Instantons and Chiral Anomalies

in Quantum Field Theory

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Abstract

Functional methods are developed in order to study the non-perturbative aspects of Yang-Mills theories. The path integral formulation of quantum mechanics is derived and then used to build a functional formalism for quantum field theory. Based on these methods, the BPST instanton solution is derived and used to study the vacuum structure of pure Yang-Mills theories. Then after discussing fermionic functional methods, the anomaly in the axial current of massless quantum electrodynamics is deduced. This derivation requires renormalization, and so a gauge invariant cut-off regulator introduced and used to to eliminate the divergences.

Contents

1 Introduction

In this report, we explore some of the non-perturbative aspects of quantum field theories, or more specifically $SU(N)$ Yang-Mills theories. Such non-perturbative effects are instantons, which are completely absent in perturbation theory. This is due to the fact that they describe the transition of the system from one vacuum to another and hence perturbation around one minimum of the potential does not reveal information about tunneling to other minima. Such tunnelling effects in quantum mechanics or quantum field theory happen via instantons. These instanton effects also turn out to be very important in the renormalization of Yang-Mills theories and many of the methods developed here are necessary to study renormalizability. Yang-Mills instantons were first discovered more that 30 years ago by Belavin, Polyakov, Schwartz and Tyupkin [1], and they still remain a key area of study in field theory. It was then realised by t' Hooft [2], Jackiw and Rebbi [3], that these same methods can be used to study the vacuum structure of Yang-Mills theories. By the end of the 70s instantons became important tools in solving unanswered problems in gauge theories, like chiral anomalies.

We build on the above developments, to construct a theory in which we can further study these fascinating objects. In Section 2, we reformulate quantum mechanics and then quantum field theory in terms of functional integrals. This formulation is a lot more powerful in tackling non-perturbative problems than the usual canonical approach. In Section 3, we then build up scalar QFT, while in sections 4 and 5 we extend the discussion to fermionic spinor fields. We then move on to discuss some of the non-perturbative aspects of Yang-Mills theories in Section 6, which we will then use to derive the BPST instanton solutions in Section 7. The final part of the report is devoted to discussing the chiral anomaly.

2 The Path Integral Formulation

In this section we will present an alternative formulation of quantum mechanics, and then quantum field theory, which relies on the direct evaluation of the time evolution operator via the method of functional integrals. This method was first pioneered and used in field theory by Richard Feynman to calculate QED amplitudes, but has since become a standard tool in studying the non-perturbative aspects of quantum field theory. We now proceed to provide the derivation of this formalism by first looking at time evolution in 'ordinary' quantum mechanics, and then introducing the functional formalism in quantum mechanics and quantum field theory respectively.

2.1 Time Evolution in Quantum Mechanics

Quantum Mechanics is a self-contained theory based on a finite number of postulates or axioms. As these axioms introduce the notation and some of the conventions used, we therefore state that in quantum mechanics:

(i) The state of a system is given by a normalised vector $|\psi\rangle$ in Hilbert space.

(ii) To each of the observables x and p, there corresponds a Hermitian operator $\hat{X}$ and $\hat{P}$ such that

$$[\hat{X}_i, \hat{P}_j] = i\hbar\delta_{ij}.$$

All other observables ω also have a corresponding Hermitian operator $\hat{\Omega} = \hat{\Omega}(\hat{X}, \hat{P})$.

(iii) Measurements of an observable ω yield one of the eigenvalues ω_i of $\hat{\Omega}$ with probability

$$P(\omega_i) = |\langle \omega_i | \psi \rangle|^2.$$

(iv) The time evolution of a system is given by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle ,$$

where $\hat{H}$ is the Quantum Hamiltonian of the system.

It is possible to find an operator $U(t, t_0)$, such that it completely gives the time evolution of the system, i.e.

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle ,$$

where we call $U(t, t_0)$ the time evolution operator. Starting from the Schrödinger equation the form of this operator is derived as follows:

$$\begin{aligned}
i\hbar\partial_t |\psi(t)\rangle &= \hat{H} |\psi(t)\rangle \\
|\psi(t + \delta t)\rangle - |\psi(t)\rangle &= -\frac{i}{\hbar}\hat{H}\delta t |\psi(t)\rangle \\
|\psi(t + \delta t)\rangle &= \left(1 - \frac{i}{\hbar}\hat{H}\delta t\right) |\psi(t)\rangle .
\end{aligned}$$

We now write $\delta t = t - t_0 = \lim_{N\to\infty} \sum^N \frac{t-t_0}{N}$ and use the above expression to get

$$\begin{aligned}
|\psi(t + \delta t)\rangle &= \lim_{N\to\infty} \left|\psi\left(t_0 + \sum^N \frac{t - t_0}{N}\right)\right\rangle \\
&= \lim_{N\to\infty} \left(1 - \frac{i}{\hbar}\hat{H}\frac{t - t_0}{N}\right)^N |\psi(t_0)\rangle \\
&= e^{-i\hat{H}(t-t_0)/\hbar} |\psi(t_0)\rangle .
\end{aligned}$$

Hence we have found that the time evolution operator $U(t, t_0)$ is a unitary operator of the form

$$U(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar}. \tag{2.1}$$

This means that if one knows the eigenvalues and eigenfunctions of the Hamiltonian, i.e. the solution to the time independent Schrödinger equation, we can gain the full time evolution of the system by (2.1).

2.2 Path Integrals in Quantum Mechanics

The general approach in canonical quantum mechanics is to find the eigenfunctions and eigenvalues of the Hamiltonian and then use these to find an expression for the time evolution operator. The path integral approach to quantum mechanics, aims to compute the time evolution directly without the need to use the Schrödinger equation. We will properly derive this result in due course, but we first write down the result we are trying to achieve. To find the kernel $\langle x|U(t)|x'\rangle$, i.e. the matrix element of $U(t)$, in one dimension using the path integral formulation, we must follow the procedure:

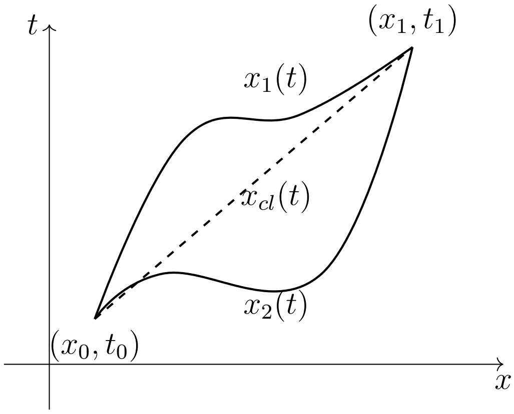


Figure 1: Schematics for the path integral formulation with $x_{cl}(t)$ being the classical path.

(i) Draw all paths $x_1(t), x_2(t) \cdots$ in the $x - t$ plane connecting (x_0, t_0) and (x_1, t_1), as shown in Figure 1;

(ii) Find the action $S[x(t)]$ for each path $x_i(t)$;

(iii) The kernel is then given by

$$K(x, x') \equiv \langle x|U(t)|x'\rangle = A \sum_i e^{iS[x_i(t)]/\hbar}, \tag{2.2}$$

where A is a normalization constant.

We now proceed to derive the above formalism and prove (2.2). Consider the Hamiltonian

$$\hat{H} = -\frac{\hat{\mathbf{P}}^2}{2m} + V(\hat{X}),$$

which has the corresponding time evolution operator

$$U(t, 0) = U(t) = e^{-i\hat{H}t/\hbar},$$

derived in Section 2.1. In the special case when $V(\hat{X}) = 0$, i.e. for a free particle, the matrix element of the propagator can be evaluated by simply using Gaussian integrals, which gives

$$\langle x'|\, e^{-it(\hat{\mathbf{P}}^2/2m)/\hbar}\, |x\rangle = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} e^{\frac{im(x'-x)^2}{2\hbar t}}. \tag{2.3}$$

Returning to the general case, we may write

$$U(t) = e^{-i\hat{H}t/\hbar} = \left(e^{-i\hat{H}t/N\hbar}\right)^N$$

for any $N \geq 0$. This means that we have expressed $U(t)$ as a product of N operators $U(t/N)$. We now define $\varepsilon = t/N$ and consider the limit as $N \to \infty$ or equivalently $\varepsilon \to 0$. In this limit we may write[1]

$$e^{-i\varepsilon(\hat{\mathbf{P}}^2/2m+V(\hat{X}))/\hbar} \simeq e^{-i\varepsilon(\hat{\mathbf{P}}^2/2m)/\hbar} \times e^{-i\varepsilon V(\hat{X})/\hbar}, \tag{2.4}$$

[1] This is a consequence of the Baker-Campbell-Hausdorff formula $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+1/2[\hat{A},\hat{B}]+\cdots}$, where $\hat{A}$ and $\hat{B}$ are arbitrary operators.

which becomes an exact relation in the above limit. Thus, it now remains to compute the kernel

$$K(x,x') = \langle x|U(t)|x'\rangle = \langle x| \underbrace{e^{-i\varepsilon(\hat{\mathbf{P}}^2/2m)/\hbar}e^{-i\varepsilon V(\hat{X})/\hbar} \times \cdots \times e^{-i\varepsilon(\hat{\mathbf{P}}^2/2m)/\hbar} \times e^{-i\varepsilon V(\hat{X})/\hbar}}_{\text{N times}} |x'\rangle\,. \tag{2.5}$$

This is done by inserting the completeness relation

$$I = \int dx\, |x\rangle\,\langle x|$$

between each term, i.e. $N-1$ times. Rename $|x_0\rangle = |x'\rangle$ and $|x_N\rangle = |x\rangle$, a general term in (2.5) then looks like

$$\langle x_n|\, e^{-i\varepsilon(\hat{\mathbf{P}}^2/2m)/\hbar}\, |x_{n-1}\rangle\, e^{-i\varepsilon V(x_{n-1})/\hbar}$$

where we have used that $\hat{X}\,|x\rangle = x\,|x\rangle$. The remaining matrix element i just the free particle kernel (2.3) from x_{n-1} to x_n in time ε. Putting these results together we have found that

$$K(x_N,x_0) = \langle x_N|U(t)|x_0\rangle = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{N/2} \prod_{n=1}^{N-1} \int dx_n\, e^{\sum_{n=1}^{N}\left(\frac{im(x_n-x_{n-1})^2}{2\hbar\varepsilon} - \frac{i\varepsilon}{\hbar}V(x_{n-1})\right)}. \tag{2.6}$$

Notice that the integrand is just a discretised version of $e^{iS[x(t)]/\hbar}$, by writing

$$e^{\sum_{n=1}^{N}\left(\frac{im(x_n-x_{n-1})^2}{2\hbar\varepsilon} - \frac{i\varepsilon}{\hbar}V(x_{n-1})\right)} = e^{\frac{i\varepsilon}{\hbar}\sum_{n=1}^{N}\left(\frac{m(x_n-x_{n-1})^2}{2\varepsilon^2} - V(x_{n-1})\right)}.$$

Thus taking the continuum limit, the kernel becomes

$$K(x,x') = \langle x|U(t)|x'\rangle = \int \mathcal{D}[x] e^{\frac{i}{\hbar}\int_0^t L(x,\dot{x})dt}, \tag{2.7}$$

where

$$\int \mathcal{D}[x] = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{N/2} \prod_{n=1}^{N-1} \int dx_n \tag{2.8}$$

is called the path integral measure. This is a functional integral, i.e. we are integrating over all possible paths $x(t)$ between x and x'. We will now show how this approach generalises to infinite degrees of freedom in quantum field theory.

2.3 Path Integrals in Quantum Field Theory

The derivation of the path integral formalism in quantum field theory is much the same as in quantum mechanics, though there are a few subtle differences. Instead of operators $\hat{X}$ and $\hat{\mathbf{P}}$ we now have the Schrödinger picture field operator $\hat{\phi}(x)$ and its conjugate $\hat{\pi}(x)$. Equivalent to $\big|x\big\rangle$ we have a complete set of eigenstates

$$\hat{\psi}(x)\big|\Psi\big\rangle = \phi(x)\big|\Psi\big\rangle$$
$$\hat{\pi}(x)\big|\Pi\big\rangle = \pi(x)\big|\Pi\big\rangle,$$

where the eigenvalues $\phi(x)$ and $\pi(x)$ are now a function of space [4]. The conjugate relation, analogous to $\big\langle x|p\big\rangle = \exp\{-ipx\}$, is given by

$$\langle\Pi|\Phi\rangle = e^{-i\int d^3x\pi(x)\phi(x)}.$$

The inner product between states is then

$$\langle\Phi|\Phi'\rangle = \int \mathcal{D}[\Pi]\langle\Phi|\Pi\rangle\langle\Pi|\Phi'\rangle = \int \mathcal{D}[\Pi]e^{-i\int d^3x\pi(x)(\phi(x)-\phi'(x))},$$

where the $\mathcal{D}[\Pi]$ denotes a functional integration over all possible functions $\pi(x)$. Thus we have found the completeness relation of these QFT states, namely

$$\int \mathcal{D}[\Phi]\big|\Phi\big\rangle\big\langle\Phi\big|.$$

As in the QM derivation, we will use this completeness relation to evaluate transition amplitudes.

Consider the vacuum matrix element $\langle 0;t_f|0;t_i\rangle$, for which we can again insert n completeness relations to give

$$\langle 0;t_f|0;t_i\rangle = \int \mathcal{D}[\Phi_1]\cdots\mathcal{D}[\Phi_n]\langle 0|e^{-i\varepsilon\hat{H}}|\Phi_n\rangle\langle\Phi_n|\cdots|\Phi_1\rangle\langle\Phi_1|e^{-i\varepsilon\hat{H}}|0\rangle,$$

where ε has to be understood as in the context of the QM derivation with its limit taken to zero. Now each separate piece becomes a Gaussian integral as before,[2] and so the vacuum matrix element becomes

$$\langle 0;t_f|0;t_i\rangle = N\int \mathcal{D}[\Phi]e^{iS[\phi]}, \tag{2.9}$$

rather unsurprisingly. This tells us that we have to integrate over all allowed field configurations Φ.

3 Bosonic Quantum Field Theory

We now proceed to formulate quantum field theory in terms of these functional integrals. This section is loosely based on [5] and [6]. It is general practice in functional field theory to introduce the so called partition function, defined as

$$Z[J] = \int \mathcal{D}[\phi]\exp\left\{i\int d^4x\,(S[\phi]+J(x)\phi(x))\right\},$$

where $J(x)$ is an ad hoc source term.[3] Thus from above we have that $Z[J]|_{J=0} = \langle 0|0\rangle$, i.e setting $J=0$ gives back the vacuum matrix element. We will see that this partition function proves to be very useful for perturbative and non-perturbative calculations, and essentially all the physics of a system is encoded in its partition function.

3.1 Free Scalar Fields

Consider the Lagrangian for a free scalar field theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{3.1}$$

[2] Even though this integral is called a Gaussian integral, it differs from previous cases as it is a functional integral. As it turns out though, it evaluates very similarly to the usual case.

[3] To gain any physical quantities, this source term will always be set to zero.

where $\phi \equiv \phi(x)$ is a real scalar field. The partition function is then given by

$$Z[J] = \int \mathcal{D}[\phi] \exp\left\{ i \int d^4x \left(\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2) + J(x)\phi \right) \right\}, \tag{3.2}$$

where $J(x)$ is the source term. We may integrate by parts in the exponent

$$\int d^4x \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) = - \int d^4x \frac{1}{2} \left(\phi \partial^\mu \partial_\mu \phi + m^2 \phi^2 \right),$$

where we have assumed that the field decays fast enough at infinity so that we can ignore boundary terms. The field can also be rewritten as

$$\phi(x) = \phi_0(x) + \varphi(x),$$

where $\phi_0(x)$ is the 'classical' field, i.e the solution to

$$(\partial_\mu \partial^\mu + m^2)\phi_0 = J. \tag{3.3}$$

This equation can be solved by finding the Green's function $G(x-x')$ for the Klein-Gordon operator

$$-(\partial_\mu \partial^\mu + m^2) G(x - x') = \delta^4(x - x'), \tag{3.4}$$

where $\delta^4(x - x')$ is the four-dimensional Dirac delta function. We solve this equation by first assuming that $G(x - x')$ can be written as a Fourier transform, i.e

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4} \tilde{G}(k) e^{ik(x-x')}.$$

Acting on this equation with the Klein-Gordon operator gives

$$-(\partial_\mu \partial^\mu + m^2) G(x - x') = \int \frac{d^4k}{(2\pi)^4} \tilde{G}(k)(k^2 - m^2) e^{ik(x-x')} = \delta^4(x - x') = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')}, \tag{3.5}$$

where we have used the Fourier definition of the delta function. The relation (3.5) shows that the Green's function takes the form

$$\tilde{G}(k) = \frac{1}{k^2 - m^2}, \tag{3.6}$$

which in position space is[4]

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2 - m^2 + i\varepsilon}. \tag{3.7}$$

Hence now that we have found the Green's function, we can write down solutions to (3.3) as

$$\phi_0(x) = - \int d^4x' \Delta_F(x - x') J(x').$$

We can use this result to rewrite the partition function (3.2) as

$$Z[J] = \int \mathcal{D}[\varphi] \exp\left\{ i \int d^4x \left[\frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2) \right] - i \int d^4x' d^4x'' \left[J(x') \Delta_F(x'' - x') J(x'') \right] \right\}.$$

[4] The factor of $i\varepsilon$ comes from the standard method of avoiding poles.

The second term in the exponential is now independent φ and hence can be taken out of the functional integral to give

$$Z[J] = \exp\left\{-\frac{i}{2}\int d^4x'd^4x''\left[J(x')\Delta_F(x''-x')J(x'')\right]\right\}\int \mathcal{D}[\varphi]\exp\left\{i\int d^4x\left[\frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2)\right]\right\}.$$

This prefactor outside the functional integral is just $Z[0]$, which will be used as a normalisation constant, i.e

$$Z[J] = N\exp\left\{-\frac{i}{2}\int d^4x'd^4x''\left[J(x')\Delta_F(x''-x')J(x'')\right]\right\}. \tag{3.8}$$

What we have done here is reduced the infinite dimensional functional integral to two integrals over four-dimensional spacetime.

The partition function (3.2) has some remarkable properties. One can realise that that the functional derivative with respect to the current $J(x_1)$ gives

$$-i\frac{\delta Z}{\delta J(x_1)} = \int \mathcal{D}[\varphi]\left(\exp\left\{iS[\varphi] + i\int d^4x J(x)\varphi(x)\right\}\varphi(x_1)\right),$$

and so evaluating this at $J=0$ we gain

$$-i\frac{\delta Z}{\delta J(x_1)}\bigg|_{J=0} = \int \mathcal{D}[\varphi]\exp\left\{iS[\varphi]\right\}\varphi(x_1) = \langle 0|\,\phi(x_1)\,|0\rangle\,.$$

This is not coincidental. We define a new generating functional $W[J]$

$$Z[J] \equiv e^{iW[J]}.$$

From (3.8) we already have an expression for this new functional

$$W[J] = -\frac{i}{2}\int d^4x'd^4x''\left[J(x')\Delta_F(x''-x')J(x'')\right].$$

We can now evaluate the functional derivatives of $W[J]$

$$\begin{aligned}-i\frac{\delta W[J]}{\delta J(x_1)} &= -\frac{i}{2}\int d^4x'd^4x''\left[\delta^4(x_1-x')\Delta_F(x''-x')J(x'') + J(x')\Delta_F(x''-x')\delta^4(x_1-x'')\right]\\ &= -i\int dx'J(x')\Delta_F(x_1-x').\end{aligned}$$

This means that the functional derivative of the partition function $Z[J]$ gives

$$\begin{aligned}-i\frac{\delta Z[J]}{\delta J(x_1)} &= -i\frac{\delta}{\delta J(x_1)}e^{iW[J]} = -i\frac{\delta W[J]}{\delta J(x_1)}Z[J]\\ &= \left(-i\int dx'J(x')\Delta_F(x_1-x')\right)Z[J],\end{aligned}$$

where it is important to note that this vanishes for $J=0$. In a similar fashion, the second functional derivative yields

$$(-i)^2\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)} = i\Delta_F(x_1-x_2)Z[J] + i\left(\int dx'J(x')\Delta_F(x_1-x')\right)\frac{\delta Z[J]}{\delta J(x_2)}$$

$$= i\Delta_F(x_1 - x_2)Z[J] - \frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)},$$

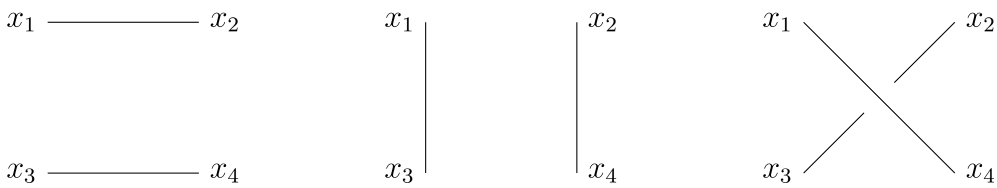
where now setting $J = 0$ gives $i\Delta_F(x_1 - x_2)$. Following this method, we find that each functional derivative of the partition function will give us a new correlation function

$$G_n(x_1, x_2, \cdots, x_n) = \frac{1}{i^n}\left(\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_n)}\right)Z[J]\bigg|_{J=0} = \langle 0|\, T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\}\, |0\rangle\,. \tag{3.9}$$

For example, the four point Green's function is given by

$$\begin{aligned}G_4(x_1, x_2, x_3, x_4) =& \frac{1}{i^4}\left(\frac{\delta^n}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta j(x_4)}\right)Z[J]\bigg|_{J=0}\\ =&\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3),\end{aligned}$$

which gives the diagrams:



Hence we have built a functional formalism which reproduces the results of the canonical approach, although this above method only works for bosonic, i.e. scalar, fields. In order to include fermions, we first have to introduce spinor fields.

4 Fermions and Spinor Fields

4.1 The Lorentz Group

Consider all isometries of Minkowski space X.

Definition. Let X be a (pseudo)Riemannian manifold with metric $\eta_{\mu\nu}$. The **isometries** of the manifold X are diffeomorphisms $F : X \to X$ such that

$$\eta_{\mu\nu}x^\nu x^\mu = x_\mu x^\mu = x_{\mu'}x^{\mu'} \quad \forall x \in X,$$

i.e preserves the metric.

This is the isometry group of X called the *Poincaré group.* A subgroup of this is the *Lorentz group* $SO(1,3)$, which consists of all those isometries that leave the origin fixed, i.e $x_\mu x^\mu = x_{\mu'}x^{\mu'}$ and $x^\mu|_{x=0} = x^{\mu'}|_{x=0}$ $\forall x \in X$.[5] We want to find representations $D[\Lambda]^a_b$ of this group which suitably describes spin-half particles. To do this we find the Lie algebra of the Lorentz group $SO(1,3)$ by writing

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu,$$

[5]Technically we are using the restricted Lorentz group $SO^+(1,3)$, which consists of all those elements of $SO(1,3)$ that preserve spacetime orientation.

where ω is infinitesimal. Imposing the $SO(1,3)$ condition implies

$$\Lambda^\mu_\sigma \Lambda^\nu_\rho \eta^{\sigma\rho} = \eta^{\mu\nu} \quad \Longrightarrow \quad \omega^{\mu\nu} + \omega^{\nu\mu} = 0,$$

that is the Lie algebra consists of all 4×4 antisymmetric matrices to be denoted $(\mathcal{M}^{\rho\sigma})^\mu_\nu$. Hence the Lie algebra $\mathcal{SO}(1,3)$ has 6 real degrees of freedom and hence 6 generators, each corresponding to one of 3 rotations and 3 boosts. Notice that we have given two group indices $[\rho\sigma]$ for the generators $(\mathcal{M}^{\rho\sigma})^\mu_\nu$ instead of one, we will however require that $[\rho\sigma]$ be antisymmetric giving us the required distinct generators. Writing the group index like this will prove to be useful for future calculations. We can also express all 4×4 antisymmetric matrices in terms of the metric as

$$\begin{aligned} (\mathcal{M}^{\rho\sigma})^{\mu\nu} &= \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu} \\ (\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu} &= \eta^{\mu\nu}\delta^{\sigma}{}_{\nu} - \eta^{\mu\nu}\delta^{\rho}{}_{\nu}. \end{aligned} \tag{4.1}$$

Now we can rewrite any arbitrary element ω^μ_ν of the Lie algebra as

$$\omega^{\mu}{}_{\nu} = \frac{1}{2}\Omega_\rho\sigma(\mathcal{M}^{\rho\sigma})^{\mu\nu},$$

where $\Omega_{\rho\sigma} \in \mathbb{R}$. The generators obey the algebra

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau}. \tag{4.2}$$

Any finite Lorentz transformation can thus be written as

$$\Lambda = e^{\frac{1}{2}\Omega_{\sigma\rho}\mathcal{M}^{\sigma\rho}}.$$

4.2 Spinors and Spinor Representations

We seek further representations of $\mathcal{SO}(1,3)$ satisfying the Lorentz algebra (4.2). To do this we first consider a closely related algebra called the *Clifford algebra*, which consists of 4×4 matrices γ such that

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_4, \tag{4.3}$$

where $\mathbb{1}_4$ is the 4-dimensional identity matrix. This is satisfied by matrices with the following properties:

$$\begin{cases} \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu & \text{if} \quad \mu \neq \nu \\ (\gamma^0)^2 = \mathbb{1}_4 & \\ (\gamma^i)^2 = -\mathbb{1}_4 & \text{for} \quad i = 1,2,3. \end{cases}$$

The simplest such representation of Clifford algebra is in 4-dimensions, for example, the matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \quad ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{4.4}$$

where σ^i are the Pauli matrices. One can also show that this representation is unique in 4-dimensions up to a transformation $M\gamma^\mu M^{-1}$, where M is any 4×4 invertible matrix. The above representation (4.4) of the Clifford algebra is called the *Weyl* or *Chiral* representation.

Now how does this relate to our Lorentz algebra? To answer this question, we have to look at the commutators of these γ-matrices

$$S^{\rho\sigma} \equiv \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2}(\gamma^\rho\gamma^\sigma - \eta^{\rho\sigma}), \tag{4.5}$$

where we used the relation (4.3). These $S^{\rho\sigma}$-matrices have the following two properties that relate them to the Lorentz algebra:

$$\begin{aligned} [S^{\rho\sigma}, \gamma^\rho] &= \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu} \quad \text{if} \quad \mu \neq \nu; \\ [S^{\mu\nu}, S^{\rho\sigma}] &= \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} + \eta^{\mu\sigma}S^{\nu\rho} - \eta^{\nu\sigma}S^{\mu\rho}. \end{aligned}$$

Hence these matrices satisfy the Lorentz algebra (4.2), and we have found a new representation of the Lie algebra $\mathcal{SO}(1,3)$. Since the γ's are 4×4 matrices, the S's are 4×4 matrices as well and so we denote them $(S^{\mu\nu})^\alpha{}_\beta$ with $\alpha, \beta = 1,2,3,4$.

At first sight, the difference between Λ and $S[\Lambda]$ is not evident. We will therefore specifically calculate their form for rotations and boosts separately:

(i) *Rotations*: The matrix S^{ij} corresponding to rotations is given by

$$S^{ij} = \frac{1}{2}(\gamma^i\gamma^j - \eta^{ij}) = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - \frac{1}{2}\delta_{ij}\mathbb{1}_4 = -\frac{i}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}.$$

By writing $\Omega_{ij} = -\varepsilon^{ijk}\theta^k$ the representation can be expressed as

$$S[\Lambda] = e^{\frac{1}{2}\Omega_{ij}S^{ij}} = e^{-\frac{1}{2}\varepsilon^{ijk}\theta^k S^{ij}} = \begin{pmatrix} e^{i\theta\cdot\sigma/2} & 0 \\ 0 & e^{i\theta\cdot\sigma/2} \end{pmatrix},$$

where $\theta = (\theta^1, \theta^2, \theta^3)$ is the rotation angle. Now consider a rotation about the x^3 axis by an angle of 2π, i.e $\theta = (0,0,2\pi)$. In this case the S-representation takes the form

$$S[\Lambda] = \begin{pmatrix} e^{i\pi\sigma^3} & 0 \\ 0 & e^{i\pi\sigma^3} \end{pmatrix} = -\mathbb{1}_4.$$

On the other hand, for the same rotation

$$\Lambda = e^{\frac{1}{2}\Omega_{\sigma\rho}\mathcal{M}^{\sigma\rho}} = \mathbb{1}_4,$$

where we have used the form of $\mathcal{M}$-matrices shown in (4.1). Hence Λ and $S[\Lambda]$ are truly different representations.

(ii) *Boosts*: The matrix S^{0i} corresponding to boosts is given by

$$S^{0i} = \frac{1}{2}(\gamma^0\gamma^i - \eta^{0i}) = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - \frac{1}{2}\delta_{0i}\mathbb{1}_4 = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.$$

We may again re-express $\Omega_{i0} = -\Omega_{0i} = \chi_i$ in term of a vector χ, which gives

$$S[\Lambda] = \begin{pmatrix} e^{\chi\cdot\sigma/2} & 0 \\ 0 & e^{-\chi\cdot\sigma/2} \end{pmatrix} = -\mathbb{1}_4.$$

This also shows us that unlike Λ, $S[\Lambda]$ is no longer unitary, i.e $S^\dagger[\Lambda]S[\Lambda] \neq \mathbb{1}$.

4.3 Dirac and Weyl Spinors

All this was done in order to introduce a field that describes fermions. Motivated by the above discussion we define a fermionic field.

Definition. A Lie algebra valued field $\psi^\alpha(x)$ is called a **Dirac spinor field** if under Lonrentz transformations it transforms as

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta \psi^\beta(x),$$

i.e under the S-representation of $SO(1,3)$.

In the following sections we will see that such fields indeed poses fermionic properties. We have seen that in the Weyl representation of the γ-matrices the S-representation of the Lorentz algebra becomes

$$S[\Lambda_{\text{rot}}] = \begin{pmatrix} e^{i\theta\cdot\sigma/2} & 0 \\ 0 & e^{i\theta\cdot\sigma/2} \end{pmatrix} \tag{4.6}$$

$$S[\Lambda_{\text{boost}}] = \begin{pmatrix} e^{\chi\cdot\sigma/2} & 0 \\ 0 & e^{-\chi\cdot\sigma/2} \end{pmatrix}, \tag{4.7}$$

which is block diagonal. This means that S is a reducible representation of the Lorentz algebra. It decomposes into two irreducible representations, which act separately on two component spinors $u_\pm(x)$ defined by

$$\psi^\alpha(x) = \begin{pmatrix} u_+^\alpha(x) \\ u_-^\alpha(x) \end{pmatrix},$$

where $u_\pm(x)$ are called *Weyl spinors*. From equations (4.6) and (4.7) we see that these Weyl spinors transform under Lorentz transformations as

$$u_\pm(x) \xrightarrow{\text{rot}} e^{i\theta\cdot\sigma/2}\, u_\pm(x)$$
$$u_\pm(x) \xrightarrow{\text{boost}} e^{\pm\chi\cdot\sigma/2}\, u_\pm(x).$$

In other words, this means that u_+ is an element of the $(1/2, 0)$ representation of the Lorentz group, while u_- is in the $(0, 1/2)$ representation. Hence the Dirac spinor field ψ is in the $(1/2, 0) \oplus (0, 1/2)$ representation of $SO(1,3)$.

The matrices (4.6) and (4.7) were only of a block diagonal form because we were working in a specific representation (4.4) for the γ-matrices. In an arbitrary representation of the Clifford algebra, these matrices are not generally block diagonal. There is an easy way of avoiding this problem and invariantly define spinors in even dimensions by using the so called fifth γ-matrix

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3.$$

Surprisingly, this new matrix also satisfies all the relations the other γ-matrices do, i.e

$$\{\gamma^5, \gamma^\mu\} = 0 \quad ; \quad (\gamma^5)^2 = \mathbb{1}.$$

We can use this to define a Lorentz invariant projection operator

$$P_\pm \equiv \frac{1}{2}(\mathbb{1} \pm \gamma^5),$$

which projects four-component spinors onto two-component spinors. For example, in the Weyl representation $\gamma^5 = \mathrm{diag}(1, 1, -1, -1)$ and so

$$P_{\pm}\psi(x) = u_{\pm}(x),$$

so in the Weyl representation $P_{\pm}$ projects Dirac spinors onto Weyl spinors. In an arbitrary representation we will still have

$$\psi_{\pm} = P_{\pm}\psi,$$

where the left hand side is a two component spinor and the right hand side is a four component Dirac spinor. We sometimes call ψ_+ left handed and ψ_- right handed spinors.

5 Fermionic Quantum Field Theory

We will now proceed to formulate the path integral partition function for fermionic fields described in section 4. Some of the material in this section is based on [7] and [4]. The first important observation is that while scalar fields commute, the fermionic fields anti-commute, i.e

$$\{\psi^{\alpha}(x), \psi^{\beta}(x')\} = \{\psi^{\dagger\alpha}(x), \psi^{\dagger\beta}(x')\} = 0.$$

This means that to describe spinor fields, we have to develop a formalism for dealing with anti-commuting variables in functional calculus.

5.1 Grassman Analysis

Definition. The **Grassman variables** θ_i form an algebra over the complex number field such that the generators satisfy the anti-commutation relation

$$\{\theta_i, \theta_j\} \equiv \theta_i\theta_j + \theta_j\theta_i = 0,$$

for all θ.

It immediately follows from this definition that Grassman numbers square to zero, i.e $\theta_i^2 = 0$. This also implies that a general function written as

$$f(\theta) = a + a_i\theta_i + \frac{1}{2}a_{ij}\theta_i\theta_j + \cdots + \frac{1}{n!}a_{i_1\cdots i_n}\theta_{i_1}\theta_{i_2}\cdots\theta_{i_n}, \tag{5.1}$$

has all anti-symmetric coefficients $a_{i_1 i_2 \cdots}$ and that this sum truncates at finite n. We can define differentiation with respect to these variables by requiring that the differentials also anti-commute, that is satisfy the condition

$$\frac{\partial}{\partial\theta_i}\theta_j + \theta_j\frac{\partial}{\partial\theta_i} = 0.$$

In remains to to formulate integration with respect to these odd variables.

We will now define such integrals for functions of Grassman numbers $f(\theta)$, called Berezinian integrals, in analogy with standard integration. For our definition to be sensible and consistent, one we must require that

(i) *Linearity*

$$\int d\theta(af(\theta) + bg(\theta)) = a\int d\theta f(\theta) + b\int d\theta g(\theta);$$

(ii) *Partial integration*

$$\int d\theta \frac{\partial}{\partial \theta} f(\theta) = 0,$$

where f, g are smooth functions and the integrals are over all space. It directly follows that we can write

$$\int d\theta = 0$$
$$\int d\theta \, \theta = 1.$$

It is interesting to note that

$$\int d\theta (af(\theta) + b) = \; b \; = \frac{\partial}{\partial \theta}(af(\theta) + b),$$

hence, in this case, differentiation is the same as integration. We can also extend integration to higher dimensions by defining the n-volume form

$$d^n\theta = d\theta_n \wedge d\theta_{n-1} \wedge \cdots \wedge d\theta_1,$$

where all of the 1-forms anti-commute, i.e

$$d\theta_i \wedge d\theta_j = -d\theta_j \wedge d\theta_i.$$

Using the expansion (5.1) for a general function $f(\theta)$ we can write down some formulas for these Berezinian integrals:

$$\int d^n\theta \, f(\theta) = \frac{1}{n!} \varepsilon_{i_1 i_2 \cdots i_n} a_{i_1 \cdots i_n} \theta_{i_1} \theta_{i_2} \cdots \theta_{i_n}$$
$$\int d^n\theta \, \theta_{i_1} \theta_{i_2} \cdots \theta_{i_n} = \varepsilon_{i_1 i_2 \cdots i_n},$$

where $\varepsilon_{i_1 i_2 \cdots i_n}$ is an n-dimensional totally anti-symmetric tensor.

It is also very interesting to look at what happens to the volume n-forms under changes of variables. Consider the transformation $\theta \to \theta_{i'} = A_{ij}\theta_j$, where A_{ij} is just a square matrix. Under such transformation, the integral behaves as

$$\int d^n\theta \, f(A\theta) = \frac{1}{n!} \varepsilon_{i_1 i_2 \cdots i_n} a_{i_1 \cdots i_n} A_{1 i_1} A_{2 i_2} \cdots A_{n i_n} = (\det A) a_{1 \cdots n} = \frac{1}{n!} (\det A) \int d^n\theta f(\theta),$$

where we have used the Leibnitz formula for calculating the determinant. Hence we have found that the volume element transforms as

$$d^n\theta' = (\det A)^{-1} d^n\theta, \tag{5.2}$$

which means that it transforms with the inverse Jacobian. This is very fascinating as for ordinary variables x we had that

$$d^n x' = (\det A) d^n x,$$

i.e they transform with the Jacobian. We also state without proof here the formula for Gaussian integrals of Grassman variables

$$\int d^n\theta d^n\bar{\theta} \, e^{\bar{\theta}_i A_{ij} \theta_j} = \det A, \tag{5.3}$$

where $\bar{\theta}_i$ are just the conjugate variables. This is also differs form the bosonic case where the integral is proportional to the square root of the determinant.

5.2 The Fermionic Oscillator

As quantum field theory is essentially described by a harmonic oscillator at every spacetime point, so it is natural to first discuss the formulation of a fermionic oscillator before attempting to tackle fermionic field theory. A ferminonic harmonic oscillator can be be described by introducing creation and annihilation operators $\hat{\mathbf{b}}^\dagger$, $\hat{\mathbf{b}}$ such that

$$\hat{\mathbf{b}}^2 = (\hat{\mathbf{b}}^\dagger)^2 = 0; \qquad \{\hat{\mathbf{b}}^\dagger, \hat{\mathbf{b}}\} = 1.$$

These operators can then be used to define states. First the vacuum state $|0\rangle$ which satisfies

$$\hat{\mathbf{b}}\,|0\rangle = 0,$$

and then an excited state $|1\rangle$ such that

$$|1\rangle = \hat{\mathbf{b}}^\dagger\,|0\rangle\,.$$

This implies that we have a two dimensional vector space of states, with basis $|0\rangle$, $|1\rangle$, such that

$$\hat{\mathbf{b}}^\dagger\,|1\rangle = 0; \qquad \hat{\mathbf{b}}\,|1\rangle = |0\rangle\,.$$

Thus if we choose the basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then the creation and annihilation operators take the form

$$\hat{\mathbf{b}}^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \qquad \hat{\mathbf{b}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is convenient, due to the nature of fermionic states, to introduce states labelled by Grassman variables

$$|\theta\rangle = |0\rangle + \theta\,|1\rangle\,; \qquad \langle\bar{\theta}| = \langle 0| + \bar{\theta}\,\langle 1|\,,$$

where we now have

$$\hat{\mathbf{b}}\,|\theta\rangle = \theta\,|\theta\rangle\,; \qquad \langle\bar{\theta}|\,\hat{\mathbf{b}}^\dagger = \bar{\theta}\,\langle\bar{\theta}|\,.$$

Our goal is to formulate path integrals in such a theory and to do so we must introduce the Hamiltonian of this system

$$\hat{\mathbf{H}} = \omega\hat{\mathbf{b}}^\dagger\hat{\mathbf{b}} = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}, \tag{5.4}$$

where the matrix form is expressed in the $|0\rangle$, $|1\rangle$ basis. We can now write the time evolution of this system from state $|\theta\rangle$ to state $|\bar{\theta}\rangle$ using (2.1)

$$\left\langle\bar{\theta}\right|e^{-i\hat{\mathbf{H}}t}\,|\theta\rangle = \langle 0|0\rangle + \bar{\theta}\theta\left\langle 1|e^{-i\omega t}|1\right\rangle = 1 + \bar{\theta}\theta e^{-i\omega t} = e^{\bar{\theta}\theta e^{-i\omega t}},$$

where for the last equality we used that the series expansion of the exponential truncates at first order. The inner product of these states can also be evaluated to give

$$\left\langle\bar{\theta}\,\middle|\,\theta'\right\rangle = e^{\frac{1}{2}\bar{\theta}\theta + \frac{1}{2}\bar{\theta}\theta' + \bar{\theta}\bar{\theta}'}. \tag{5.5}$$

In analogy with the bosonic case, we need to introduce a completeness relation

$$\int d\bar{\theta}d\theta e^{\bar{\theta}\theta}\,|\theta\rangle\left\langle\bar{\theta}\right| = |\,0\,\rangle\langle\,0| + |\,1\,\rangle\langle\,1| = 1, \tag{5.6}$$

where we used that mixed terms like $|\,0\,\rangle\langle\,1|$ do not contribute to the integral. We will use this to construct the path integral for fermionic fields similarly to the bosonic case.

5.3 Free Spinor Fields

Having seen how a fermionic oscillator works, we now proceed to formulate the theory for free fermionic fields. We want to express the time evolution of fermionic systems in terms of path integrals. As in the bosonic case, we insert N completeness relations (5.6) into the kernel and discretise time $T = (N+1)\varepsilon$ to give

$$\langle \bar{\theta}_N | e^{-i\hat{H}\varepsilon} | \theta_0 \rangle = \int \left(\prod_{k=1}^{N} d\bar{\theta}_k d\theta_k e^{\bar{\theta}_k \theta_k} \right) \langle \bar{\theta}_k | e^{-i\hat{H}\varepsilon} | \theta_{k-1} \rangle \langle \bar{\theta}_{k-1} | e^{-i\hat{H}\varepsilon} | \theta_{k-2} \rangle \cdots \langle \bar{\theta}_1 | e^{-i\hat{H}\varepsilon} | \theta_0 \rangle ,$$

in the limit as $N \to \infty$, $\varepsilon \to 0$ One typical term in this expression evaluates to

$$\begin{aligned} \langle \bar{\theta}_k | e^{-i\varepsilon\hat{H}} | \theta_{k-1} \rangle &= \langle \bar{\theta}_k | (1 - i\varepsilon\hat{H} + \mathcal{O}(\varepsilon^2)) | \theta_{k-1} \rangle \\ &= (1 - i\varepsilon\omega\bar{\theta}_k\theta_{k-1}) \langle \bar{\theta}_k | \theta_{k-1} \rangle \\ &= e^{-i\omega\bar{\theta}_n\theta_{n-1} - \frac{1}{2}\theta_n\bar{\theta}_n - \frac{1}{2}\bar{\theta}_{n-1}\theta_{n-1} + \bar{\theta}_n\bar{\theta}_{n-1}}, \end{aligned}$$

where we have used (5.5) and the specific form of the Hamiltonian (5.4). In the continuum time limit this becomes

$$\lim_{\varepsilon \to 0} e^{-i\omega\bar{\theta}_n\theta_{n-1} - \frac{1}{2}\theta_n\bar{\theta}_n - \frac{1}{2}\bar{\theta}_{n-1}\theta_{n-1} + \bar{\theta}_n\bar{\theta}_{n-1}} = e^{i\varepsilon\bar{\theta}(t)\left(\frac{i}{2}\partial_t - \omega\right)\theta(t)}.$$

Thus taking the product of all the terms gives

$$K(\bar{\theta}_N, \theta_0) = \int \prod_{k=1}^{N} d\bar{\theta}_k d\theta_k e^{i\bar{\theta}_k}\theta_k e^{iS[\theta,\bar{\theta}]} = \int \mathcal{D}[\theta]\mathcal{D}[\bar{\theta}] e^{iS[\theta,\bar{\theta}]}.$$

Thus we have found the kernel for fermionic quantum mechanics. Similarly to the scalar field case, this readily generalises to field theory, where the path integral formula becomes

$$K(\psi, \bar{\psi}) = N \int \mathcal{D}[\bar{\psi}]\mathcal{D}[\psi] e^{iS[\bar{\psi},\psi]}. \tag{5.7}$$

We see that the fermionic path integral looks very much the same as the bosonic one. Essentially, the only difference is that in the fermionic case one has to take care about the non-commuting Grassman variables when manipulating the above formula.

6 Gauge and Yang-Mills Theories

In this section we give a short introduction to Yang-Mills theories, which will also give us a chance to fix some of the conventions used later on. Yang-Mills theories form a big part of physics as the standard model is a $SU(N)$ Yang-Mills theory, hence to our best knowledge nature is best described by Yang-Mills theories and Gauge fields.

6.1 Basics of Yang-Mills Theories

We first define some of terms we use to build up the theory.

Definition. A **gauge theory** is a field theory in which the Lagrangian is invariant under a group continuous local transformations called the gauge group.

Definition. A **Yang-Mills theory** is a gauge theory based on the gauge group $SU(N)$, or more generally any compact Lie group.

We will be using the following conventions regarding the $SU(N)$ group:

(i) *Trace of generators:* $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$;

(ii) *Commutator of generators:* $[T^a, T^b] = if^{abc}T^c$;

(iii) *Cartan metric:* $g_{ab} = \delta_{ab}$;

where T^a are the generators, F^{abc} are the structure constants and g_{ab} is the Cartan metric of $SU(N)$. The Yang-Mills field strength is defined as

$$F_{\mu\nu} \equiv F^a_{\mu\nu}T^a = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \tag{6.1}$$

which implies

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc}A^a_\mu A^b_\nu.$$

Here A^a_μ is the Yang-Mills gauge field which by definition transforms under $SU(N)$ transformations as

$$A_\mu \longrightarrow A_{\mu'} = UA_\mu U^\dagger - iU\partial_\mu U^\dagger,$$

where $U = e^{i\theta^a T^a} \in SU(N)$ with $\theta^a \in \mathbb{R}$. The Lagrangian density for a pure $SU(N)$ Yang-Mills theory is defined as

$$\mathcal{L} \equiv -\frac{1}{2g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4g^2}F^a_{\mu\nu}F^{a,\mu\nu},$$

where g is the gauge coupling constant. The action therefore takes the form

$$S = \int d^4x\mathcal{L} = -\int d^4x\frac{1}{4g^2}F^a_{\mu\nu}F^{a,\mu\nu}.$$

6.2 Euclidean Formulation of Field Theory

Quantum field theory can be reformulated in terms of a Euclidean metric $\delta_{\mu\nu}$ instead of the Minkowski metric $\eta_{\mu\nu}$. The transformation which brings η to the form of δ is called Wick rotation. In terms of coordinates this transformation can be written as[6]

$$x^\mu = (x^0, \mathbf{x}) \longrightarrow x^{\mu'} = (ix^0, \mathbf{x}),$$

and thus the norm transforms as

$$x_\mu x^\mu = (x^0)^2 - \mathbf{x}^2 = (-ix^{0'})^2 - \mathbf{x}'^2 = -((x^{0'})^2 + \mathbf{x}^2).$$

Consequently, after Wick rotation the pure Yang-Mills action becomes $S \to -iS_E$, where[7]

$$S^E = \int d^4x\mathcal{L}^E = \int d^4x\frac{1}{4g^2}F^a_{\mu\nu}F^{a,\mu\nu}.$$

In Euclidean coordinates we also have the simplification that since the metric is $\delta_{\mu\nu}$, upper and lower spacetime indices are no longer distinct, i.e. $x_{\mu'} = x^{\mu'}$. Wick rotation turns out to be very useful in non-perturbative field theory, not only because of the simplified metric but because many quantities in the path integral formalism are mathematically ill-defined using the Minkowski metric.

[6] We will usually use Greek indices for both Minkowskian and Euclidean coordinates, though for Euclidean coordinates we write $\mu = 1, 2, 3, 4$. It will always be clear from the context which indices we are using.

[7] The integration is over Euclidean variables as well.

7 Instantons and Θ-Vacua in Yang-Mills Theories

In this section we are solely using Wick rotated coordinates, that is throughout this whole section $\mu = 1, 2, 3, 4$ and $x_\mu = \delta_{\mu\nu} x^\nu$. Consider a pure $SU(N)$ Yang-Mills theory, with Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2} F^a{}_{\mu\nu} F^{a,\mu\nu}, \tag{7.1}$$

where a is the group index for the fundamental representation, g is the coupling constant and $F_{\mu\nu}$ is the Yang-Mills field strength tensor (6.1). Notice the factor of $1/g^2$ in the Lagrangian, which is due to the fact that we are now using a different definition for the field strength tensor

$$F^a{}_{\mu\nu} \equiv \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu - f^{abc} A^b{}_\mu A^c{}_\nu.$$

i.e. we have absorbed the dimensionless constant g into A_μ. This redefinition also removes the factor of g from the covariant derivative, which now takes the form

$$D_\mu = \partial_\mu + iA_\mu.$$

The Lagrangian (7.1), leads to the equations of motion

$$D_\mu F_{\mu\nu} = 0.$$

The euclidean action corresponding to the Lagrangian (7.1) is given by

$$S^E = \int d^4x \mathcal{L}^E = \frac{1}{4g} \int d^4x \left(F^a_{\mu\nu}\right)^2. \tag{7.2}$$

Our goal is to find non-perturbative solutions to this equation called instantons. We consider the specific case of an $SU(2)$ Yang-Mills theory, but whenever possible we state the general $SU(N)$ result.

Definition. An **instanton** is a non-perturbative solution to the euclidean equation of motion such that it has a finite non-zero action.

In our $SU(N)$ Yang-Mills theory this translates to the condition that

$$\lim_{|\mathbf{x}|\to\infty} A_\mu(x) \equiv A_\mu^\infty = 0. \tag{7.3}$$

We can express this in a more general form as under gauge transformations

$$A_\mu \to U A_\mu U^{-1} + iU\partial_\mu U^{-1},$$

where $U \equiv U(x) = \exp\{i\theta^a(x)T^a\} \in SU(N)$. Thus, since A_μ vanishes at infinity, we can rewrite A_μ^∞ up to a gauge transformation as

$$A_\mu^\infty = iU^{-1}\partial_\mu U = iU^\dagger \partial_\mu U = -iU\partial_\mu U^\dagger, \tag{7.4}$$

where we have used that $U^\dagger U = \mathbb{1}$. This precisely means that A_μ approaches a pure gauge at infinity. We can thus define an effective vacuum manifold $\mathcal{V}_{\text{eff}} \equiv \{A^\infty\}$, i.e the set of 'values' of A_μ at infinity, which is isomorphic to $SU(2)$, i.e. $\mathcal{V}_{\text{eff}} \simeq SU(2)$. This means that instanton solutions are characterised by the map

$$A_\mu^\infty : S^3_\infty \to SU(2),$$

or more precisely the third homotopy group $\pi_3(SU(2))$, which is given by

$$\pi_3(SU(2)) = \mathbb{Z}.$$

Hence each instanton solution can be identified with a winding number or topological charge Q.

Proposition. *The topological **winding number** Q of a field configuration A_μ is given by*

$$
\begin{aligned}
Q =& \frac{1}{24\pi^2} \oint_{S^3} d\Omega_\mu \varepsilon_{\mu\nu\rho\sigma} Tr\{(U^{-1}\partial_\nu U)(U^{-1}\partial_\rho U)(U^{-1}\partial_\sigma U)\} \\
=& -\frac{1}{16\pi^2} \int_{\mathbb{R}^4} d^4x \, Tr(F_{\mu\nu} \star F_{\mu\nu})
\end{aligned} \tag{7.5}
$$

*where the $\star F_{\mu\nu}$ denotes the **Hodge dual** of $F_{\mu\nu}$, i.e*

$$
\star F^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^{a,\rho\sigma},
$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is again the totally anti-symmetric Levi-Civita tensor [8].

Proof. To prove this proposition we start with expanding the trace in expression (7.5)

$$
\begin{aligned}
\mathrm{Tr}\{F_{\mu\nu} \star F_{\mu\nu}\} =& 2\mathrm{Tr}\{\varepsilon_{\mu\nu\rho\sigma}(\partial_\mu A_\nu \partial_\rho A_\sigma + 2\partial_\mu A_\nu A_\rho A_\sigma + A_\mu A_\nu A_\rho A_\sigma)\} \\
=& 2\partial_\mu \mathrm{Tr}\left\{\varepsilon_{\mu\nu\rho\sigma}\left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma\right)\right\},
\end{aligned}
$$

where for the second equality we have used the cyclic property of the trace. We can now use Stoke's theorem on the integral (7.5) as we are integrating a total derivative. This reduces the integral over all 4-dimensional euclidean space to an integral over the boundary, i.e the 3-sphere at infinity S^3_∞,

$$
Q = \frac{1}{12\pi^2} \oint_{S^3} d\Omega_\mu \mathrm{Tr}\left\{\varepsilon_{\mu\nu\rho\sigma}\left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma\right)\right\}, \tag{7.6}
$$

where $d\Omega_\mu$ is the area element on the 3-sphere.Using the properties of the fields this expression for the topological charge can also be written as

$$
Q = \frac{1}{24\pi^2} \oint_{S^3} d\Omega_\mu \varepsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\{(U^{-1}\partial_\nu U)(U^{-1}\partial_\rho U)(U^{-1}\partial_\sigma U)\}.
$$

This shows, that to every point x^μ at infinity, there corresponds a group element $U \in SU(N)$, and so this integral represents how many times the 3-sphere S^3 is 'wrapped around' the group $SU(N)$[8] [8]. Now, recall the topological result that $S^{2n-1} = SU(N)/SU(N-1)$ and so consequently for $N = 2$, $\pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$ [9]. $\square$

It is important to note that during this proof we have found a conserved current

$$
K_\mu = 2\left\{\varepsilon_{\mu\nu\rho\sigma}\left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\mu A_\sigma\right)\right\}.
$$

What is interesting about this current is that it does not come from Noether's theorem, but rather comes from purely topological considerations. Such currents are called *Chern-Simmons currents*, which play an important role in the mathematical description of non-perturbative phenomena.

We have seen that instanton solutions are characterised by their winding number Q, such that each instanton with a certain winding number cannot be continuously deformed into one with a different

[8]Note that this is in no way a complete proof so the $\square$ at the end is not justified. Essentially we only gave motivation for the definition of the winding number.

winding number. This follows from the gauge invariance of (7.5). Now one may naturally ask what the difference in the action is between instantons with different Q. To answer this consider the action (7.2) for our theory. We can rewrite this by noticing that $\mathrm{Tr}(FF) = \mathrm{Tr}(F \mp \star F)^2/2 \mp \mathrm{Tr}(F \star F)$ and so

$$S^E = \frac{1}{4g^2} \int d^4x F^a_{\mu\nu} F^a_{\mu\nu} = \frac{1}{2g^2} \int d^4x \left(\frac{1}{2} \mathrm{Tr}(F \mp \star F)^2 \pm \mathrm{Tr}(F \star F) \right) \tag{7.7}$$

$$\geq \pm \frac{1}{2g^2} \int d^4x \mathrm{Tr}(F \star F) = \frac{8\pi^2}{g^2} \times (\mp Q). \tag{7.8}$$

Here to go from (7.7) to (7.8) we have used the fact that $\mathrm{Tr}(T^a T^b) = \delta_{ab}/2$, and so $\mathrm{Tr}(F \mp \star F)^2 \geq 0$. The inequality (7.8) is saturated if F is (anti)-self dual, i.e $F = \pm \star F$, where the (+)-sign corresponds to a self dual and the (−)-sign corresponds to an anti-selfdual fields. The corresponding instanton solutions are also called instantons and anti-instantons respectively, where instantons have $Q > 0$, while anti-instantons have $Q < 0$. Thus we have found that the action of instanton solutions is given by

$$S^E = \frac{8\pi^2}{g^2} |Q|, \tag{7.9}$$

and so it only depends on the absolute value of the winding number.

7.1 Self Duality and The BPST Instanton

We now proceed to find explicit instanton solutions for our Lagrangian (7.1), where we will now choose a specific gauge group $SU(2)$. We largely follow the structure presented in the review [10], but other sources include [11], [8] and [12]. As seen, this requires us to solve the (anti)-selfduality equation

$$F^a_{\mu\nu} = \pm \star F^a_{\mu\nu} = \pm \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^a_{\rho\sigma}. \tag{7.10}$$

This is alone sufficient, since $F_{\mu\nu}$ by construction satisfies the Bianchi identity $D_\mu \star F_{\mu\nu} = 0$, and so if $F_{\mu\nu}$ is (anti)-selfdual, then it automatically satisfies the field equations $D_\mu F_{\mu\nu} = 0$.

Solutions to this equation can be found using a suitable ansatz. We try to find the solution with $Q = 1$ called the BPST instanton. To do this we notice that any $U \in SU(2)$ can be written as $U = A\mathbb{1}_2 + iB^i\sigma^i$, with $A^2 + \mathbf{B}^2 = 1$ to ensure unit determinant. Since A and B^i are otherwise arbitrary, we can make them variables and write a general $U(x)$ as

$$U(x) = \frac{ix^i\sigma_i + x^4\mathbb{1}_2}{r},$$

where $r = |x|$. The winding number of this $U(x)$ can be explicitly found using (7.5), which gives that $Q = 1$. Using this form of U, which we rename U_1 for obvious reasons, we can now substitute into (7.4) which gives that

$$A_4^\infty = -\frac{x^i\sigma^i}{r^2} \quad ; \quad A_i^\infty = \frac{1}{r^2}(x^4\sigma^i + \epsilon_{ijk}x_j\sigma_k),$$

where $i = 1, 2, 3$. We can simplify this expression by introducing the t' Hooft symbol $\eta^a_{\mu\nu}$ defined as

$$\eta^a_{ij} = \varepsilon_{aij} \quad ; \quad \eta^a_{i4} = \delta_{ai},$$

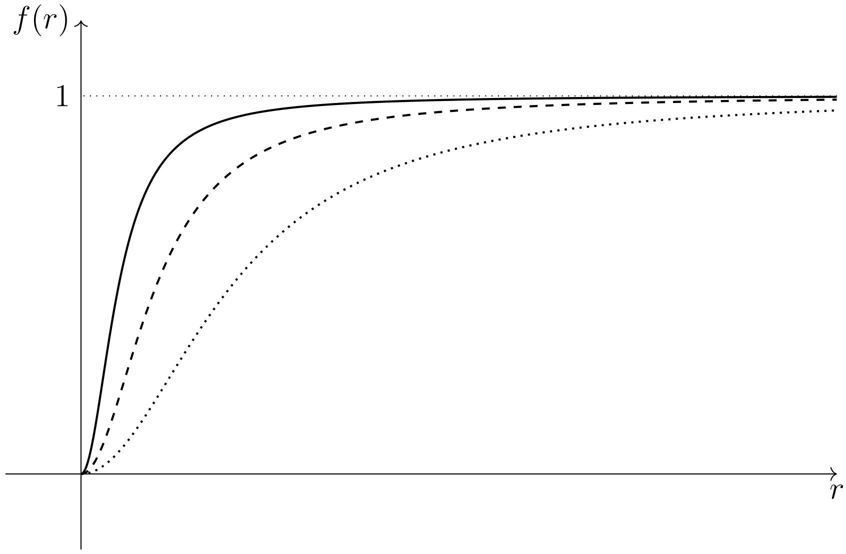


Figure 2: The radial profile of the BPST instanton solution (7.17) for $f(r)$ as a function of the four-radius $r = \sqrt{x_\mu x^\mu}$. The tree plots are for parameter values $\rho = 0.5, 1, 2$ top to bottom respectively.

where $i, j = 1, 2, 3$. This symbol has some nice properties, some of which we now state without proof

$$\star\eta^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\eta^a_{\rho\sigma} = \eta^a_{\mu\nu}; \tag{7.11}$$

$$\varepsilon_{abc}\eta^b_{\mu\rho}\eta^c_{\nu\sigma} = \delta_{\mu\nu}\eta^a_{\rho\sigma} - \delta_{\mu\sigma}\eta^a_{\rho\nu} - \delta_{\rho\nu}\eta^a_{\mu\sigma} + \delta_{\rho\sigma}\eta^a_{\mu\nu}; \tag{7.12}$$

$$\varepsilon_{\mu\nu\rho\sigma}\eta^a_{\sigma\lambda} = \delta_{\nu\lambda}\eta^a_{\mu\rho} - \delta_{\mu\lambda}\eta^a_{\nu\rho} - \delta_{\rho\lambda}\eta^a_{\mu\nu}. \tag{7.13}$$

Now, in terms of these symbols, the expression for A^∞ takes a much nicer form

$$A^\infty_\mu = \eta^a_{\mu\nu}\frac{x^\nu}{r^2}. \tag{7.14}$$

The asymptotic form (7.14) of A_μ suggests an ansatz

$$A^a_\mu(x) = 2\eta^a_{\mu\nu}\frac{x^\nu}{r^2}f(r^2), \tag{7.15}$$

where $f(r^2)$ is an arbitrary scalar function. The original boundary condition (7.3) implies the following boundary conditions on $f(r^2)$:

(i) $\lim_{r\to\infty} f(r^2) = 1$;

(ii) $\lim_{r\to 0} f(r^2) \propto r^{n\geq 2}$;

where the second condition guarantees a non-singular solution. Substituting or ansatz (7.15) into the selfduality equation (7.10) gives

$$\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc}a^b_\mu A^c_\nu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc}a^b_\mu A^c_\nu\right).$$

By direct calculation, using the relation (7.12), we get that the field strength is given by

$$F^a_{\mu\nu} = 4\left[\eta^a_{\mu\nu}\frac{f(f-1)}{r^2} + \frac{\eta^a_{\nu\rho}x_\mu x^\rho - \eta^a_{\mu\rho}x_\nu x^\rho}{r^4}\left(f(f-1) + r^2 f'\right)\right],$$

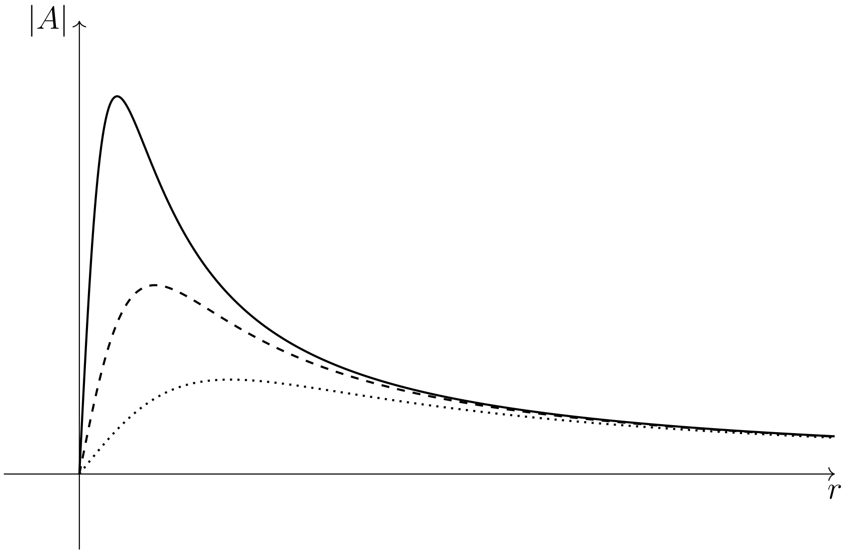


Figure 3: The radial profile of the BPST instanton solution (7.18) for $|A|$ as a function of the four-radius $r = \sqrt{x_\mu x^\mu}$. The tree plots are for values $\rho = 0.5, 1, 2$ top to bottom respectively.

where f' denotes a differential with respect to r^2. To calculate the dual field strength, we have to use the relations (7.11) and (7.13) giving

$$\star F^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^a_{\rho\sigma} = 4\left[-\eta^a_{\mu\nu}f' - \frac{\eta^a_{\nu\rho}x_\mu x^\rho - \eta^a_{\mu\rho}x_\nu x^\rho}{r^4}\left(f(f-1) + r^2 f'\right)\right].$$

Thus comparing these two expressions for the field and its dual, we arrive at a first order differential equation for $f(r^2)$ given by

$$r^2\frac{df}{dr^2} = f(f-1), \tag{7.16}$$

with boundary conditions (i) and (ii) given above. The non-trivial, i.e. $f \neq 0$, solutions to this equation are the functions

$$f(r^2) = \frac{r^2}{r^2+\rho^2}, \tag{7.17}$$

where ρ is an arbitrary constant. The corresponding solution for the field A_μ can now be deduced from (7.15) giving

$$A^a_\mu = 2\eta^a_{\mu\nu}\frac{x^\nu - x^\nu_0}{(x-x_0)^2+\rho^2}, \tag{7.18}$$

where we used translational invariance to make the solution more general. Hence we have successfully found a $Q = 1$ instanton solution to our pure $SU(2)$ Yang-Mills theory called the *BPST instanton*[9]. The solutions (7.17) and (7.18) for $f(r^2)$ and $|A|$ are shown in Figure 2 and 3.
We can also construct the BPST or one-instanton field strength

$$F^a_{\mu\nu} = -4\eta^a_{\mu\nu}\frac{\rho^2}{\left((x-x_0)^2+\rho^2\right)^2}, \tag{7.19}$$

[9]BPST stands for the physicists Alexander Belavin, Alexander Polyakov, Albert Schwarz and Yu. S. Tyupkin who first derived this solution in 1975 in reference [1].

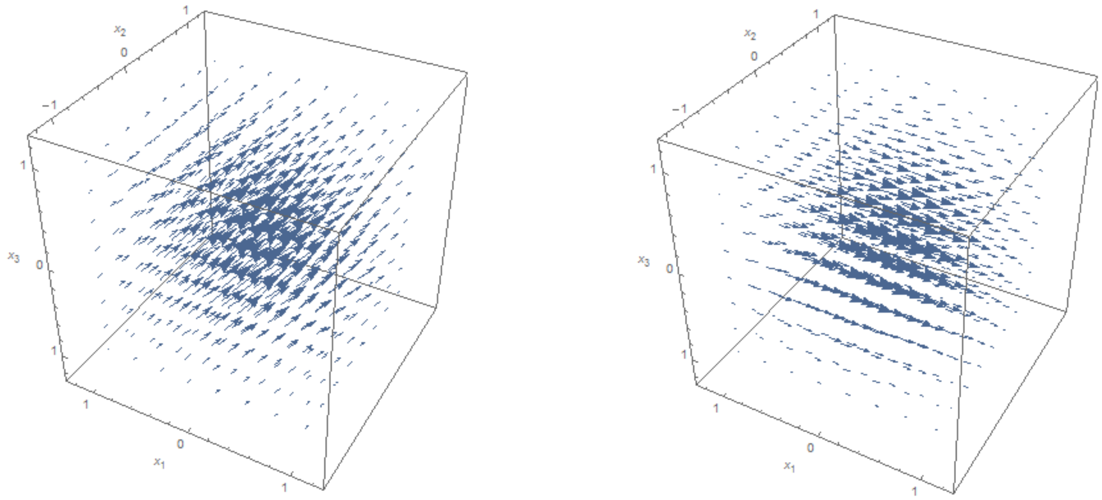


Figure 4: The E^3_i [left] and B^3_i [right] fields (7.21) plotted as a function of the spatial coordinates x^i for parameter value $\rho = 1$. As the Hamiltonian is $\mathcal{H} \propto E^2 + B^2$, this shows that the energy density is localised in space.

which is manifestly selfdual due to the properties of the η's. The anti-instanton solution with $Q = -1$ can also be explicitly found by solving the anti-selfduality equation. An identical calculation will then give that the one anti-instanton solution is

$$A^a_\mu = 2\bar{\eta}^a_{\mu\nu}\frac{x^\nu}{r^2 + \rho^2},$$

where $\bar{\eta}^a_{\mu\nu}$ are the anti-selfdual t' Hooft symbols defined as

$$\bar{\eta}^a_{ij} = \varepsilon_{aij} \quad ; \quad \bar{\eta}^a_{i4} = -\delta_{ai},$$

for $i, j = 1, 2, 3$. Both of these solutions have finite action

$$S^E_1 = \frac{8\pi^2}{g^2},$$

given by (7.9). We can also find the Hamiltonian of this system, which is given by[10]

$$\mathcal{H} = \frac{1}{2}(\partial_0 A^a_i)^2 + \frac{1}{4}F^a_{ij}F^{a,ij} = \frac{1}{2}(E^a_i)^2 + \frac{1}{2}(B^a_i)^2, \tag{7.20}$$

where we have defined the analogues of the electric and magnetic fields E^a_i and B^a_i as

$$E^a_i \equiv \partial_0 A^a_i\,; \qquad B^a_i \equiv -\frac{1}{2}\varepsilon_{i\rho\sigma}F^a_{\rho\sigma}. \tag{7.21}$$

These vector fields are shown in Figure 4 for some specific values of the group index. Thus we see that the energy density is localised and well defined as expected.

[10]It is important to note, that in order to get this expression we have fixed the gauge $A_4 = 0$.

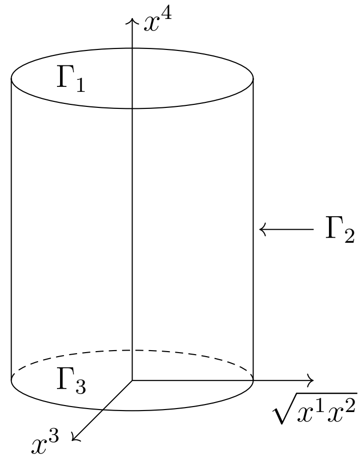


Figure 5: The three dimensional cylinder C^3 embedded into 4D ambient space with axis along the x^4 coordinate. Two of the axes have been collapsed into one for visualization. The surfaces of integration are labelled Γ_i.

7.2 Θ-Vacua and CP Violation

We now proceed with showing how instantons are related to the vacua of Yang-Mills theories, which will also give us a more intuitive picture of what instantons actually represent. Consider the integral (7.6) for the instanton winding number. As we have seen the application of Stoke's theorem gave an integral over the 3-sphere at spatial infinity. Since the topological winding number Q is homotopy and gauge invariant, we have some freedom to manipulate this integral. Consider continuously deforming the sphere S^3 into the cylinder C^3 such that the axis of the cylinder lies on the x_4, i.e. euclidean time, axis. The continuity of this transformation ensures that Q wont change due to its homotopy invariance. Also, due to gauge invariance we can choose a gauge in which[11] $A^4 = 0$. So the integral (7.6) splits into two parts

$$Q = \frac{1}{24\pi^2} \oint_{S^3} d\Omega_\mu K_\mu = \frac{1}{24\pi^2} \left[\int_{\Gamma_1} d\Omega_\mu K_\mu + \int_{\Gamma_2} d\Omega_\mu K_\mu + \int_{\Gamma_3} d\Omega_\mu K_\mu \right],$$

where the surfaces Γ_i are shown in Figure 5 [4].
Our gauge choice ensures that the integral over Γ_2 vanishes and so

$$Q = \frac{1}{24\pi^2} \left[\int_{\Gamma_1} d\Omega_\mu K_\mu - \int_{\Gamma_3} d\Omega_\mu K_\mu \right] = Q_1 - Q_2,$$

where the minus sign comes from the outward norm convention for the surface element $d\Omega_\mu$. Thus we have found that an (anti)-instanton solutions interpolate between different degenerate vacua in time $x^4 = t \to -\infty$ to $x^4 = t \to \infty$. For example, the BPST instanton has $Q = 1$ and so it interpolates between neighboring vacua.

This also shows that Yang-Mills theories have an infinite set of degenerate vacuum states $|Q\rangle$ labelled by their topological winding. A system then can tunnel between these vacuum states via instantons. It is important to note that by the state $|Q_n\rangle$ we mean the field configuration

$$A_\mu^{(n)} = iU_n^{-1} \partial_\mu U_n. \tag{7.22}$$

[11] We can always find such a gauge transformation as we can just write $A_{\mu'} \to UA_\mu U^{-1} + iU\partial_\mu U^{-1}$ and solve for U.

This raises the question which vacuum to pick for perturbative or non-perturbative calculations as we have seen that each vacuum solution has a distinct topological charge and so different action via (7.2). This also implies that these vacua $|Q\rangle$ are not gauge invariant. We therefore look for a gauge invariant vacuum state that can be used as a true vacuum state for Yang-Mills theories.

Proposition. *The* Θ-***vacuum*** *state defined by*

$$|\Theta\rangle = \sum_n e^{in\theta} |Q_n\rangle \,, \tag{7.23}$$

$\theta \in \mathbb{R}$, *is gauge invariant.*

Proof. Consider the functional integral for the transition between states $|\Theta\rangle \to |\Theta'\rangle$

$$\langle\Theta'|e^{-i\hat{H}t}|\Theta\rangle = \sum_{n,n'} e^{i(n-n')\theta} e^{in'(\theta-\theta')} \langle Q_{n'}|e^{-i\hat{H}t}|Q_n\rangle \,,$$

where we have used definition (7.23). As we know that tunnelling between the two vacua $|Q_n\rangle$ and $|Q_{n'}\rangle$ happens via instantons with charge $Q = Q_n - Q_{n'}$ we can rewrite this as

$$\begin{aligned}
\langle\Theta'|e^{-i\hat{H}t}|\Theta\rangle =& 2\pi\delta(\theta-\theta') \sum_Q e^{-iQ\theta} \langle Q_n + Q|e^{-i\hat{H}t}|Q_n\rangle \\
\approx& \sum_Q e^{-iQ\theta} \int \mathcal{D}[A_Q] \exp\left\{ i \int d^4x \mathcal{L} \right\} \\
=& \int \mathcal{D}[A_\mu] \exp\left\{ i \int d^4x \left(\mathcal{L} + \frac{\theta}{16\pi^2} \mathrm{Tr}(F_{\mu\nu} \star F_{\mu\nu}) \right) \right\} ,
\end{aligned}$$

where we have used the definition of the delta function and (7.5) [7]. This expression is manifestly gauge invariant and so we have found the unique vacuum of our pure Yang-Mills theory. □

Due to this gauge invariant vacua of the theory, the above proof motivates us to add a so called Θ-term to the $SU(N)$ Yang-Mills Lagrangian

$$\mathcal{L}_\theta \equiv -\frac{\theta}{24\pi^2} \partial_\mu K_\mu,$$

where θ is a gauge invariant real parameter. Even more interestingly this θ-term, allowed by gauge invariance, violates the CP symmetry of $SU(3)$ quantum chromodynamics. This apparent breaking of the CP symmetry by the θ-vacuum term is called the strong CP problem.

8 Chiral Anomalies in Yang-Mills Theories

In this section we study the general behaviour or Yang-Mills theories under quantization. We will see that some symmetries of the classical theory might be lost during the quantization process, which problematically implies inconsistencies in some theories.

Definition. Consider a gauge field theory with action S which is invariant under some symmetry group G. We say that the symmetry G is **anomalous** if this G-symmetry is not present in the fully quantized theory.

The form of the anomaly depends on the group G, which may be continuous, discrete, global or local. If G is also the gauge group of the theory, then the the anomaly is called gauge anomaly.

Noether's theorem states that each continuous global symmetry of a theory results in a conserved current j^a_μ. On the other hand, if this symmetry is anomalous, then the resulting current will have a non-zero divergence, i.e.

$$\partial^\mu j^a_\mu = \mathcal{A}^a,$$

where we call the object $\mathcal{A}^a$ the anomaly. Our goal is to study systems with such anomalies, and derive the divergence of the Noether current.

8.1 QED Symmetries and Pion Decay

Here, we focus on one particular class of anomalies called chiral anomalies. Such anomalies can be found in theories with fermions coupled to other fields. Chiral anomalies were first calculated by Steinberger in 1949 [13], though not in the same context. Consider the fermionic term of the QED Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\not{\partial} - \not{A} - m)\psi, \tag{8.1}$$

where ψ is a 4-component Dirac spinor field and the $O_\mu \gamma^\mu = \not{O}$ for any field O_μ. In (8.1) the photons are coupled to the $U(1)$ symmetry of the Dirac Lagrangian, and so it is natural to ask what other symmetries this theory has

(i) *Vector symmetry*: This is the above mentioned global $U(1)$ symmetry meaning that the Lagrangian (8.1) is invariant under transformations $\psi \to e^{-i\alpha}\psi$ resulting in a Noether current

$$j^\mu_V = \bar{\psi}\gamma^\mu\psi.$$

(ii) *Axial symmetry*: The Lagrangian (8.1) is also invariant under the transformation $\psi \to e^{i\alpha\gamma^5}\psi$, sometimes referred to as chiral symmetry. This results in the Noether current

$$j^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi,$$

called the axial or chiral current. It is important to note that the divergence of the current is

$$\partial_\mu J^\mu_A = 2im\bar{\psi}\gamma^5\psi, \tag{8.2}$$

and so chiral symmetry only hold in the massless limit.

The first observation of an anomaly came through the process of a neutral pion decaying into two photons. The pion does not couple directly to electromagnetism, but such decay processes can be induced at the one loop level using a virtual fermion loop as shown in Figure 6. Steinberger, in 1949 [13], calculated the rate of such process using a proton loop to be $\sim 10^{16}s^{-1}$, which proved satisfactory at the time. Problems arose nearly twenty years later, when other methods became available for calculating such decay rates. These new methods gave the result, e.g the Sutherland-Veltman theorem [14, 15], that the decay rate should vanish in the massless limit. One reason for this disagreement is that the Sutherland-Veltman theorem largely relies on the conserved Noether current (8.2), but as we will now show this current gains corrections due to non-perturbative effects.

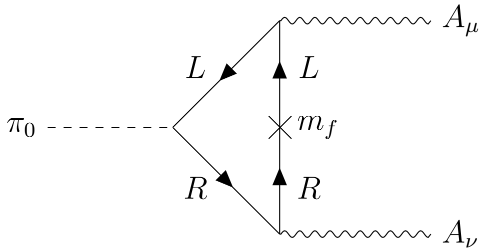


Figure 6: Feynman diagram for a pion decaying to two photons at one loop. The diagram should vanish when the mass of the fermion tends to zero.

8.2 Abelian Chiral Anomaly

Let Greek indices be Euclidean again, i.e $\mu = 1, 2, 3, 4$. Consider a theory in which massless fermions are coupled to photons in $3+1$-dimensions, with action

$$S = -\int d^4x \left(\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi \right), \tag{8.3}$$

where $D_\mu = \partial_\mu + iA_\mu$. The sourceless partition function is then given by

$$Z = \int \mathcal{D}[A]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] e^{iS[A,\psi,\bar{\psi}]}. \tag{8.4}$$

Our goal now is to find the divergence in the axial current for this theory and thus calculate the chiral anomaly. We will do this via the Fujikawa method by which one can calculate the chiral anomaly using the measure of the partition function [16]. The method relies on the realization that the invariance of the action is not sufficient to carry a symmetry into the quantum theory. This is because we can see from (8.4) that the integral measure must also be invariant. Thus finding the chiral anomaly requires that we examine how the path integral measure transforms under chiral transformations, which in turn requires a more careful definition of the measure. Consider the Dirac operator $\not{D} = D_\mu \gamma^\mu$, with eigenfunction expansion

$$\not{D}\phi_n = \lambda_n \phi_n\,; \qquad \bar{\phi}_n \not{D} = \lambda_n \bar{\phi}_n. \tag{8.5}$$

We can then expand the spinor field in terms of these eigenfunctions as

$$\psi(x) = \sum_n \theta_n \phi_n(x)\,; \qquad \bar{\psi}(x) = \sum_n \bar{\theta}_n \bar{\phi}_n(x), \tag{8.6}$$

where θ_i and $\bar{\theta}_i$ are Grassman variables so that the fields anti-commute. Using these expansions we can now precisely define the path integral measure for fermions as

$$\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] = \prod_n d\theta_n d\bar{\theta}_n, \tag{8.7}$$

where the $d\theta_i$ are Grassman 1-forms described in Section 5.1. Consider a change of variables[12]

$$\psi(x) \to \psi'(x) = \psi(x) + \epsilon(x);$$

[12] In functional integration, a change of variables actually means a change of function on which the integral depends, in this case ψ.

$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x) + \bar{\epsilon}(x), \tag{8.8}$$

where $\epsilon(x) = i\alpha(x)\gamma^5\psi(x)$ and $\bar{\epsilon}(x) = \bar{\psi}i\alpha(x)\gamma^5$. This corresponds to an infinitesimal space-dependent chiral transformation with local parameter $\alpha(x)$. We can thus express this change of variables more compactly as

$$\psi'(x) = (1 + i\alpha\gamma^5)\psi(x),$$

which after substitution of the eigenfunction equation (8.6) becomes

$$\sum_m \theta'_m \phi_m(x) = (1 + i\alpha\gamma^5) \sum_p \theta_p \phi_p(x).$$

Due to the orthonormality of the eigenfunctions, we may multiply this equation by $\phi_n(x)$ and integrate over all space giving

$$\theta'_n = \sum_m \delta_{nm}\theta'_m = \sum_m \theta'_m \int d^4x \phi_m(x)\phi_n^\dagger(x) \tag{8.9}$$

$$= \int d^4x \phi_n^\dagger(x) \sum_p (1 + i\alpha\gamma^5)\phi_p(x)\theta_p \tag{8.10}$$

$$= \sum_p (\delta_{np} + C_{np})\theta_p, \tag{8.11}$$

where we have defined the C-matrix as

$$C_{np} = i \int d^4x \phi_n^\dagger(x)\alpha(x)\gamma^5\phi_p(x).$$

Hence due to the property of Jacobians of odd variables (5.2) we have found that the measure under the change of variables (8.8) transforms as

$$\mathcal{D}[\psi']\mathcal{D}[\bar{\psi}'] = \frac{1}{[\det(\mathbb{1} + C)]^2}\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}],$$

i.e. with the square inverse of the Jacobian. Thus we have reduced the problem to calculating a determinant. We can expand the determinant using a corollary of the Jacobi formula,[13] namely

$$\det(\mathbb{1} + C) = e^{\mathrm{Tr}(\ln(\mathbb{1}+C))} = e^{\mathrm{Tr}(C) - \frac{1}{2}C^2 + \mathcal{O}(\alpha^4)}.$$

To first order in the parameter α the determinant is then

$$[\det(\mathbb{1} + C)]^{-2} = e^{-2i\int d^4x \alpha(x) \sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)}. \tag{8.12}$$

The partition function after the chiral transformation (8.8) can now be written as

$$Z = \int \mathcal{D}[\psi']\mathcal{D}[\bar{\psi}'] e^{S[\psi,\bar{\psi}] - 2i\int d^4x\alpha(x)\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)},$$

from which we can extract the divergence in the chiral current by variation of the parameter $\alpha(x)$

$$\partial_\mu j_A^\mu = 2im\bar{\psi}\gamma^5\psi + 2i\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x). \tag{8.13}$$

[13] The corollary states that $\det(e^{tB}) = e^{\mathrm{Tr}(tB)}$, where B is a square matrix.

Hence it remains to evaluate the sum[14]

$$A(x) = \sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x), \tag{8.14}$$

which is divergent. This divergence is due to the trace over infinite modes of the Dirac operator, and hence to get any physical quantities we need to regularize.

8.3 Gauge Invariant Regularization

To regulate the sum (8.14) we introduce a regulator of the form $f(\lambda_n^2/\Lambda^2)$, where λ_n are the eigenvalues of the Dirac operator in (8.5) and Λ is a UV cut-off. This method we follow originates from Fujikawa [16], and was then further generalised by Umezawa [17]. Here f is an arbitrary smooth function such that

$$f(\infty) = f'(\infty) = f''(\infty) = \cdots = 0, \qquad f(0) = 1.$$

We can now write down the regulated sum as

$$\begin{aligned}
A(x) &= \lim_{\Lambda\to\infty} \sum_n \phi_n^\dagger \gamma^5 f(\lambda_n^2/\Lambda^2)\phi_n \\
&= \lim_{\Lambda\to\infty} \sum_n \phi_n^\dagger \gamma^5 f(\not{D}^2/\Lambda^2)\phi_n \\
&= \lim_{\Lambda\to\infty} \sum_n \int \frac{d^4k d^4k'}{(2\pi)^4(2\pi)^4} e^{-ik'x}\tilde{\phi}_n^\dagger(k')\gamma^5 f(\not{D}^2/\Lambda^2)e^{ikx}\tilde{\phi}_n(k) \\
&= \lim_{\Lambda\to\infty} \mathrm{Tr}\left(\int \frac{d^4k}{(2\pi)^4} e^{-ikx}\gamma^5 f(\not{D}^2/\Lambda^2)e^{ikx}\right),
\end{aligned}$$

where we have written the Dirac modes as a Fourier transform and used that $\sum_n \tilde{\phi}_n^\dagger(k)\tilde{\phi}_n(k') = (2\pi)^4\delta^4(k-k')$. The square of the Dirac operator can be expanded to give the identity

$$\not{D}^2 = D^2 + S^{\mu\nu}F_{\mu\nu},$$

where $S^{\mu\nu}$ is the spinor representation of the Lorentz algebra defined in (4.5). Thus $A(x)$ now becomes

$$\begin{aligned}
A(x) &= \lim_{\Lambda\to\infty} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\gamma^5 f((D^2 + S^{\mu\nu}F_{\mu\nu})/\Lambda^2)e^{ikx} \\
&= \lim_{\Lambda\to\infty} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \gamma^5 f\left(\frac{1}{\Lambda^2}(ik_\mu + D_\mu)^2 + S^{\mu\nu}F_{\mu\nu}\right),
\end{aligned}$$

where we used the relation $D_\mu[g(x)\exp(ikx)] = \exp(ikx)(ik_\mu + D_\mu)g(x)$. We can now Taylor expand f around k^2, which gives

$$A(x) = \lim_{\Lambda\to\infty} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^2}\gamma^5\left(f(k^2/\Lambda^2) + \frac{B}{\Lambda^2}f'(k^2/\Lambda^2) + \frac{B^2}{2!\Lambda^4}f''(k^2/\Lambda^2)\right),$$

[14] The notation $A(x)$ might be seen as confusing, but all sources use this notation, so we adopt it in this report.

where $B(x) = (k_\mu + D_\mu)^2 + S^{\mu\nu}F_{\mu\nu} - k_\mu^2$. Notice that due to the properties of traces of γ-matrices, the first non vanishing term in the trace is of $\mathcal{O}(\Lambda^{-4})$, on the other hand this is the last non-vanishing term in the limit $\Lambda \to \infty$. Hence the infinite sum reduces to just one term

$$\begin{aligned}
A(x) &= \lim_{\Lambda\to\infty} \mathrm{Tr}\left[\gamma^5 \left(\frac{1}{\Lambda^2} S^{\mu\nu}F_{\mu\nu}\right)^2\right] \frac{1}{8} \int \frac{d^4k}{(2\pi)^4} f''(k^2/\Lambda^2) \\
&= \lim_{\Lambda\to\infty} \mathrm{Tr}(F \star F) \int \frac{d^4k}{(2\pi)^4} f''(k^2) \\
&= -\frac{1}{16\pi^2} \mathrm{Tr}(F \star F),
\end{aligned}$$

where we have used some properties of the γ-matrices. This means that we have found the divergence in the axial current in Monkowski space is

$$\partial_\mu j_A^\mu = \mathcal{A} = -\frac{1}{8\pi^2} \mathrm{Tr}(F \star F) = -\frac{1}{16\pi^2} F^{a,\mu\nu} \star F^a_{\mu\nu}, \tag{8.15}$$

where the factor of i comes from expressing the result in Minkowski space. This is of course non-zero even in the massless limit. We have thus shown that chiral anomalies are due to non-perturbative instanton-like effects.

9 Conclusion

In this report we have discussed two main aspects of non-perturbative Yang-Mills theories, namely instantons and chiral anomalies. We explicitly found the one-istanton solution to an $SU(2)$ pure Yang-Mills theory, and discussed the consequences they have for the vacuum structure of these theories. Further analysis can be done in this area, by explicitly finding higher winding instanton solutions and explore their behaviour. One can also look at multiple instantons in one system, which will lead to the theory of instanton gases. We have also shown that chiral anomalies, at least in the Abelian case, can be studied using path integral methods. Our derivation was based on the invariance of the path integral formalism, which lead us to look more closely at the measure. This method can also be expanded to non-Abelian Yang-Mills theories, where chiral anomalies turn out to be a bigger problem. We were also forced to regulate the chiral anomaly, which was done using a gauge invariant cut-off regulator. Other methods of renorming chiral theories can be explored, and consequently one may also show that the ABJ anomaly is independent of the choice of regulator. In conclusion, we have seen a far from complete introduction to non-perturbative QFT. These same methods developed in this report can be used to study more complicated theories like the standard model.

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