorem in the region \( \Omega \) (Fig. 1) using the functions \( \vec{H}_\perp \) and \( \vec{H}_\perp \). Thus,
\[
\int_\Omega (\nabla \vec{H}_\perp \cdot \nabla \vec{H}_\perp) dv = \int_{\partial \Omega} \left( \frac{\partial \vec{H}_\perp}{\partial n} - \vec{H}_\perp \frac{\partial \vec{H}_\perp}{\partial n} \right) ds.
\]  
The integral over the inner boundary vanishes due to the condition (23), and so we deduce that
\[
(-\tilde{\omega}^2 + \omega^2) \int_\Omega \vec{H}_\perp \cdot \vec{H}_\perp dv = \int_{\partial [L\cup R\cup B\cup T]} \left( \frac{\partial \vec{H}_\perp}{\partial n} - \vec{H}_\perp \frac{\partial \vec{H}_\perp}{\partial n} \right) ds,
\]  
where \( L,R,B,T \) are the surfaces on the left, right, bottom and top parts of the boundary of the elementary cell (Fig. 1). By using the Bloch-Floquet conditions (25) and (27) we can reduce the integrals appearing on the right-hand side of Eq. (31). Thus
\[
\int_{\partial [L\cup R]} \left( \frac{\partial \vec{H}_\perp}{\partial n} - \vec{H}_\perp \frac{\partial \vec{H}_\perp}{\partial n} \right) ds = (1 - e^{i\beta d} e^{-i\vec{\beta}d}) \int_R \left( \frac{\partial \vec{H}_\perp}{\partial z} - \vec{H}_\perp \frac{\partial \vec{H}_\perp}{\partial z} \right) ds.
\]  
Similarly, we find that
\[
\int_{\partial [B\cup T]} \left( \frac{\partial \vec{H}_\perp}{\partial n} - \vec{H}_\perp \frac{\partial \vec{H}_\perp}{\partial n} \right) ds = (1 - e^{i\beta d} e^{-i\vec{\beta}d}) \int_T \left( \frac{\partial \vec{H}_\perp}{\partial z} - \vec{H}_\perp \frac{\partial \vec{H}_\perp}{\partial z} \right) ds.
\]  
We then have the following exact relation between the \( \omega \) and \( \tilde{\omega} \):
\[
(-\tilde{\omega}^2 + \omega^2) \int_\Omega \vec{H}_\perp \cdot \vec{H}_\perp dv = \int_R \left( \frac{\partial \vec{H}_\perp}{\partial z} - \vec{H}_\perp \frac{\partial \vec{H}_\perp}{\partial z} \right) ds.
\]  
We now look at what happens to this relation when both \( k \) and \( \beta \) are small. First we introduce the auxiliary static fields \( u_1 \) and \( u_2 \), which satisfy
\[
\nabla^2 u_1 = 0, \quad u_1(x,y,z+d) = u_1(x,y,z) + d \quad \text{in} \quad \Omega
\]
\[
\frac{\partial u_1}{\partial n} = 0 \quad \text{on} \quad B \cup T,
\]
\[
\nabla^2 u_2 = 0, \quad u_2(x,y+d,z) = u_2(x,y,z) + d \quad \text{in} \quad \Omega
\]
\[
\frac{\partial u_2}{\partial n} = 0 \quad \text{on} \quad L \cup R.
\]  
These two static problems possess their own static effective properties, which are defined by the average flux of each field through the side of the unit cell. The field \( u_1 \) corresponds to a static problem where there exists a unit applied field in the \( z \) axis. We can define the static effective property as
\[
\sigma_{zz}^{\text{eff}} = \frac{1}{d^2} \int_L \frac{\partial u_1}{\partial z} ds.
\]  
In a similar way the problem (36) corresponds to interaction with a model applied field in the \( y \) direction, thus
\[
\sigma_{yy}^{\text{eff}} = \frac{1}{d^2} \int_B \frac{\partial u_2}{\partial y} ds.
\]  
We would like to formally relate these static properties to the dynamic refractive index.

We note that we can use the following approximation for \( \vec{H}_\perp \):
\[
\vec{H}_\perp(x) = e^{i(\beta u_1(x) + k u_2(x))} + O(\beta^2 + k^2)
\]
\[
= 1 + i \beta u_1(x) + i k u_2(x) + o(|\beta| + |k|).
\]  
It can be seen that this approximation will satisfy all the boundary conditions (including the Bloch-Floquet condition) exactly, as well as satisfying the Helmholtz equation to first order. The perturbed problem for \( \vec{H}_\perp \) can be approximated in a similar way:
\[
\vec{H}_\perp(x) = e^{i(\beta u_1(x) + k u_2(x))} + o(|\beta| + |k|).
\]  
The integral on the left-hand side of Eq. (34) is then
where $\mu(\Omega)$ is the measure (or volume) of the region $\Omega$.

The integrals on the right-hand side of Eq. (34) can be evaluated. The result for the integral over the surface $L$ is

\begin{equation}
(1 - e^{i\beta d - i\beta d}) \int_L \left( \frac{\partial H_z}{\partial z} - \frac{\partial H_z}{\partial y} \right) ds
= (\beta^2 - \beta^2) d \int_L \frac{\partial u_1}{\partial z} ds + O(\beta^3 + k^3).
\end{equation}

In a similar way we can show that

\begin{equation}
(1 - e^{i\beta d - i\beta d}) \int_B \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_z}{\partial y} \right) ds
= (k^2 - k^2) d \int_B \frac{\partial u_2}{\partial y} ds + O(\beta^3 + k^3).
\end{equation}

Combining these with Eq. (41) we obtain

\begin{equation}
(\omega^2 - \omega^2) \mu(\Omega) = (\beta^2 - \beta^2) d \int_L \frac{\partial u_1}{\partial z} ds
+ (k^2 - k^2) d \int_B \frac{\partial u_2}{\partial y} ds + O(\beta^3 + k^3).
\end{equation}

Because $\tilde{\omega} = \omega + \delta \omega$ we know that

\begin{equation}
\omega^2 - \omega^2 = (\omega + \delta \omega)(\omega - \delta \omega) \approx 2 \omega \delta \omega.
\end{equation}

Similarly we have for $\tilde{\beta}$ and $\tilde{k}$:

\begin{equation}
\beta^2 - \beta^2 \approx 2 \beta \delta \beta, \quad k^2 - k^2 \approx 2 k \delta k,
\end{equation}

and so Eq. (44) becomes

\begin{equation}
\mu(\Omega) \omega \delta \omega \beta \delta \beta \delta k d \int_L \frac{\partial u_1}{\partial z} ds + k \delta k d \int_B \frac{\partial u_2}{\partial y} ds
+ O(\beta^3 + k^3).
\end{equation}

The integrals on the right-hand side correspond to the average flux through each side of the central unit cell, in response to a unit applied field in the relevant direction. They are related to the static effective properties $\sigma_{zz}^{\text{eff}}$ and $\sigma_{yy}^{\text{eff}}$ via Eqs. (37) and (38), and so we have

\begin{equation}
\mu(\Omega) \omega \delta \omega \beta \delta \beta \delta k d \int_L \frac{\partial u_1}{\partial z} ds + k \delta k d \int_B \frac{\partial u_2}{\partial y} ds
+ O(\beta^3 + k^3).
\end{equation}

Thus we can define the dynamic effective refractive index in terms of the static quantities for the same material:
the coordinates as $x_1$, $x_2$ and $x_3$). Then, rather than letting the Bloch vector tend towards zero, we introduce a small parameter $\eta$, which characterises the pitch of the array, i.e., the distance between the centers of two adjacent cylinders (see Fig. 2).

We are looking for propagating modes $(E_\eta, H_\eta)$, defined by Eqs. (1) and (2), in a metallic waveguide of cross section $\Omega_f$ filled with a periodic assembly of ferromagnetic rods. We assume that the wavelengths of these modes to be large compared to the size of the inclusions (the diameter of the rods) but otherwise they may be in resonance with $\Omega_f$. Since the tangential trace $n/\nabla \mathbf{H}_\eta$ of the magnetic field $\mathbf{H}_\eta$ on the boundary $\partial \Omega_f$ is an unknown of the spectral problem, we choose an electric field formulation: for a given propagation constant $\beta > 0$ we are therefore looking for pairs $(k^2_\beta, E_\eta)$. $E_\eta \in \{\mu \in L^2(\Omega_f, \mathbb{C}^2), \text{curl}_\beta \mu = 0\}$ such that

$$ (P^E_\eta) \begin{cases} e^{-1}_\eta \text{curl}_\beta \mu^{-1}_\eta(x) \text{curl}_\beta E_\eta(x) = k^2 E_\eta(x) \quad &\text{in } \Omega_f \\ n \nabla E_\eta = 0 \quad &\text{on } \partial \Omega_f \end{cases} $$

(A1)

where $d_\beta \{\text{curl}_\beta, \text{div}_\beta\}$ is defined as $d_\beta E_\eta(x_1, x_2) = d_\beta E_\eta(x_1, x_2) e^{i\beta x_3} e^{-i\beta x_3}$ and $k_\eta$ denotes the spectral parameter. The permittivity and permeability

$$ e_\eta(x) = \frac{x_3}{\eta}, \quad \mu_\eta(x) = \mu \left( \frac{x_3}{\eta} \right), $$

need only be $\eta$-periodic coercive and bounded matrices ($Y = \mathbb{R}$). Here, they are piecewise constant functions. The spectrum of the operator associated with the spectral problem $P^E_\eta$ consists of a discrete set of isolated eigenvalues in $[\beta^2(e_\eta \mu_\eta^{-1}) + \infty)$. The main idea of homogenization theory (see, for example, Refs. 11, 21, 23, and 25) is to select two scales in the study: a microscopic one (the size of the basic cell $\eta Y$) and a macroscopic one (the size of the whole waveguide cross-section of shape $\Omega_f$). From a physical point of view, one can say that the modulus of the propagating field is forced to oscillate due to rapid changes in the permittivity and the permeability within the microstructure. In fact, the smaller the diameter (of order $\eta$) of the rods, the faster the modulus of the field $E_\eta$ oscillates. Hence, we assume that $E_\eta$, solution of the problem $(P^E_\eta)$ has a two-scale expansion of the form

$$ \forall x \in \Omega_f, \quad E_\eta(x) = E_\eta \left( \frac{x}{\eta} \right) + \eta \mathbf{E}_\eta \left( \frac{x}{\eta} \right) + \eta^2 \mathbf{E}_2 \left( \frac{x}{\eta} \right) + \cdots, $$

where $E_\eta : \Omega_f \times Y \rightarrow \mathbb{C}^2$ is a smooth function of 4 variables, independent of $\eta$, such that $\forall x \in \Omega_f, \quad E_\eta(x, \cdot) \equiv Y$ periodic.

Our goal is to characterize the propagating modes when $\eta$ tends to zero. If the coefficients $E_\eta$ do not increase “too much” when $\eta$ tends to zero, the limit of $\mathbf{E}_\eta$ will be $\mathbf{E}_h$, the rougher approximation of $\mathbf{E}_\eta$. Hence, we make the assumption that for all $x \in \mathbb{R}^3$, $E_\eta(x, \eta) = o(x/\eta^2)$, so that the expansion (also denoted by the German word “ansatz”) still makes sense in the neighborhood of 0. If the above expansion is relevant, we can state the following fundamental result.23,24

When $\eta$ tends to zero, $E_\eta$ solution of the problem $(P^E_\eta)$, converges weakly in $L^2(\Omega_f)$ to the solution $E_{\text{hom}}$ of the spectral problem $(P^E_{\text{hom}})$ defined as follows: for a given $\beta > 0$, we look for pairs $(k^2_{\text{hom}}, E_{\text{hom}})$, $E_{\text{hom}} \neq 0$, such that

$$ (P^E_{\text{hom}}) \begin{cases} e^{-1}_{\text{hom}} \text{curl}_\beta \mu^{-1}_{\text{hom}} \text{curl}_\beta E_{\text{hom}}(x) = k^2 E_{\text{hom}}(x) \quad &\text{in } \Omega_f \\ n \nabla E_{\text{hom}} = 0 \quad &\text{on } \partial \Omega_f \end{cases} $$(A2)

where the homogenized matrices of permittivity and permeability are defined by

FIG. 2. The cross section of a microstructured fibre represented by a fixed set $\Omega_f$ filled with a cladding $\Omega_{\eta'}$. For a given $\eta < \eta'$, $\Omega_{\eta'} \subset \Omega_{\eta} \subset \Omega_f$ (homogenization setting).
The relative permittivity and permeability matrices of the homogenized problem are equal to

\[
\chi_{\text{hom}} = \begin{pmatrix}
\langle \chi(y) \rangle_Y & 0 & 0 \\
0 & \langle \chi(y) \rangle_Y & 0 \\
0 & 0 & \langle \chi(y) \rangle_Y
\end{pmatrix},
\]

where \( \phi_{ij} = (\partial \phi / \partial y_j) e_i + (\partial \phi / \partial y_i) e_2 \) and \( \text{div}_{i,j} \Phi = \partial \Phi / \partial y_i + \partial \Phi / \partial y_j \). Analogously \( W_Y = (W_1, W_2) \) satisfies the two following scalar problems (\( L_j \)) of electrostatic type:

\[ (K_j): - \text{div}_{i,j} [\mu_i(y) (\text{grad}_{i,j} V_j (y - y_j))] = 0, \quad j \in \{1, 2\}, \]

where \( \text{grad}_{i,j} = (\partial \phi / \partial y_j) e_i + (\partial \phi / \partial y_i) e_2 \) and \( \text{div}_{i,j} \Phi = \partial \Phi / \partial y_i + \partial \Phi / \partial y_j \).

The relative permittivity and permeability matrices of the homogenized problem are equal to

\[
\chi_{\text{hom}} = \begin{pmatrix}
\langle \chi(y) \rangle_Y & 0 & 0 \\
0 & \langle \chi(y) \rangle_Y & 0 \\
0 & 0 & \langle \chi(y) \rangle_Y
\end{pmatrix}.
\]

In the transverse case, and for a scattering problem, Feltbacq has observed that taking the limit \( \eta \to 0 \) in an array of infinitely conducting rods leads to a concentration effect on the boundary \( \partial \Omega \) associated to effective transmission conditions for the homogenized field.\(^\text{12}\) As a corollary, the limits \( \eta \to 0 \) and \( e \to + \infty \) do not commute. We emphasize here that such an effect does not occur since we deal with a spectral problem (no transmission conditions). Nevertheless, when \( \beta > 0 \), we note that \( \chi_{\text{hom},33} \neq + \infty \) when we readily start the homogenization process from a set of metallic cylinders. This can be indeed classified as a noncommuting limit effect. We note that there is no apparent discontinuity in the effective properties when \( \beta \to 0 \). There is in fact no contradiction with the results given in the main text. This can be seen by linking the two-scale homogenization together with the Bloch-wave homogenization. This final step requires melting the two previous theories in a common mathematical frame, which was originally developed in the classical book by Benoissans, Lions and Papanicolaou.\(^\text{22}\) The idea is to introduce the so-called shifted Maxwell operator defined for \( k_{\text{block}} \in \mathbb{Z}' = [0; 1/2 \pi]^2 \) (\( \mathbb{Y}' \) is the reciprocal cell which is known as first Brillouin zone in solid state physics) as

\[
A_{\beta}(k_{\text{block}}) = (\text{curl}_B + i k_{\text{block}} \mathbb{Y}) \mu^{-1}(y)(\text{curl}_B + i k_{\text{block}} \mathbb{Y}),
\]

and to look for a given \( \beta > 0 \), for \( (\Lambda, E_{k_{\text{block}}}) \), \( E_{k_{\text{block}}} \in \mathbb{U}_{k_{\text{block}}} \in L^2(\mathbb{R}^2, \mathbb{C}^3) \), \( \text{curl}_{k_{\text{block}}} \mathbb{U} = 0 \), \( u_{k_{\text{block}}} (y + 2 \pi \mathbf{p}) = e^{2 \pi i k_{\text{block}} \mathbf{p} \cdot \mathbf{m}} u_{k_{\text{block}}} (y) \) such that

\[
(\mathbb{P}_{k_{\text{block}}}) : A_{\beta}(k_{\text{block}}) E_{k_{\text{block}}} = \Lambda E_{k_{\text{block}}} \text{ in } \mathbb{R}^2.
\]

The spectrum of the operator associated to the spectral problem \( (\mathbb{P}_{k_{\text{block}}}) \) consists of a discrete set of isolated eigenvalues \( \Lambda_n = \{ \beta^2 / (\epsilon \mu) \} + \infty \} \) (arranged in increasing orders and each eigenvalue is repeated as many times as its multiplicity) which are associated to so-called Bloch eigenfunctions \( E_{k_{\text{block}}} \). If we define the \( m \)th Bloch coefficient of \( u \in L^2(\mathbb{R}^2, \mathbb{C}^3) \) (continuous Bloch wave decomposition)\(^\text{22}\) by

\[
[u]_m^{\mathbb{P}_{k_{\text{block}}}}(k_{\text{block}}) = \frac{1}{2 \pi \eta} \int_{\mathbb{R}^2} u(z) e^{-2 \pi i k_{\text{block}} \mathbf{p} \cdot \mathbf{m}} E_{k_{\text{block}}} e^{-\mathbb{P}_{k_{\text{block}}}^*} \eta(z) d\mathbf{z},
\]

then Eq. (A1) can be set as an infinite discrete set of algebraic equations in the following way for \( m \in \mathbb{N}^* \):

\[
A_{\beta}(\eta k_{\text{block}})[E_{k_{\text{block}}}^m(z)]^\eta(z) = \eta^{-2} \Lambda_m(\eta \mathbf{z}) [E_{k_{\text{block}}}^m(z)]^\eta(z).
\]

The fundamental mode \( [E_{k_{\text{block}}}^m(z)]^\eta(z) \) provides us with the
homogenized field. We now take a Taylor expansion for its associated eigenvalue in a neighborhood of the origin
\[ \eta^2 \left( \Lambda_1(0) + \eta \frac{\partial \Lambda_1(0)}{\partial k_{\text{bloch},p}} z_p + \frac{\eta^2}{2} \frac{\partial^2 \Lambda_1(0)}{\partial k_{\text{bloch},p} \partial k_{\text{bloch},q}} z_p z_q \right) 
+ 0(\eta^3 z^3) \left( e^{-1} E_{k_{\text{bloch}}}(z) \right)^\eta(z) = \omega^2 \left( E_{k_{\text{hom}}}(z) \right)^\eta(z), \]
where \( p, q = 1, 2 \). Taking the \( L^2 \) weak limit in the previous expression when \( \eta \) tends to 0 we obtain
\[ \frac{1}{2} \frac{\partial^2 \Lambda_1(0)}{\partial k_{\text{bloch},p} \partial k_{\text{bloch},q}} z_p z_q E_{k_{\text{hom}}}(z) = \omega^2 \langle \varepsilon \rangle \gamma E_{k_{\text{hom}}}(z), \]
where \( u^*(z) = (1/2 \pi) \int_{\mathbb{R}^2} u(x) e^{i \pi x \cdot z} dx \) is the Fourier transform of \( u(x) \). To prove this, one has to note that the weak limit of \( \eta \langle u \rangle \) in \( L^2 \) is nothing but \( \langle \varepsilon \rangle u^* \). If we take the inverse Fourier transform in the previous equation, we retrieve the homogenized problem (A2). The tensor \( \mu_{\text{hom,pq}} e_{\text{hom,pq}}^{-1} (\hat{\partial}^2 \Lambda_1 / \hat{\partial} k_{\text{bloch},p} \hat{\partial} k_{\text{bloch},q}(0)) \) defines the “effective mass” of the first band on the dispersion diagram associated to the Bloch spectrum of the Maxwell operator for an infinite array of cylinders. The effect of non-commuting limit discussed in the main text is solely due to the parabolic shape of the acoustic band in the neighborhood of the origin. The right quantity to look at in oblique incidence is therefore the effective mass and no more the slope of the acoustic curve at the origin of the Bloch diagram as noticed in Ref. 16.

1. L. Lorenz, Wiedemannische Annalen 11, 70 (1869).