Holographic Entanglement Entropy

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Abstract

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In this dissertation we study various aspects of the holographic entanglement entropy proposals. We give a compact introduction to AdS space and CFT, which we use to state and explore the AdS/CFT conjecture. We focus on a bottom-up approach to the duality and calculate some boundary observables. Also, introducing some concepts from quantum information theory we combine them with the AdS/CFT correspondence and give a concise description of the RT and HRT proposals. We verify the validity of these proposals by holographically proving some entropy inequalities. At the end we provide a holographic derivation of the RT conjecture and discuss some of the subtleties involved.

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1 Introduction

The AdS/CFT correspondence conjectured two decades ago by Maldecina in [1] revealed a deep connection between two seemingly very different physical theories. It points to a duality between string theory in Anti-de Sitter (AdS) space and Conformal Field Theory (CFT) a dimension lower. Since the early days of the correspondence many more details have been discovered which uncovered a large holographic landcape with dualies between various variations of the general theories of both sides. Most problems solved although related more to the dynamical matching of degrees of freedom or to the calculation of quantities through the correspondence. In fact, it turns out when the field theory side of the correspondence is strongly coupled the gravity side largely simplifies.

In this dissertation we attempt to discuss more the fundamental aspects of the AdS/CFT duality, that is the emergence of the AdS spacetime itself. It was originally uncovered in [2, 3] that the AdS geometry on the gravity side is strongly connected to the entanglement structure in the boundary CFT. This seems higly non-trivial as it connects pure geometric objects on one side with the specific QFT states on the other. The argument was later extended in [4] to include a lot wider range of QFT states. These proposals established a new field of study within the holographic landscape called holographic entaglement entropy. Rather tellingly called as such because it relates the entanglement entropy of field theory states with the geometry of the dual AdS space.

Our aim is to explore some aspects of these proposals. The discussion will not be exhaustive nor self-contained, we rather give an introduction to various aspects of the holographic entanglement entropy conjecture. Apart from the original proposals [2, 3, 4], many great reviews exist [5, 6, 7] which proved very useful in writing parts of this dissertation. Due to its complexity we first have to discuss various aspects of holography and quantum information theory before moving on to combine these two.

The outline of the dissertation goes as follows: In Section 2, we introduce the two sides of the AdS/CFT correspondence separately. We first define AdS space and discuss its geometric properties after which we move on to study conformal field theories. Section 3 is dedicated to the AdS/CFT correspondence. First, motivating the duality we discribe its many forms and comment on their validity. We then look at how to calculate boundary observables from the bulk theory, for example CFT correlators. As an interlude, in Section 4 we give an introduction to quantum information theory and define some of the main quantities involved in the holographic entaglement entropy formula. We also set up a path integral framework in which to calculate some of these quantities in general QFT's. Finally, in Section 5 we precisely state the holographic entanglement entropy proposals and explore many of its properties. We then provide a holographic derivation of the conjecture which highlights some of the subtleties involved.

2 Background: AdS and CFT

The aim of this section is to provide the background needed to formulate the AdS/CFT correspondence while fixing some of the notation and conventions used throughout this dissertation.

2.1 Anti-de Sitter Space

Anti-de Sitter space is a member of a larger family of geometric objects called maximally symmetric spacetimes. It therefore instructive to discuss what such spacetimes are as this provides us with a nice view of the origins of AdS space. We consider vacuum solutions to Einstein's equations in d dimensions

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.1}$$

where R and $R_{\mu\nu}$ are the Ricci scalar and tensor respectively, $g_{\mu\nu}$ is the Riemann metric, $T_{\mu\nu}$ is the energy momentum tensor and Λ is the cosmological constant. That is, we are looking for solutions with $T_{\mu\nu} = 0$. Einstein's equation above may also be rederived from the action

$$S[g_{\mu\nu}, \phi] = S_{\mathrm{EH}}[g_{\mu\nu}] + S_{\mathrm{Matter}}[g_{\mu\nu}, \phi],$$

where $S_{\mathrm{Matter}}[g_{\mu\nu}, \phi]$ is the action for the collection of matter fields ϕ and

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \frac{1}{16\pi G} \int d^d x \sqrt{-g}\,(R - 2\Lambda)$$

is the Einstein-Hilbert action. Thus we see that vacuum solutions to Einstein's equation are precisely the solutions to the variational problem $\delta S_{\mathrm{EH}} = 0$.

Maximally symmetric spaces are ones which have the highest possible number of symmetries. From the viewpoint of differential geometry, symmetries are formulated in term of Killing vectors. One can show that a manifold of dimension d has at most $d(d+1)/2$ Killing vectors and so maximally symmetric spaces are ones which saturate this bound. It can also be shown that if a space is maximally symmetric it must have a Riemann tensor given by

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)} (g_{\nu\sigma} g_{\mu\rho} - g_{\nu\rho} g_{\mu\sigma}),$$

i.e. it has a constant curvature. For a Lorentzian g there are three distinct cases here depending on the sign of the Ricci scalar. For $R = 0$ we have a flat space, which we will refer to as Minkowski space. In the case of constant positive curvature $R > 0$ we say the space is de-Sitter, while if $R < 0$ we call it anti de-Sitter. With each choice of sign there comes a corresponding choice of sign for the cosmological constant enforced by Einstein's equation.

Contracting (2.1) with $g^{\mu\nu}$ we find the relation

$$R = \frac{2d\Lambda}{d-2},$$

which implies that $\Lambda = 0$ for Minkowski, $\Lambda > 0$ for de Sitter and $\Lambda < 0$ for anti-de Sitter space.

For future convenience from now on we will discuss AdS_{d+1}, i.e. $d+1$ dimensional AdS space. AdS_{d+1} spacetime can be embedded into $d+2$ dimensional Minkowski space via

$$\bar{\eta}_{MN}X^M X^N = -(X^0)^2 + \sum_{i=1}^{d}(X^i)^2 - (X^{d+1})^2 = -L^2 \tag{2.2}$$

with an induced metric

$$ds^2 = \bar{\eta}_{MN}dX^M dX^N = -(dX^0)^2 + \sum_{i=1}^{d}(dX^i)^2 - (dX^{d+1})^2,$$

where $M, N \in \{0, 1, \cdots, d+1\}$ and $\bar{\eta} = \mathrm{diag}(-, +, +, \cdots, +, -)$. We will also refer to the embedding radius L as the AdS radius. It is important to note at this point that the the embedding (2.2) manifestly has an $SO(d, 2)$ symmetry and thus the isometry group of AdS_{d+1} is also $SO(2, d)$. This provides a nice check since we know that the group $SO(2, d)$ is $(d+2)(d+1)/2$ dimensional[1] and thus we again see that AdS space is indeed maximally symmetric.

Solving the embedding (2.2) we can find multiple coordinate systems which cover either the whole or only part of AdS_{d+1}. Here, we give two examples of such coordinates in each case defining the coordinates and giving the metric.

- Global Coordinates $(\rho, \tau, \Omega_{d-1})$:

$$\begin{cases} X^0 &= L\cosh\rho\cos\tau & \Omega_i \in \mathbb{R} \text{ s.t. } \sum_i \Omega_i^2 = 1 \\ X^i &= L\,\Omega_i \sinh\rho & \rho \in \mathbb{R}_+, \quad i \in \{1, 2, \cdots, d\} \\ X^{d+1} &= L\cosh\rho\sin\tau & \tau \in [0, 2\pi) \end{cases} \tag{2.3}$$

$$ds^2 = L^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2\right) \tag{2.4}$$

- Poincaré Coordinates (z, x^i, t):

$$\begin{cases} X^0 &= \frac{1}{2z}(z^2 + L^2 + \vec{x}^2 - t^2) & z \in \mathbb{R}_+ \\ X^j &= \frac{Lx^i}{z} & j \in \{1, 2, \cdots, d-1\} \\ X^d &= \frac{1}{2z}(z^2 - L^2 + \vec{x}^2 - t^2) & \vec{x} \in \mathbb{R}^{d-1} \\ X^{d+1} &= \frac{Lt}{z} & t \in \mathbb{R} \end{cases} \tag{2.5}$$

$$ds^2 = \frac{L^2}{z^2}(dz^2 + d\vec{x}^2 - dt^2) \tag{2.6}$$

[1] The difference from the formula above is due to the fact that we are now working in $d+1$ dimensions.

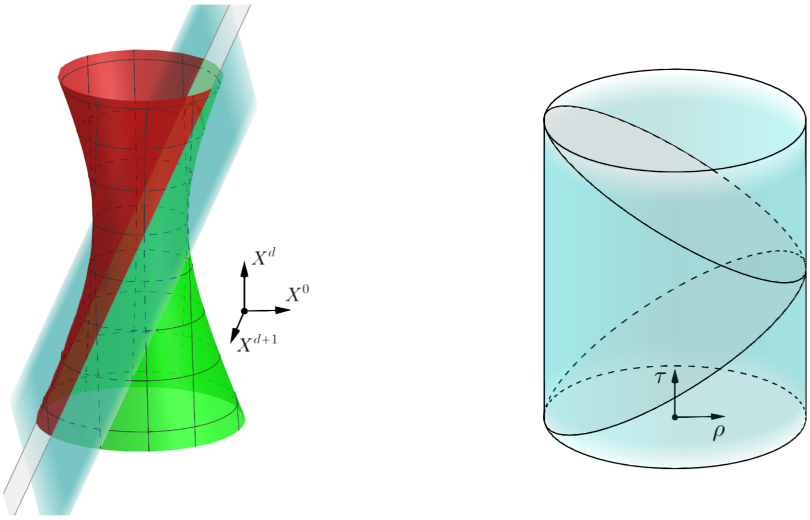

FIGURE 2.1: The two Poincaré patches divided by the hypersurface $X^0 = X^d$ depicted in the embedded AdS space [Left]. The Poincaré patch shown in global coordinates [Right].

The global coordinates unsurprisingly cover the whole of AdS_{d+1}. On the other hand, two Poincaré charts are needed to cover all of AdS_{d+1}, namely $z > 0$ and $z < 0$. These two charts divide the space by the $X^0 = X^d$ hypersurface as seen in Figure 2.1. Thus as defined above for $z > 0$ the Poincaré coordinates cover only half of AdS_{d+1} called the *Poincaré patch*. This patch is shown within the greater AdS space in Figure 2.1.

Let us now have a closer look at the above coordinates. Seemingly, there are two interesting limits to examine in the Poincaré coordinates. The limit $z \to \pm\infty$ is precisely the hypersurface dividing the two Poincaré patches and thus this is a coordinate singularity. On the other hand, the limit $z \to 0$ is a real pole of the metric which we call the AdS boundary, or $\partial\mathrm{AdS}_{d+1}$. In global coordinates this boundary is located at $\rho \to \infty$. From (2.6) we see that the the metric in the Poincaré patch for fixed z is conormally Minkowski, this will be important later on.

In either case, we can evaluate the Ricci scalar and hence the cosmological costant for this *AdS* geometry, which we find to be

$$R = -\frac{d(d+1)}{L^2} \qquad \text{and} \qquad \Lambda = -\frac{d(d-1)}{2L^2}.$$

It is also important to note that $\partial\mathrm{AdS}_{d+1}$ is timelike and thus it can be reached in finite time. This means that to specify dynamics in AdS space one not only needs initial data on a Cauchy slice but also boundary conditions at $z \to 0$. This has consequences that will be pivotal in latter sections.

2.2 Conformal Field Theory

Conformal Field Theories (CFT's) are field theories which are invariant under scaling and local angle preserving transformations called conformal transformations. A conformal transformation is a coordinate map $x^\mu \to x'^\mu$ such that[2]

$$\eta'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma} = \Omega^{-2}(x)\eta_{\mu\nu}, \tag{2.7}$$

i.e. the metric remains invariant up to a Weyl transformation. These transformations form a group, called the conformal group which turns out to be $SO(2,d)$. Setting $\Omega = 1$ gives the Poincaré group and thus it is a subgroup of the larger conformal group. Considering infinitesimal transformations $x^\mu \to x'^\mu = x^\mu + \varepsilon\xi^\mu(x) + \mathcal{O}(\varepsilon^2)$ and $\Omega \equiv e^{\varepsilon K(x)} = 1 + \varepsilon K(x) + \mathcal{O}(\varepsilon^2)$, and imposing (2.7) gives the coformal Killing equation

$$2K\eta_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu. \tag{2.8}$$

It tuns out that this has very different solutions in $d = 2$ and $d > 2$ dimensions. We consider the $d > 2$ case for which the most general solution is given by

$$\xi^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \sigma x^\mu + b^\mu x^2 - 2b_\nu x^\nu x^\mu \tag{2.9}$$

$$K(x) = \frac{1}{d}\partial_\mu\xi^\mu, \tag{2.10}$$

where $\omega^\mu{}_\nu = -\omega^\nu{}_\mu$. The conformal Killing vectors $\xi = \xi^\mu\partial_\mu$ give the generators for the Lie algebra of the conformal group called the conformal algebra. Apart from the Poincaré generators of the Poincaré algebra, there are two new generators of the conformal algebra generating

- Dilations: $D = x^\mu\partial_\mu$;
- Special Conformal Transformations: $K_\mu = x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu$.

Under conformal transformations a primary scalar field theory operator $\mathscr{O}_\Delta$ transforms as

$$\mathscr{O}_\Delta(x) \to \mathscr{O}'_\Delta(x') = \Omega^{-\Delta}\mathscr{O}_\Delta(x) \approx \mathscr{O}_\Delta(x) + \delta\mathscr{O}_\Delta(x),$$

where we refer to Δ as the conformal dimension of $\mathscr{O}_\Delta$. From the above infinitesimal transformations we can deduce that

$$\delta\mathscr{O}_\Delta(x) = -K(x)\Delta\mathscr{O}_\Delta(x) - \xi^\mu\partial_\mu\mathscr{O}_\Delta(x),$$

where $\xi^\mu(x)$ and $K(x)$ are given above in (2.9) and (2.10).

CFT's represent a very special class of field theories as the above constraint on the metric also largely restricts any correlation function in the theory. As an example, take two scalar primary operators $\mathscr{O}_1(x_1)$, $\mathscr{O}_2(x_2)$ and consider their two point correlator $\langle\mathscr{O}_1(x_1)\,\mathscr{O}_2(x_2)\rangle \equiv f(x_1, x_2)$. As our theory is conformally invariant we know that the two point function is also

[2] $\eta_{\mu\nu}$ is the flat Minkowski metric.

invariant under the conformal group, i.e. it satisfies the conformal ward identity

$$\delta \langle \mathscr{O}_1(x_1)\, \mathscr{O}_2(x_2)\rangle = \langle \delta\mathscr{O}_1(x_1)\, \mathscr{O}_2(x_2)\rangle + \langle \mathscr{O}_1(x_1)\, \delta\mathscr{O}_2(x_2)\rangle = 0.$$

This allows us to restrict the form of $f(x_1, x_2)$ as follows. Translation symmetry implies that $f(x_1, x_2) \to f(x_1 - x_2) \equiv f(x_{12})$, while Lorentz symmetry adds an extra constraint $f(x_{12}) \to f(x_{12}^2)$. Now, due to the extra dilation and special conformal transformation we can also impose the conformal ward identities for these transfromations. Invariance under dilations implies that

$$\delta_D f(x_{12}^2) = (x_1^\mu \partial_{1\mu} + x_2^\mu \partial_{2\mu} + \Delta_1 + \Delta_2) f(x_{12}^2) = 0,$$

which restricts $f(x_{12}^2)$ to the form

$$f(x_{12}^2) = \frac{A}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}}.$$

The ward identity for the special conformal transformation, given by

$$\delta_{K_\mu} f(x_{12}^2) = (x_1^2 \partial_1^\mu - 2x_1^\mu \Delta_1 - 2x_1^\mu x_1^\nu \partial_{1\nu} + x_2^2 \partial_2^\mu - 2x_2^\mu \Delta_2 - 2x_2^\mu x_2^\nu \partial_{2\nu}) f(x_{12}^2) = 0,$$

further restricts the two point function, which must therefore be of the form

$$\langle \mathscr{O}_1(x_1)\, \mathscr{O}_2(x_2)\rangle = \begin{cases} \frac{A}{(x_{12}^2)^\Delta} & \text{if} \quad \Delta_1 = \Delta_2 = \Delta \\ 0 & \text{if} \quad \Delta_1 \neq \Delta_2. \end{cases} \tag{2.11}$$

Thus we can see that in a CFT the form of the scalar two point functions is fully fixed. This restricted form of the two point CFT correlator will provide a useful check for the AdS/CFT correspondence.

3 The AdS/CFT Correspondence

The most general observation one can make about the AdS/CFT correspondence is that it draws an equivalence between two very different theories. On one side of the correspondence there is a special class of strongly coupled field theories, while on the other side there is quanatum gravity in asymptotically AdS spacetimes. This is a higly non-trivial statement on its own, however the field theory exists in one less dimension then the quantum gravity theory, which further highlights the significance of the duality. The connection between the two theories was first discovered in [1], where a lot more specific version of the duality was considered, namely the duality between $\mathcal{N} = 4$ Super Yang-Mills Theory in 4 dimensions and string therory in $\mathrm{AdS}_5 \times S^5$. Since then, we know that considerably weaker statements of the correspondence also exist which motivated the alternative name, Gauge/Gravity correspondence. In this section we explore the landscape of this correspondence and use it to calculate some observable field theory quantities from gravity theroies. There are plenty of original papers, e.g. [1, 8], covering what we will discuss, but as we are taking a more retrospective viewpoint, the most useful resources for compiling this section were the notes [9, 10], which both follow a bottom-up approach like us, and the books [11, 12].

3.1 Motivating AdS/CFT

The original motivation for the duality came from the special behaviour of some gauge thories in the large N limit.[1] It was realised in [13] that some gauge theories largly simplify in such a limit, which again seems hihly non-trivial since one would expect the degrees of freedom to quickly diverge. It turns out that if we consider the specific large N limit of $SU(N)$ theories in which $N \to \infty$, the Yang-Mills coupling $g \to 0$ and g^2N remains fixed, non-planar diagrams are supressed which results in big simplifications. Not only do planar diagrams dominate, but in the above limit, also called the *'t Hooft large N limit*, the subdominant digrams arrange according to topology, which hints a connection to perturbative string theory.

There are also motivations from the gravity side of the correspondence. For example, the structure of boundary obervables in AdS gravity theories resembles field theory correlaors. Also, the Bekenstein–Hawking entropy, since it is proportional to the area of some horizon, scales like a local field theory on this horizon.

3.2 The AdS/CFT Dictionary

In what follows we give an introductory tour of the AdS/CFT landscape. There is no derivation of the correspondence, rather the above motivations together with other considerations from the string theory prespective point towards the equivalence of objects on either side of the duality. There are, however, many calculational checks one can perform to verify

[1]Here N refers to the gauge group dimension, i.e. $SU(N)$ for a Yang-Mills Theories.

the validity of AdS/CFT. We now state two versions of the duality and comment on their validity and use.

The AdS/CFT Correspondence: The most general framework of the correpondence goes as follows. On the gravity side of the correspondence we have string theory on an asymptotically AdS_{d+1} spacetime with a dynamical metric, thus a theory of quantum gravity. On the field theory side there is a field theory in d dimensions which approaches a CFT_d in the UV limit.

The AdS space is usually referred to as the *bulk*. This is a very general and not so useful statement of the duality. There are a lot more precise statements one can make about more specific theories on either side of the correspondance. The one first dicdovered, for example, draws an equivalence between $\mathcal{N} = 4$ Super Yang-Mills theory and string therory in $\text{AdS}_5 \times S^5$. In this case the limits of validity are well studied and uderstood.

The $\text{AdS}_5/\text{CFT}_4$ Correspondence: $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills theory in 4 dimensions is dynamically equivalent to type IIB superstring theory on $\text{AdS}_5 \times S^5$. The free paramaters of the two theories are related via

$$g_{\text{YM}}^2 = 2\pi g_s \qquad \text{and} \qquad 2g_{\text{YM}}^2 N = \frac{L^4}{l_s},$$

where g_{YM} is the Yang-Mills coupling, g_s is the string coupling, L is the AdS curvature and l_s is the string length. The dynamical equivalence means that these two theories contain precisely the same information and therefore decribe the same physics.

Even though the above statement is very robust it is still hard to do specific calculations due to the complicated theories on both sides. Thus weaker, but more useful versions of the correspondance can be thought of by invoking varous limits of the involved parameters. The most useful limit turns out to be one where we take the weak string coupling $g_s \ll 1$ limit in which case the string theory reduces to 'classical' string theory meaning that tree level diagrams dominate. The corresponding limit of the field theory is where $g_{\text{YM}} \ll 1$ with $g_{\text{YM}}^2 N$ fixed, i.e. the 't Hooft large N limit. In this limit things largely simplify. We have already seen that the field theory in this limit is dominated by planar diagrams and we have also mentioned that the gravity side reduces to 'classical' string theory. In this limit there is only one free parameter on both sided of the duality. These are the t'Hooft coupling $\lambda = g_{\text{YM}}^2 N$ for the field theory and the ratio L^4/l_s for the string theory, which are related by $2\lambda = L^4/l_s$. From here on, we will be implicitly working in this t'Hooft large N limit.

In any case, there are broad ranging equivalences we can draw between objects in AdS and CFT without having to refer to the particular version of the correspondence. These identify some features of the duality and provide a dictionary between the two theories, thus it is most commonly referred to as the AdS/CFT dictionary. In what follows we list some of the elements of this dictionary:

$\underline{\mathbf{AdS}_{d+1}}$	$\longleftrightarrow$	$\underline{\mathbf{CFT}_d}$
Isometry group $SO(d,2)$	$\longleftrightarrow$	Symmetry group $SO(d,2)$
- Translations in radial direction	$\longleftrightarrow$	Dilatation -
Boundary conditions	$\longleftrightarrow$	Sources in $\mathscr{Z}_{\text{CFT}}$
Bulk fields	$\longleftrightarrow$	Sources of operators
- Dynamical metric $h_{\mu\nu}$	$\longleftrightarrow$	Stress-energy tensor $T_{\mu\nu}$ -
- Scalar field ϕ	$\longleftrightarrow$	Scalar operator $\mathscr{O}$ -
- Mass m	$\longleftrightarrow$	Conformal dimension Δ -
Black holes	$\longleftrightarrow$	Finite temperature

When we say that boundary conditions in the bulk act as sources in the CFT partition function, what we mean is the relation

$$\mathscr{Z}_{\text{String}}[\phi_0(x^\mu)] = \left\langle e^{\int d^dx\, \phi_0(x^\mu)\mathscr{O}(x^\mu)} \right\rangle_{\text{CFT}}, \tag{3.1}$$

where $\phi_0(x^\mu)$ represents boundary conditions for a corresponding bulk field $\phi(x^\mu, z)$ and $\mathscr{O}$ is the CFT operator dual to ϕ. This equivalence of the partition functions forms a pivotal part of the duality. Relation (3.1) also gives exact meaning to the latter parts of the dictionary. For example, if ϕ is a scalar bulk field, then its boundary condition $\phi_0(x^\mu)$ sources a scalar CFT operator $\mathscr{O}$ precisely as written in (3.1). The above equivalence further simplifies in the large t'Hooft coupling $\lambda \gg 1$ limit[2], as in this limit the string partition function is dominated by it's saddle point of the classical action, i.e.

$$\mathscr{Z}_{\text{String}}[\phi_0(x^\mu)] \approx e^{-S_{Cl}[\phi_0]}.$$

Now, some conceptual comments are due here. We will often refer to the CFT as living on the boundary of AdS. This, though a useful picture, can sometimes be vague. We have seen in Section 2.1 that the boundary of AdS_{d+1} has naturally the structure of Minkowski space, the spacetime of our CFT. This together with the fact that the boundary conditions of bulk fields source CFT operators give the special connection between bulk and boudary.

In general we differenciate between two conceptually different, but equivalently useful ways of looking at AdS/CFT [14].

- Top-Down: In this approach we start with a well know duality, for example the $\text{AdS}_5/\text{CFT}_4$ correspondence. We then have well defined theories on both sides with specific parameters and known Lagrangians. We can then ask questions about the specific dynamis of the theories and use the correspondence to calculate new quantities or explore new regimes within these theories.
- Bottom-Up: Somewhat opposite to the top-down approach, we are less interested in the specific aspects of the theories on either side. Since we usually use this approach to holographically calculate CFT observables, we only consider a minimal AdS model. By minimal we mean that we only introduce objects into AdS space that are specifically

[2] Even though we are taking the $\lambda \gg 1$ limit, we are still considering λ to be fixed and thus we are still working in the t'Hooft large N limit.

needed for the calculation. For example, to calculate a general CFT correlator it is enough to consider a scalar field in AdS.

Both approaches have their place in the Holographic landscape, we will however concentrate on the Bottom-Up approach as this is better suited to discussing holographic entanglement entropy.

3.3 Boundary Observables

We now move on to calculating field theory correlation functions in the bulk. This will further clarify the subtelties involving the duality and give a check on its correctness. Consider a massive scalar field $\phi(x^\mu, z)$ living in the bulk $\mathrm{AdS}_{\mathrm{d+1}}$ spacetime. The bulk action is then given by[3]

$$S_{\mathrm{AdS}} = -C \int d^d x\, dz\, \sqrt{-g}\left(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2\right),$$

where $m = 1, 2, \cdots, d+1$, m is the mass of ϕ and C is a shorthand for the normalisation which we will ususally drop. Recall that the AdS metric in Poincaré coordinates is given by

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu).$$

Thus the above action gives rise to the Klein-Gordon equation

$$\left(z^2\partial_z^2\phi - (d-1)z\,\partial_z + z^2\partial^\mu\partial_\mu - m^2L^2\right)\phi(x,z) = 0.$$

We want to find the behaviour of this solution near the AdS boundary $z \to 0$. Considering a plane wave ansatz in the x directions, i.e. $\phi(x,z) = \exp(ip^\mu x_\mu)\phi(z)$, gives

$$\left(z^2\partial_z^2\phi - (d-1)z\,\partial_z - z^2p^2 - m^2L^2\right)\phi(z) = 0,$$

where near the boundary the term with p^2 is subleading and so can be ignored. Hence for $z \to 0$ we have two independent solutions given by

$$\phi(x,z) = (\phi_+(x) + \mathcal{O}(z^2))z^{\Delta_+} + (\phi_-(x) + \mathcal{O}(z^2))z^{\Delta_-},$$

where

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2}. \tag{3.2}$$

These exponents also satisfy the relationship $\Delta_+ + \Delta_- = d$. Since there are two solutions, we have to pick one to serve as the boundary condition and hence as the source of the dual CFT operator. The standard boundary condition is to fix ϕ_-, which is the leading behaviour[4] of ϕ as $z \to 0$. Then, conformal invariance of ϕ on the boundary requires that under dilations, i.e. under $x^\mu \to \lambda x^\mu$ and $z \to \lambda z$, we must have

$$\phi_- \to \lambda^{-\Delta_-}\phi_-,$$

[3] Notice, that we are considering the probe limit in which the scalar field does not contribute to the energy-momentum tensor.

[4] It is the leading behaviour because $\Delta_+ \geq \Delta_-$.

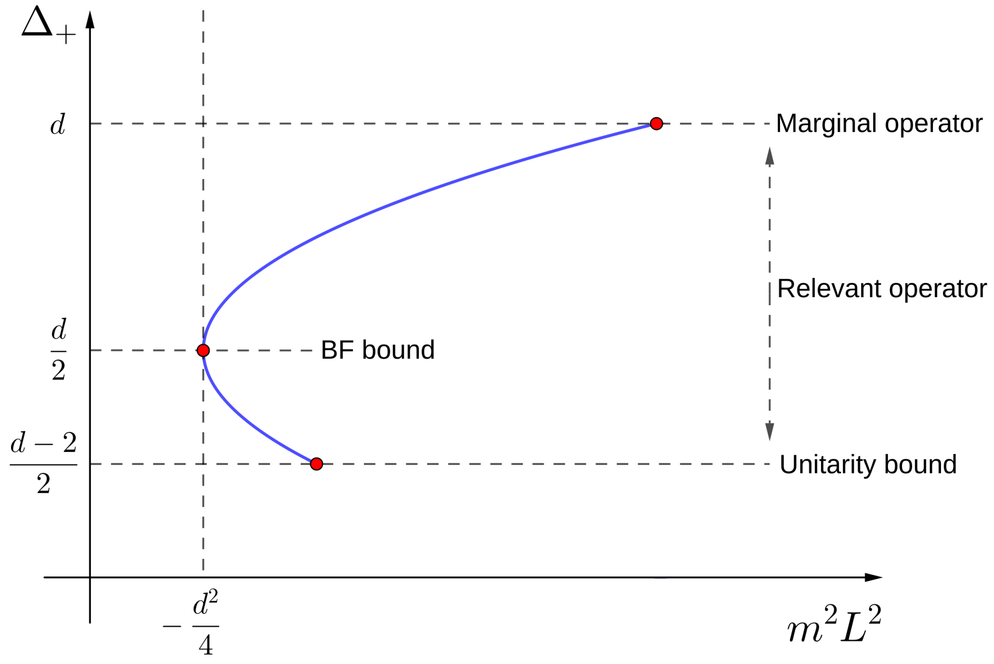

FIGURE 3.1: The conformal scaling dimension Δ_+ of a CFT operator $\mathcal{O}$ as a function of the mass of the corresponding AdS field.

that is ϕ_- has conformal scaling dimension Δ_-. For a CFT operator $\mathcal{O}_{\Delta_+}$ of dimension Δ its source must have dimension $d-\Delta$ so that the term $\int \phi_0 \mathcal{O}$ is conformally invariant. Thus we conclude that ϕ_- is the source for the field theory operator $\mathcal{O}$ of dimension $\Delta_+ = d - \Delta_-$.

Now, consider again the relation (3.2) which determines the scaling dimension of $\mathcal{O}_{\Delta_+}$. We can see that this only allows us to construct operators with dimensions greater than d, that is it seems like this procedure only allows us to give a bulk description of irrelevant CFT operators, or at best, marginal ones. What partly resolves this issue is that unlike in flat spacetimes, in AdS the mass squared of the scalar field is allowed to take values down to

$$\frac{-d^2}{4L^2} \leq m^2,$$

called the Breitenlohner-Freedman (BF) bound, without introducing instabilities in the theory [15]. This in turn allows for dual desciptions of CFT operators down to dimensions of $\Delta_+ \geq d/2$. Thus we can distinguish between the following cases for the CFT operators:

- $m^2 < 0,\ \Delta_+ < d$: Relevant oprator, $\phi \to 0$ as $z \to 0$;
- $m^2 = 0,\ \Delta_+ = d$: Marginal operator, $\phi \to \phi_-$ as $z \to 0$;
- $m^2 > 0,\ \Delta_+ > d$: Irrelevant operator, $\phi \to \infty$ as[5] $z \to 0$.

We still, however, have to face up to the fact that we did not yet reach the unitarity bound of $d/2-1$. To describe all relevant operators down to this limit we must return to the choice of the original boundary conditions. Recall, that we said that the standard boundary condition is to fix ϕ_- and leave ϕ_+ free. Invoking alternate boundary conditions, in which we instead

[5] If $\phi_- \neq 0$.

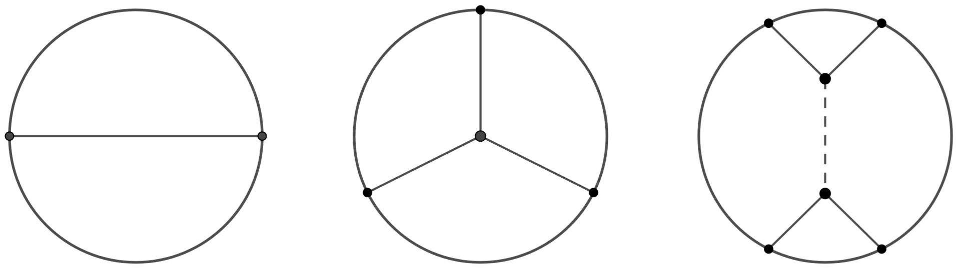

FIGURE 3.2: Examples of tree level Witten diagrams for the two point [Left], three point [Centre] and four point [Right] CFT correlator. We see that 3pt functions require an interaction in the bulk.

fix ϕ_+, reverses the role of ϕ_+ and ϕ_-, and alows us to reach the unitarity bound. This procedure of reversing the role of the two leading order boundary terms is called *alternative quantization*. The above discussion on the dimension of CFT operators is summarised in Figure 3.1.

3.3.1 Correlation Functions

Let us now turn our attention to the bulk calculation of CFT correlators. Consider the central bulk-boundary relationship (3.1). We have already seen that in the large t'Hooft coupling limit the string theory partition function can be approximated via

$$\mathcal{Z}_{\text{String}}[\phi_0(x^\mu)] \approx e^{-S_{\text{Cl}}[\phi_0]}, \tag{3.3}$$

which is a semiclassical saddle point approximation. On the other side, one can rewrite the field theory partition function as the sum of connected diagrams, i.e.

$$\left\langle e^{\int \phi_0 \mathcal{O}} \right\rangle_{\text{CFT}} = e^{-\mathcal{W}_{\text{CFT}}[\phi_0]}, \tag{3.4}$$

where $\mathcal{W}[\phi_0]$ generates the connected correlators of the theory. Substituting back into the main bulk-boundary formula (3.1) we get the simple relation

$$\mathcal{W}[\phi_0] = S_{\text{Cl}}[\phi_0], \tag{3.5}$$

where based on our discussion above we identitfy $\phi_0 = \phi_-$. That is we deduced that the classical action of the solution $\phi(x,z)$ in the bulk is the generating function of connected scalar CFT correlators. We can now calculate the scalar n-point function of a CFT$_d$ from the bulk via functional derivatives, that is

$$\begin{aligned}
\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle &= \frac{\delta^n\,\mathcal{W}[\phi_-]}{\delta\phi_-(x_1)\,\delta\phi_-(x_2)\cdots\delta\phi_-(x_n)}\bigg|_{\phi_-(x_i)=0} \\
&= \frac{\delta^n\,S_{\text{Cl}}[\phi_-]}{\delta\phi_-(x_1)\,\delta\phi_-(x_2)\cdots\delta\phi_-(x_n)}\bigg|_{\phi_-(x_i)=0}.
\end{aligned}$$

What this means is that calculating correlation functions in the bulk amounts to evaluating tree level diagrams on the gravity side of the duality. These diagrams in AdS space are often referred to as *Witten diagrams*, some are depicted in Figure 3.2. These diagrams come with a set of standard Feynman rules which help evaluate them. Thus we must find the approprite propagators for the problem. They are obtained as greens functions of the differential operator $(\Box - m^2)$ as usual, but now with the difference that we have to worry about boundary conditions due to the timelike AdS boundary. We can construct a *bulk-to-boundary propagator* by solving the equation $(\Box - m^2)\phi = 0$, with the boundary condition $\phi(z, x) = \phi_-(x)z^{d-\Delta}$ as $z \to 0$, where we have redefined $\Delta_+ \equiv \Delta$. This problems is of course solved by an integral equation

$$\phi(z, x, x') = \int d^d x' K(z, x, x')\phi_-(x'),$$

where $K(z, x, x')$ is the sought after bulk-to-boundary propagator. One can think of it as the response in the bulk to a change in the boundary conditions. Analogously to standard field theory we may also define a *bulk-to-bulk propagator* $G(x, z, x', z')$ as the solution to

$$(\Box - m^2)G(x, z, x', z') = \frac{1}{\sqrt{-g}}\delta(z - z')\delta(x - x'),$$

which is the standard Green's function. We can then think of this as a measure of the response in the bulk to a change in the bulk. We can then use these propagators to calcuate the above witten diagrams and thus holographically calculate CFT correlators. Also, it is important to mention that for one to talk about 3pt correlators, bulk interactions must be included, which further complicate calculations.

The specific form of the propagators is of course highly theory dependent and in most cases requires us to introduce some sort of consistent renormalisation of the divergences. Such renormalisation scemes within a holographic context is referred to as *holographic renormalisation* and involves adding boundary terms to the action. These boundary terms will not affect the equations of motion, but they allow us to compensate the infinities in an organised fashion. For example the two point scalar CFT correlator can be holographically calculated, as in [8, 12, 11], to give the same result as we got based on purely CFT symmetry arguments, i.e. (2.11). This provides a robust check on the validity of the AdS/CFT conjecture.

4 Quantum Entanglement

We now take a detour and discuss a very different aspect of quantum theory, namely quantum information theory. The main beauty in the formulation of holographic entanglement entropy is how it brings fundamental aspects of quantum theory, like entanglement, into the AdS/CFT landscape. Thus to properly discuss such holographic properties we must first study the entanglement structure of general quantum theories. In what follows we discuss entanglement first in discrete and then in continuous quantum systems. We then define entanglement entropy and state its properties. Once equipped with such quantities we go on to give a path integral representation of the entanglement entropy in QFT. The early parts of this section are loosely based on the classic quantum information book [16].

4.1 Entanglement in Discrete Quantum Systems

Consider a quantum system with a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, for example we can imagine a system of two qubits. Then a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|\psi\rangle = \sum_{i,j} c_{ij} \, |e_i\rangle_A \otimes |e_j\rangle_B \,, \tag{4.1}$$

where $|e_i\rangle_A$ and $|e_j\rangle_B$ are some basis for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. We say a state $|\psi\rangle$ is *separable* if the coefficients c_{ij} in (4.1) factorise into the from $c_{ij} = c_i^A c_j^B$, thus for a separable state we have that

$$|\psi\rangle = \sum_{i,j} c_i^A \, |e_i\rangle_A \otimes c_j^B \, |e_j\rangle_B = |\chi\rangle_A \otimes |\chi\rangle_B \,, \tag{4.2}$$

i.e. it can be written az a tensor product of two states $|\chi\rangle_A \in \mathcal{H}_A$ and $|\chi\rangle_B \in \mathcal{H}_B$. If a state $|\psi\rangle$ does not admit such a separation we call it *entangled*.

As an example we may take the above mentioned system of two qubits in which case $\mathcal{H}_A$ and $\mathcal{H}_B$ are spanned by orthonormal basis $\{|0\rangle_A \,, |1\rangle_A\}$ and $\{|0\rangle_B \,, |1\rangle_B\}$ respectively. Then the state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(\, |0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B \, \big)$$

does not factorise as (4.2) and so $|\psi\rangle$ is an entangled state.

This discussion for the case of two Hilbert spaces clearly generalises to any number of spaces $\mathcal{H} = \otimes_\alpha \mathcal{H}_\alpha$, for example a lattice system or a spin chain. In this case a state $|\psi\rangle \in \otimes_\alpha \mathcal{H}_\alpha$ can be written as

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_n} \, |e_{i_1}\rangle_{\alpha_1} \otimes |e_{i_2}\rangle_{\alpha_2} \otimes \cdots \otimes |e_{i_n}\rangle_{\alpha_n} \,, \tag{4.3}$$

where each $|e_{i_j}\rangle_{\alpha_j}$ is a basis for $\mathcal{H}_{\alpha_j}$. A state is similarly called *separable* if in (4.3) the coefficients factorise as $c_{i_1,i_2,\cdots,i_n} = c_{i_1}^{\alpha_1} c_{i_2}^{\alpha_2} \cdots c_{i_n}^{\alpha_n}$, else the state is referred to as *entangled*.

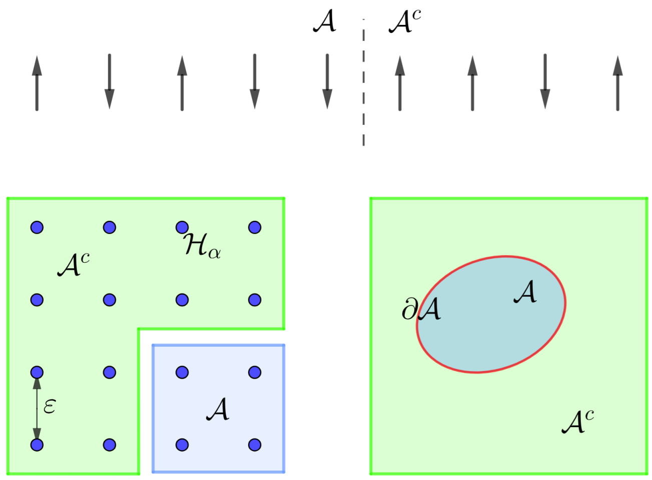

FIGURE 4.1: The bipartitioning of quantum systems into two regions $\mathcal{A}$ & $\mathcal{A}^c$, namely a sipn chain [Top], lattice [Left] and continuum QFT [Right].

Our aim is to analyse and quantify the quantum entanglement between two subsystems of a larger quantum system. For example consider a lattice system with a Hilbert space at each lattice point and divide it into two separate spacial regions $\mathcal{A}$ and $\mathcal{A}^c$ such that their union gives the whole lattice, i.e. $\mathcal{H} = \otimes_\alpha \mathcal{H}_\alpha = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_{\mathcal{A}^c}$. Further examples of such bipartitioning of quantum systems is shown in Figure 4.1. We want to figure out how to quantify the dependence of the degrees of freedom in region $\mathcal{A}$ on the ones in region $\mathcal{A}^c$ for a given state $|\psi\rangle \in \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_{\mathcal{A}^c}$.

So far we have considered quantum mechanics from the perspective of states in a Hilbert space. There exists an equivalent formulation in terms of density matrices which will prove the most powerful approach for this problem. The *density matrix* for a pure state $|\psi\rangle \in \otimes_\alpha \mathcal{H}_\alpha$ is simply defined as the operator

$$\rho = |\psi\rangle \langle\psi| \, . \tag{4.4}$$

Since we are interested in the relation of two regions $\mathcal{A}$ and $\mathcal{A}^c$, we also define the *reduced density matrix* given by

$$\rho_\mathcal{A} = \mathrm{Tr}_{\mathcal{A}^c}(|\psi\rangle \langle\psi|), \tag{4.5}$$

i.e. we remove the degrees of freedom in $\mathcal{A}^c$ by tracing over the Hilbert spaces $\mathcal{H}_{\mathcal{A}^c}$. Using the powerful theorem of Schmidt decomposition from linear algebra we can find further properties of deduced density matrices.

The Schmidt Decomposition Theorem: Let $|\psi\rangle \in \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_{\mathcal{A}^c}$ be a pure state in a bipartitioned system as above. Then there exist orthonormal basis $|e_i\rangle_\mathcal{A}$ and $|e_i\rangle_{\mathcal{A}^c}$ for $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_{\mathcal{A}^c}$ respectively such that

$$|\psi\rangle = \sum_i \lambda_i \, |e_i\rangle_\mathcal{A} \otimes |e_i\rangle_{\mathcal{A}^c} \, , \tag{4.6}$$

with $\sum_i \lambda_i^2 = 1$ and $\lambda_i \geq 0$.

This has clear implications for the reduced density matrix. If $|\Psi\rangle$ is a pure state as above then by (4.6)

$$\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{A}^c}\big(\,|\psi\rangle\,\langle\psi|\,\big) = \sum_i \lambda_i^2\, \mathrm{Tr}_{\mathcal{A}^c}\big(\,|e_i\rangle_{\mathcal{A}}\ {}_{\mathcal{A}}\langle e_i| \otimes |e_i\rangle_{\mathcal{A}^c}\ {}_{\mathcal{A}^c}\langle e_i|\,\big) = \sum_i \lambda_i^2\, |e_i\rangle_{\mathcal{A}}\ {}_{\mathcal{A}}\langle e_i|\,, \quad (4.7)$$

and similarly $\rho_{\mathcal{A}^c} = \sum_i \lambda_i^2\, |e_i\rangle_{\mathcal{A}^c}\ {}_{\mathcal{A}^c}\langle e_i|$. Thus we see that the the eigenvalues of $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{A}^c}$ coincide as they are both λ_i^2.

We are now equipped with the tools necessary to quantify the entanglement of a state is such bipartitioned quantum systems. We define the *entanglement entropy* as the *von Neumann entropy* of the reduced density matrix, i.e.

$$S_{\mathcal{A}} = -\mathrm{Tr}_{\mathcal{A}}\big(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}\big), \quad (4.8)$$

which measures the entanglement between regions $\mathcal{A}$ and $\mathcal{A}^c$ for a state $|\psi\rangle \in \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{A}^c}$. For later convenience we also define the so called *Rényi entropies*

$$S_{\mathcal{A}}^{(q)} = \frac{1}{1-q} \log \mathrm{Tr}_{\mathcal{A}}\big(\rho_{\mathcal{A}}^q\big), \quad (4.9)$$

where $q \in \mathbb{Z}_+$. These are the moments of the reduced density matrix with an extra normalisation factor. One powerful property of the Rényi entropies is that

$$S_{\mathcal{A}} = \lim_{q\to 1} S_{\mathcal{A}}^{(q)}, \quad (4.10)$$

which proves to be a useful trick in computations of entanglement entropy. This relation can be easily proved by writing[1]

$$\begin{aligned}
\lim_{q\to 1} S_{\mathcal{A}}^{(q)} =& \lim_{q\to 1} \frac{1}{1-q} \log \mathrm{Tr}_{\mathcal{A}}\big(\rho_{\mathcal{A}}^q\big) \\
=& \lim_{q\to 1} \frac{1}{1-q} \log \mathrm{Tr}_{\mathcal{A}}\big(\rho_{\mathcal{A}}\rho_{\mathcal{A}}^{q-1}\big) \\
=& \lim_{q\to 1} \frac{1}{1-q} \log \mathrm{Tr}_{\mathcal{A}}\big(\rho_{\mathcal{A}} e^{(q-1)\log \rho_{\mathcal{A}}}\big) \\
=& \lim_{q\to 1} \frac{1}{1-q} \log \mathrm{Tr}_{\mathcal{A}}\big(\rho_{\mathcal{A}} + (q-1)\rho_{\mathcal{A}} \log \rho_{\mathcal{A}} + \mathcal{O}(q-1)^2\big) \\
=& \lim_{q\to 1} \frac{1}{1-q} \log \big(1 + (q-1)\mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}) + \mathcal{O}(q-1)^2\big) \\
=& S_{\mathcal{A}},
\end{aligned}$$

and so (4.10) indeed holds.

[1] One might notice that the Rényi entropies in (4.9) are only defined for integer values of q, and so the limit there makes little sense. This will later be defined more rigorously by analytic continuation of $S_{\mathcal{A}}^{(q)}$ to non integer values of q which we also assume to be doing here.

Recall that we have shown in (4.7) that the density matrices $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{A}^c}$ have the same eigenvalues. This implies that for a pure state the Rényi entropies $S^{(q)}_{\mathcal{A}}$ and $S^{(q)}_{\mathcal{A}^c}$ are equal, i.e.

$$S^{(q)}_{\mathcal{A}} = \frac{1}{1-q}\log \mathrm{Tr}_{\mathcal{A}}\big(\rho^q_{\mathcal{A}}\big) = \frac{1}{1-q}\log\left(\sum_i \lambda_i^q\right) = S^{(q)}_{\mathcal{A}^c}, \tag{4.11}$$

where λ_i are the eigenvalues of $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{A}^c}$. It is also important to note that the Rényi and von Neumann entropies are defined in terms of traces of the reduced density matrix and hence remain unchanged under unitary transformations supported on either $\mathcal{A}$ or $\mathcal{A}^c$. Thus the only way to change the entanglement entropy of a state is to act with a unitary transformation supported on $\mathcal{A}\cup\mathcal{A}^c$.

4.2 Properties of Entanglement Entropy

Let us now turn to explore some of the properties of quantum entanglement entropy which will provide a useful check in the holographic setup that will follow. Recall that the entanglement entropy in a bipartite quantum system is given by

$$S_{\mathcal{A}} = -\mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}}\log\rho_{\mathcal{A}}).$$

There are several interesting properties which follow, but we will only state the ones most relevant to our future analysis. Consider a quantum system partitioned into distinct regions $\mathcal{A}_i$, such that $\mathcal{A} = \cup_i\mathcal{A}_i$ with the Hilbert space of the whole system takes the form $\mathcal{H} = \otimes_i\mathcal{H}_{\mathcal{A}_i}$. Then the entanglement entropy between these regions satisfies the following properties:

- Positivity: $(S_{\mathcal{A}} \geq 0)$;
- Subadditivity: $(S_{\mathcal{A}_1\cup\mathcal{A}_2} \leq S_{\mathcal{A}_1} + S_{\mathcal{A}_2})$;
- Araki-Lieb Inequality: $(S_{\mathcal{A}_1\cup\mathcal{A}_2} \geq |S_{\mathcal{A}_1} - S_{\mathcal{A}_2}|)$;
- Strong Subadditivity: $(S_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3} + S_{\mathcal{A}_2} \leq S_{\mathcal{A}_1\cup\mathcal{A}_2} + S_{\mathcal{A}_2\cup\mathcal{A}_3})$.

The proof of many of these properties is quite subtle and rely on many lemmas from functional analysis, they can mostly be found in [16]. The reason why we mention this is because it will turn out that the proof of these inequalities is substantially simpler in the holographic context and thus we leave the proof to latter sections. It is however important to emphasise that the above relations hold in any quantum system whether continuous or not and hence provide a very robust check on the 'quantumness' of any theory.

4.3 Entanglement in QFT

We now proceed to formulate entanglement entropy as defined above for quantum field theories. One might think of this as taking the limit $\varepsilon \to 0$ in Figure 4.1. The states of quantum mechanics $|\psi\rangle$ now become wavefunctionals $\Phi[\phi(\vec{x})]$, where $\phi(\vec{x})$ is the set of fields that characterise the system on a Cauchy slice Σ parametrised by spatial coordinates $\vec{x}$. Thus we can follow the construction of entanglement and Rényi entropies from the previous section. We will now have spatial regions $\mathcal{A}, \mathcal{A}^c \subset \Sigma$ separated by an entangling surface $\partial\mathcal{A}$, which is the boundary of region $\mathcal{A}$ as shown in Figure 4.1. In the continuum setup, the trace over Hilbert spaces in $\mathcal{A}^c$ in (4.5) amounts to integrating over all field configurations in that region. With a reduced density matrix constructed it should now be straightforward to also define the entanglement entropy, but here we run into some complications. Since the reduced density matrix is now a continuous operator, taking its logarithm is tricky. Luckily we have already developed some machinery to deal with this, namely relation (4.10) which states that

$$S_{\mathcal{A}} = \lim_{q \to 1} \frac{1}{1-q} \log \mathrm{Tr}_{\mathcal{A}}\big(\rho_{\mathcal{A}}^q\big). \tag{4.12}$$

This solves our problem as we are now taking the logarithm of a trace. In what now follows we will make the above arguments more precise.

Our strategy is as follows: We first discuss the causal structure of entanglement entropy and set up a path integral framework for the calculation of matrix elements of the reduced density matrix. We then apply the above trick and calculate the entanglement entropy by suitable analytic continuation of the Rényi entropies.

4.3.1 Entanglement Entropy and Causality

In order to deal with time dependent states, we have to consider the causal structure of entanglement entropy in QFT. States are defined by field data on a Cauchy slice Σ and so the spacetime is defined by $\mathcal{B} = D^-[\Sigma] \cup D^+[\Sigma]$, where $D^-[\Sigma]$ and $D^+[\Sigma]$ are the past and future domains of dependence of Σ respectively. Entanglement is formulated in terms of spatial subregions $\mathcal{A}, \mathcal{A}^c \subset \Sigma$, where $\mathcal{A} \cup \mathcal{A}^c = \Sigma$ with boundary $\partial\mathcal{A}$. The domain of causal dependence of $\mathcal{A}$ is similarly given by the subregion $D^\pm[\mathcal{A}]$ in which the reduced density matrix can be uniquely evolved. It is important to realise that the domains $D^\pm[\mathcal{A}]$ and $D^\pm[\mathcal{A}^c]$ do not make up the whole spacetime as illustrated in Figure 4.2. We have to consider the regions of spacetime which can be causally influenced by the boundary $\partial\mathcal{A}$, denoted $J^\pm[\partial\mathcal{A}]$. Thus the spacetime decomposes into causal parts

$$\mathcal{B} = D^\pm[\mathcal{A}] \cup D^\pm[\mathcal{A}^c] \cup J^\pm[\partial\mathcal{A}].$$

Following our previous discussion, the eigenvalues of the density matrix can only be changed via unitary transformations supported on $D[\mathcal{A}] \cup D[\mathcal{A}^c]$. This implies that the entanglement entropy is not sensitive to the particular choice of Cauchy slice Σ, i.e. we are allowed to choose any other Cauchy slice Σ' without affecting the entropy as long as $D^\pm[\mathcal{A}] = D^\pm[\mathcal{A}']$. A lot longer discussion on this issue of causality can be found in [17].

4.3.2 Path Integral Formalism

In the path integral formalism of QFT, states are represented by functional integrals over field configurations. We start by writing down a Euclidean path integral representation of the vacuum state of our QFT. We denote by $\phi(\vec{x}, t_E)$ the collection of fields defining our theory, but for simplicity we will think of this as a single scalar field. Let $|\phi\rangle$ be a basis for the Hilbert space of states on a constant-time Cauchy slice $\Sigma_{t_E=0}$. We take it to be the

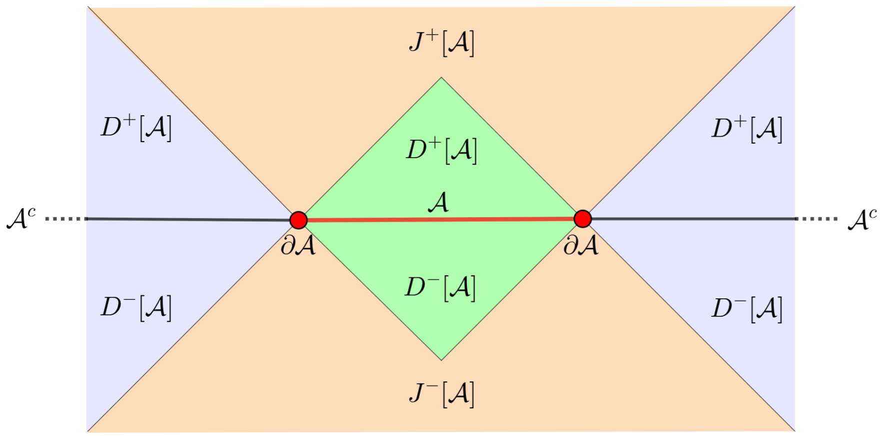

FIGURE 4.2: The causal structure of the bipartitioning of a cauchy slice Σ into two regions $\mathcal{A}$ and $\mathcal{A}^c$ such that $\mathcal{A} \cup \mathcal{A}^c = \Sigma$. The figure covers the spacetime $\mathcal{B}$ and the time direction is perpendicular to Σ.

eigenbasis formed by the field operator $\hat{\phi}(\vec{x}, t_E)$ at $t_E = 0$, that is

$$\hat{\phi}(\vec{x}, t_E = 0) |\phi\rangle = \phi(\vec{x}) |\phi\rangle .$$

We can then represent the vacuum wavefunctional $\Phi[\phi(\vec{x})]$ as the Euclidean path integral

$$\Phi[\phi(\vec{x})] = \langle 0|\phi\rangle = \int_{\phi(\vec{x},t_E=-\infty)=0}^{\phi(\vec{x},t_E=0)=\phi(\vec{x})} [\mathcal{D}\phi] e^{-S_E[\phi]}, \tag{4.13}$$

where we are integrating over the Euclidean half plane $t_E \leq 0$. This allows us in turn to write down a path integral representation for the matrix elements of the density matrix as

$$\rho_{\phi\,\phi'} \equiv \langle\phi|\rho|\phi'\rangle = \langle\phi|0\rangle \langle 0|\phi'\rangle = \Phi^*[\phi]\Phi[\phi'], \tag{4.14}$$

where we get $\Phi^*[\phi]$ by integrating from $t_E = \infty$ to $t_E = 0$.

It is important to note that what we have found here is the density matrix for the vacuum state. For simplicity we will continue to explicitly work with this vacuum density matrix even though this Euclidean formalism readily generalises to states with no non-trivial time dependence. These states have a Euclidean path integral representation precisely like (4.13), but now with some specific boundary condition on the lower limit, i.e.

$$\langle\phi'|\phi\rangle = \int_{\phi(\vec{x},t_E=t')=\phi'(\vec{x})}^{\phi(\vec{x},t_E=0)=\phi(\vec{x})} [\mathcal{D}\phi] e^{-S_E[\phi]}.$$

To get the reduced density matrix $\rho_\mathcal{A}$ we need to remove the degrees of freedom in $\mathcal{A}^c$. This can be done by glueing the two path integrals in (4.14) together at $t_E = 0$ along $\mathcal{A}^c$ with a cut along $\mathcal{A}$ as shown in Figure 4.3. Denoting the field values along $\mathcal{A}$ for the two limits as $\phi(t_E = 0^+)|_\mathcal{A} = \phi_+$ and $\phi(t_E = 0^-)|_\mathcal{A} = \phi_-$ the path integral for the reduced density matrix

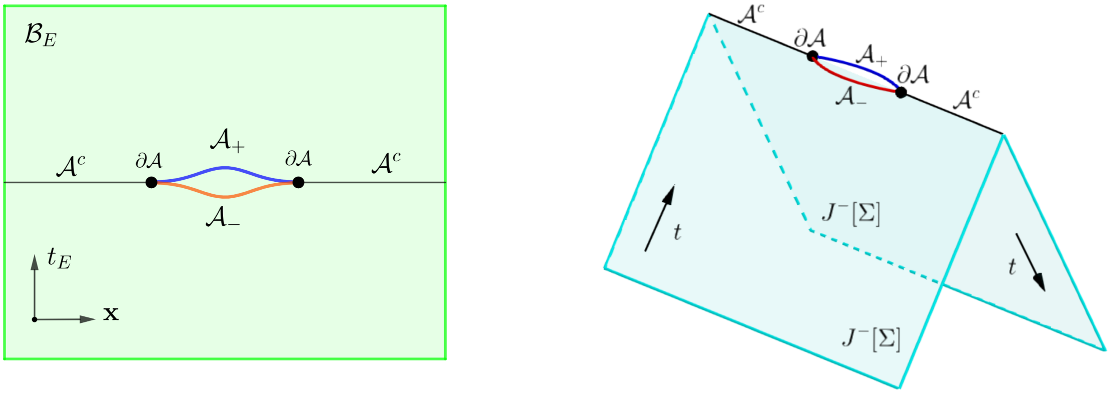

FIGURE 4.3: The path integral construction for the computation of the matrix elements of the reduced density matrix. For time-idependent states: Euclidean half spacetimes glued together along $\mathcal{A}^c$ with a slit cut through $\mathcal{A}$ [Left]. For time-dependent states: Two copies of $J^-[\Sigma]$ glued together along $\mathcal{A}^c$ with a slit cut through $\mathcal{A}$ [Right].

is given by

$$(\rho_{\mathcal{A}})_{\phi_+\phi_-} = \int_{t_E=-\infty}^{t_E=\infty} [\mathcal{D}\phi] e^{-S_E[\phi]} \delta_E(\phi_\pm), \tag{4.15}$$

where $\delta_E(\phi_\pm) = \delta(\phi(t_E = 0^+)|_{\mathcal{A}} - \phi_+)\delta(\phi(t_E = 0^-)|_{\mathcal{A}} - \phi_-)$ is a delta function that conveniently extracts the matrix elements.

As we have emphasized, the above Euclidean construction only works for time-independent states, or more generally states with a time reflection symmetry. In these cases, integrating over the whole of Euclidean spacetime is compatible with causality as there is no non-trivial information about the state in the future domain of the Cauchy slice it is defined on. For general time dependent states one cannot simply integrate over all time. This is due to the fact that to define a density matrix $\rho(t)$ at time t, we cannot use field data at a time $t' > t$ without violating causality. Based on our previous discussion on causality in Section 4.3.1, we know that instead on the half Euclidean plane what we have now instead is $J^-[\Sigma_{t=0}]$, the causal past of the Cauchy slice our states are defined on. One can then find a path integral representation for the reduced density matrix by first evolving forward in $J^-[\Sigma_{t=0}]$ until Σ and then retracing the evolution backwards on another copy of $J^-[\Sigma_{t=0}]$. The rest of the construction follows on from the time independent case, but now with the geometry consisting of two copies of $J^-[\Sigma_{t=0}]$ glued along Σ with a slit cut along $\mathcal{A}$ as shown in Figure 4.3. Denoting the forward and backwards evolving fields as ϕ_R and ϕ_L respectively[2] we have that

$$(\rho_{\mathcal{A}})_{\phi_+\phi_-} = \int_{J^-[\Sigma]} [\mathcal{D}\phi_R]\,[\mathcal{D}\phi_L] e^{iS[\phi_R]-iS[\phi_L]} \delta(\phi_\pm), \tag{4.16}$$

where now the delta function takes the form $\delta(\phi_\pm) = \delta(\phi_L(t = 0^+)|_{\mathcal{A}} - \phi_+)\delta(\phi_R(t = 0^-)|_{\mathcal{A}} - \phi_-)$. The method to compute path integrals via such a setup is called the *Schwinger-Keldysh formalism.*

[2]The fields ϕ_R and ϕ_L live on two distinct copies of $J^-[\Sigma]$.

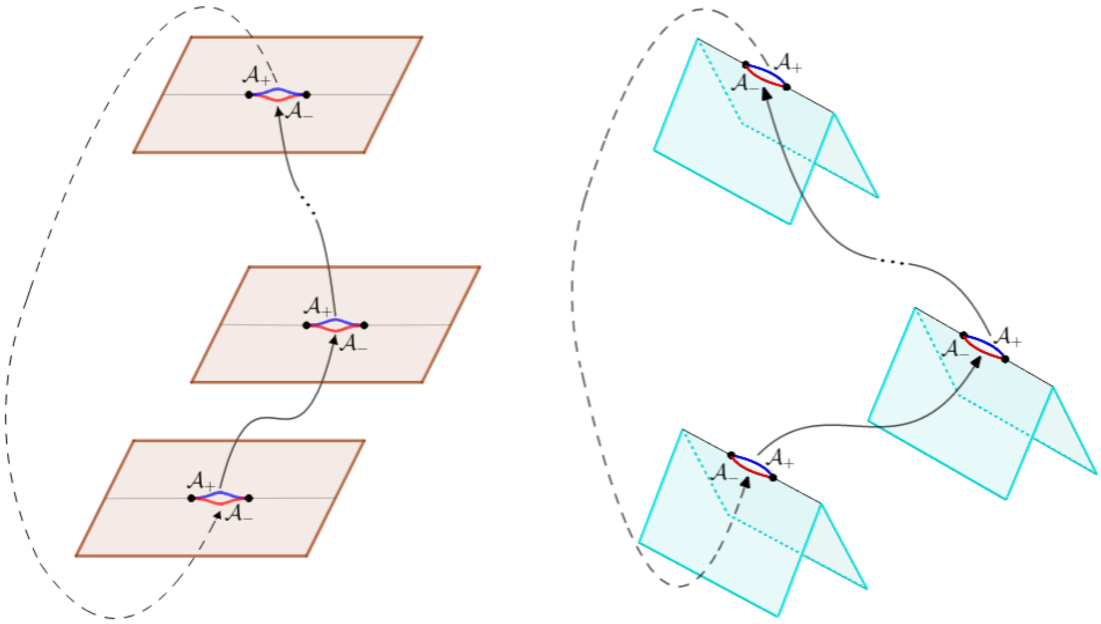

FIGURE 4.4: The replica path integral construction used to compute the powers of the reduced density matrix in the case of time-independent [Left] and time-dependent [Right] states. The arrows denote identifications of the boundary conditions. The solid arrows are the identifications used for the powers while the dashed arrows are the identifications used for taking the trace.

4.3.3 Replica Construction

Having found an expression for the reduced density matrix we are motivated by (4.12) to calculate the powers of it. The method used to calculate these powers is called the *replica method* as it essentially copies the above computation q times. To start with, consider time-independent states and take q copies of the above path integral construction (4.15), i.e.

$$(\rho_{\mathcal{A}})_{\phi_{1+}\,\phi_{1-}}(\rho_{\mathcal{A}})_{\phi_{2+}\,\phi_{2-}}\cdots(\rho_{\mathcal{A}})_{\phi_{q+}\,\phi_{q-}}.$$

Since we are performing matrix multiplication, we are required to integrate over intermediate boundary conditions while identifying the k^{th} $(+)$ with the $(k+1)^{\text{st}}$ $(-)$ boundary condition as shown in Figure 4.4. This can be achieved by taking q copies of (4.15) as above and inserting delta functions which generate the required identifications, i.e.

$$(\rho^q_{\mathcal{A}})_{\phi_+\,\phi_-} = \int \prod_{j=1}^{q-1} d\phi_+^{(j)}\,\delta(\phi_+^{(j)} - \phi_-^{(j+1)}) \int \prod_{k=1}^{q} [\mathcal{D}\phi^{(k)}] e^{-\sum_{k=1}^{q} S_E[\phi^{(k)}]}\,\delta_E(\phi_\pm). \tag{4.17}$$

All that remains for the computation of the Rényi entropies is taking the trace of (4.17). It should be clear from Figure 4.4 that this simply involves identifying the remaining two boundaries, i.e. $\phi_-^{(1)}$ and $\phi_+^{(q)}$. Thus the trace of the q^{th} power of the reduced density matrix for time-independent states is given by

$$\mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}}^q) = \int d\phi_+^{(q)}\, \delta(\phi_+^{(q)} - \phi_-^{(1)}) \int \prod_{j=1}^{q-1} d\phi_+^{(j)}\, \delta(\phi_+^{(j)} - \phi_-^{(j+1)}) \\ \times \int \prod_{k=1}^{q} [\mathcal{D}\phi^{(k)}] e^{-\sum_{k=1}^{q} S_E[\phi^{(k)}]}\, \delta_E(\phi_\pm), \tag{4.18}$$

where we have inserted an extra delta function to make the trace identification.

For states with general time-dependence this replica construction works very similarly as in Figure 4.4. The only difference is that we now have to keep track of the time direction on each copy of $J^-[\Sigma_{t=0}]$. This means that we identify the k^{th} (+,L) boundary with the $(k+1)^{\text{st}}$ (-,R) boundary, where we have denoted the time direction by L and R as before. Taking the trace again amounts to identifying the remaining two boundaries.

In either time dependent or independent case one can equivalently think of the above replica construction in more geometric terms. From Figure 4.4 we see that the path integral is nothing more then integrating over fields living on a new spacetime manifold $\mathcal{B}_q$ which is constructed by taking q copies of $\mathcal{B}$ (or $J^-[\Sigma_t]$ for time dependent states) and making the same identifications of the boundaries as before. We will refer to this new manifold $\mathcal{B}_q$ as the q-fold branched cover of $\mathcal{B}$. Thus we can equivalently think of (4.18) as a path integral on $\mathcal{B}_q$ and write

$$\mathrm{Tr}(\rho_{\mathcal{A}}^q) \equiv \frac{\mathcal{Z}[\mathcal{B}_q]}{\mathcal{Z}[\mathcal{B}]^q} = \frac{1}{\mathcal{Z}[\mathcal{B}]^q} \int_{\mathcal{B}_q} [\mathcal{D}\phi] e^{-S_E[\phi]}, \tag{4.19}$$

where we have divided by a factor of $\mathcal{Z}[\mathcal{B}]^q$ to properly normalise the entropy.[3] Consequently the Rényi entropies can be calculated via

$$S_{\mathcal{A}}^{(q)} = \frac{1}{1-q} \log(\mathrm{Tr}(\rho_{\mathcal{A}}^q)) = \frac{1}{1-q} \log\left(\frac{\mathcal{Z}[\mathcal{B}_q]}{\mathcal{Z}[\mathcal{B}]^q}\right). \tag{4.20}$$

Having found a path integral representation of the Rényi entropies it is now time to invoke the limit (4.10) to calculate the entanglement entropy. This, however, requires an analytic continuation of $S_{\mathcal{A}}^{(q)}$, defined for $q \in \mathbb{Z}$, to real values of q, which in general is not guaranteed to uniquely exist. We can evade the issue using Carlson's theorem from complex analysis which guarantees a unique analytic continuation given that $S_{\mathcal{A}}^{(q)}$ behaves sub-exponentially at $q \to \pm i\infty$.

It is also important to note that by construction, the replica path integral (4.19) carries a $\mathbb{Z}_q$ symmetry due to the cyclicity of the trace. One can think of this symmetry as the reshuffling of the replica copies of the spacetime $\mathcal{B}$ in (4.18).

[3] Notice that so far, for example in (4.15), we have ignored the normalisation of the path integrals. This factor of $1/\mathcal{Z}[\mathcal{B}]^q$ is precisely inherited from a normalisation of $1/\mathcal{Z}[\mathcal{B}]$ in (4.15).

5 Holographic Entanglement Entropy

Now that we are equipped with all the necessary tools, we move on to give a holographic description of entanglement entropy. Having discussed features of the AdS/CFT correspondence and quantum information theory one might ask how quantum entanglement is captured in a holographic setup. The problem was originally addressed Ryu and Takayanagi (RT) [2, 3], who gave a holographic prescription for calculating the entanglement entropy of time-independent states. This was then generalised by Hubney, Rangamani and Takayanagi (HRT) [4] to include general time dependent states. In this section we present some of the central aspects of the holographic entanglement entropy formula. We first state the RT and HRT prescriptions and discuss their properties, which we then use to give a holographic proof of strong subadditivity. Then, we attempt deriving the RT prescription from the AdS/CFT correspondence and finally we consider some examples.

5.1 The Hubney-Rangamani-Takayanagi Prescription

The Hubney-Rangamani-Takayanagi Prescription: Consider a QFT_d living on $\mathcal{B}_d$, which is the boundary geometry of some asymptotically AdS_{d+1} bulk spacetime $\mathcal{M}_{d+1}$, and let $\Sigma \subset \mathcal{B}_d$ be a Cauchy slice on this boundary geometry. Consider spatial subregions $\mathcal{A}, \mathcal{A}^c \subset \Sigma$ such that $\mathcal{A} \cup \mathcal{A}^c = \Sigma$ and denote the boundary of $\mathcal{A}$ as $\partial\mathcal{A}$. We are then required to find a bulk codimension-2 surface satisfying the following constraints:

- It must be an extremal surface anchored on $\partial\mathcal{A}$. More precisely we need to find a surface $\mathcal{E}_\mathcal{A}$ which extremises the bulk area functional with boundary condition $\mathcal{E}_\mathcal{A}\big|_\mathcal{B} = \partial\mathcal{A}$;
- If there are multiple such surfaces we only need to consider extremal surfaces which can be smoothly deformed to $\mathcal{A}$. This means that there must exist a bulk codimension-1 smooth surface $\mathcal{R}_\mathcal{A} \subset \mathcal{M}_{d+1}$ interpolating between $\mathcal{A}$ and $\mathcal{E}_\mathcal{A}$;
- If there are still more than one extremal surfaces $\mathcal{E}_\mathcal{A}$ left satisfying the above constraints we should pick the one with the smallest area.

The entanglement entropy in QFT_d is then given by the area of $\mathcal{E}_\mathcal{A}$ in Planck units, that is

$$S_\mathcal{A} = \frac{\mathrm{Area}(\mathcal{E}_\mathcal{A})}{4G_N^{(d+1)}}, \tag{5.1}$$

where $G_N^{(d+1)}$ is Newton's constant in $d+1$ dimensions.

This is the covariant HRT proposal for time-dependent states put forward in [4]. If the state in question in the QFT_d is time-independent then it suffices to find a minimal surface instead of an extremal one, this is the original RT proposal [2, 3].

In many cases it is more useful to formulate the extremal area condition in terms of more geometric objects. We adopt local coordinates normal to the surface $\mathcal{E}_\mathcal{A}$ and denote by $\mathbf{n}_1^N$

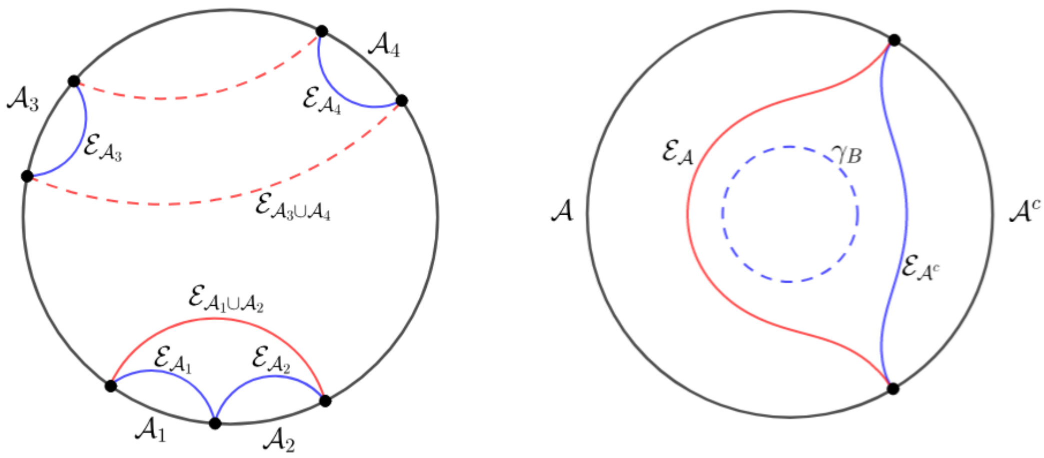

FIGURE 5.1: Various examples of RT or HRT surfaces. The extremal surface of two either connected or disconnected regions [Left]. The extremal surface of a black hole setup, where we see that the bifurcation surface γ_B plays an important role in the homology constraint. [Right]

and $\mathbf{n}_2^N$ the two normal basis vectors[1] with normalisation $\mathbf{n}_1^N\mathbf{n}_2^M g_{NM} = -1$ and $\mathbf{n}_a^N\mathbf{n}_a^M g_{NM} = 0$, the indices $\{a, b\}$ run over the two normal directions. Using this basis, and the projector $\gamma_{AB} + \mathbf{n}_A^1\mathbf{n}_B^2 + \mathbf{n}_A^2\mathbf{n}_B^1$, we define the extrinsic curvature of $\mathcal{E}_{\mathcal{A}}$ as

$$K^a_{NM} = \gamma_N^P\,\gamma_M^Q\nabla_P\,\mathbf{n}_Q^a.$$

In terms of these quantities the condition of extremality can be rephrased as the vanishing of the trace of the extrinsic curvature, i.e.

$$\gamma^{NM}K^a_{NM} \equiv K^a = 0 \quad\iff\quad \mathcal{E}_{\mathcal{A}} \text{ extremal.}$$

If we are talking about time-independent states, we will have a spacelike and a timelike normal coordinate, say x and t respectively. In this case what we want is the minimal area constraint of the RT proposal. In this formulation this reduces to the requirement that the trace of the spacelike extrinsic curvature should vanish, i.e. $K^x = 0$ [5, 18].

5.2 Holographic Entropy Inequalities

The RT and HRT prescriptions for entanglement entropy look very different in appearance to the quantum information definition via the von Neumann entropy. It is therefore beneficial to check whether the properties of $S_{\mathcal{A}}$ discussed in Section 4.2 are indeed satisfied in the holographic setup. The holographic proof of these inequalities provides a further insight into the workings of the RT proposal and nicely demonstrates the link between bulk geometry and entanglement. We consider a time-independent state within the framework of the RT proposal. It is also possible to prove these properties for general time-dependent states [19], but this involves an alternate formulations of the HRT proposal and is significantly more

[1]There are two normal basis vectors since $\mathcal{E}_{\mathcal{A}}$ is a codimansion-2 surface.

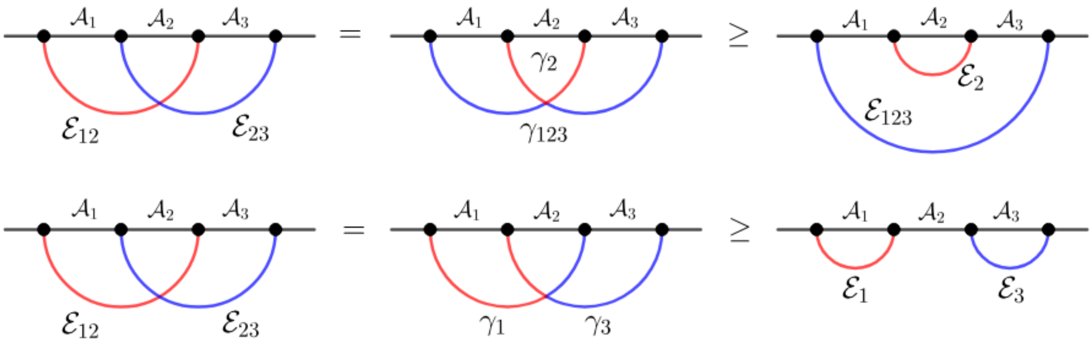

FIGURE 5.2: The surfaces used for the proof of strong subadditivity in the holographic setup. In each case $\mathcal{E}$ is an extremal while γ is a general bulk codimesion-2 surface.

involved, thus we will concentrate on the simpler but more insightful RT setup. Consider the above holographic setup with a QFT$_d$ living on $\mathcal{B}$ which has a constant time Cauchy slice $\Sigma \subset \mathcal{B}$. But now, let Σ be partitioned into multiple non overlapping subsets[2] $\mathcal{A}_i \subset \Sigma$. Then, the entanglement entropy properties can be holographically proved as follows. For each property we give a holographic proof and also discuss the saturation of some inequalities from the holographic viewpoint.

- Positivity ($S_\mathcal{A} \geq 0$):
  Proof. This follows directly form the RT formula in which $S_\mathcal{A}$ is given as the area of some spacelike surface which is positive by construction.
  Saturation. $S_\mathcal{A} = 0$ if and only if either $\mathcal{A} = \emptyset$ or $\mathcal{A} = \Sigma$.

- Subbaditivity ($S_{\mathcal{A}_1 \cup \mathcal{A}_2} \leq S_{\mathcal{A}_1} + S_{\mathcal{A}_2}$):
  Proof. Consider the minimal RT surfaces $\mathcal{E}_1$ and $\mathcal{E}_2$ for $S_{\mathcal{A}_1}$ and $S_{\mathcal{A}_2}$ respectively. From Figure 5.1 we see that the surface $\gamma_{12} = \mathcal{E}_1 \cup \mathcal{E}_2$ is a surface anchored on $\partial(\mathcal{A}_1 \cup \mathcal{A}_2)$ and thus must have an area greater that or equal to the minimal RT surface $\mathcal{E}_{12}$ for $\mathcal{A}_1 \cup \mathcal{A}_2$. Thus we have that

$$\text{Area}(\mathcal{E}_1) + \text{Area}(\mathcal{E}_2) = \text{Area}(\gamma_{12}) \geq \text{Area}(\mathcal{E}_{12})$$

  and so this directly implies that

$$S_{\mathcal{A}_1 \cup \mathcal{A}_2} \leq S_{\mathcal{A}_1} + S_{\mathcal{A}_2}$$

  via the RT formula. □
  Saturation. $S_{\mathcal{A}_1 \cup \mathcal{A}_2} = S_{\mathcal{A}_1} + S_{\mathcal{A}_2}$ if and only if $\mathcal{E}_{12} = \mathcal{E}_1 \cup \mathcal{E}_2$. From Figure 5.1 we can see that this happens if the two regions $\mathcal{A}_1$ and $\mathcal{A}_2$ are disconnected and are far apart on the scale of their own size. In this case the minimal RT surface is the one that does not join them which is precisely $\mathcal{E}_1 \cup \mathcal{E}_2$.

- Strong Subadditivity ($S_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3} + S_{\mathcal{A}_2} \leq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_2 \cup \mathcal{A}_3}$):
  Proof. We follow a similar cutting and glueing procedure. Take three regions $\mathcal{A}_{1,2,3}$ and consider the minimal RT surfaces $\mathcal{E}_{12}$ and $\mathcal{E}_{23}$. From Figure 5.2 we see that there are

[2]We let the range of i be general here, which is fine for the holographic proofs.

two ways to cut and glue the surfaces, one giving γ_{123} and γ_2, while the other giving γ_1 and γ_3. Each γ is a surface anchored at the same boundary points as the corresponding $\mathcal{E}$, which is a minimal surface. This immediately implies the two inequalities

$$\begin{aligned} S_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3} + S_{\mathcal{A}_2} &\leq S_{\mathcal{A}_1\cup\mathcal{A}_2} + S_{\mathcal{A}_2\cup\mathcal{A}_3} \\ S_{\mathcal{A}_1} + S_{\mathcal{A}_3} &\leq S_{\mathcal{A}_1\cup\mathcal{A}_2} + S_{\mathcal{A}_2\cup\mathcal{A}_3}, \end{aligned}$$

the first of which is strong subadditivity. □

Saturation: $S_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3}+S_{\mathcal{A}_2} = S_{\mathcal{A}_1\cup\mathcal{A}_2}+S_{\mathcal{A}_2\cup\mathcal{A}_3}$ if and only if $\gamma_2 = \mathcal{E}_2$ and $\gamma_{123} = \mathcal{E}_{123}$.

The holographic proofs above all relied on very simple geometric relations like the area of a surface. This demonstrates the main power of the RT formula which is that it reformulates a highly quantum-like phenomenon in the boundary in terms of purely geometric objects in the bulk.

5.3 Derivation of the RT Proposal

In what follows, we provide a derivation of the RT proposal. Heuristically, the proof is structured as follows: We first use some basic entries from the AdS/CFT dictionary to find an expression of the Renyi entropies in terms of bulk partition functions. We then use the saddle point approximation in the bulk to evaluate the entropy as the on-shell value of a this semiclassical action. We follow the notation of previous sections and consider a CFT$_d$ living on a boundary geometry $\mathcal{B}$, with a bipartite Cauchy slice $\mathcal{A}, \mathcal{A}^c \subset \Sigma$. Dual to this, there is a bulk spacetime $\mathcal{M}$ with conformal boundary $\partial\mathcal{M} = \mathcal{B}$. As we have seen in Section 4.3.3, to calculate the entanglement entropy one has to evaluate the CFT partition function on the branched replica geometry $\mathcal{B}_q$.

The first step in the holographic derivation is the basic bulk to boundary relation in AdS/CFT that relates the partition function of the boundary CFT to the partition function of the bulk string theory, namely[3]

$$\mathcal{Z}_{\text{String}}[\phi_0] = \left\langle e^{\int d^dx\, \phi_0(x^\mu)\mathscr{O}(x^\mu)} \right\rangle_{\text{CFT}}. \tag{5.2}$$

Since the replica method requires us to calculate the partition function on the branched cover geometry $\mathcal{B}_q$ the bulk to boundary relation (5.2) shows that we need to find a bulk geometry $\mathcal{M}_q$ such that $\partial\mathcal{M}_q = \mathcal{B}_q$. This means that we need to find $\mathcal{M}_q$ that solves Einstein's equation such that $\partial\mathcal{M}_q = \mathcal{B}_q$. Let the bulk theory be given by the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\,\left(R - 2\Lambda\right), \tag{5.3}$$

where R is the Ricci scalar and Λ is the cosmological constant as before. Then the above statement precisely means that $\delta S_{\text{EH}} = 0$ on $\mathcal{M}_q$ with the boundary condition $\partial\mathcal{M}_q = \mathcal{B}_q$.

As before in AdS/CFT, we may use a saddle point method to semiclassically approximate the AdS partition function

$$\mathcal{Z}_{\text{String}}[\mathcal{M}_q] \approx e^{-S_{\text{Classical}}[\mathcal{M}_q]},$$

[3]In our case the only source turned on is the metric tensor.

where $\mathcal{M}_q$ is the stationary point of the EH action as above. This together with (4.20) allows us to calculate the Renyi entropy as

$$\begin{aligned} S_{\mathcal{A}}^{(q)} &= \frac{1}{1-q}\log\left(\frac{\mathcal{Z}[\mathcal{B}_q]}{\mathcal{Z}[\mathcal{B}]^q}\right) \\ &= \frac{1}{1-q}\log\left(\frac{\mathcal{Z}[\mathcal{M}_q]}{\mathcal{Z}[\mathcal{M}]^q}\right) \\ &\approx \frac{1}{1-q}\big(\, q\, S_{\text{Cl}}[\mathcal{M}] - S_{\text{Cl}}[\mathcal{M}_q]\,\big). \end{aligned} \tag{5.4}$$

Now, in order to calculate the entanglement entropy one has to take the $q \to 1$ limit of this expression. At this point our geometric description breaks down, we cannot simply talk about 1.5 replica spacetime copies. This is where the great insight of Lewkowycz-Maldacena (LM) comes to the help, who realised in [20] that the continuation to $q \in \mathbb{R}$ is much easier in the gravitational context.

The LM derivation conveniently splits into three parts. A kinematic part in which we implement the analytic continuation, a dynamic part in which we ensure that the solution satisfies Einstein's equations and a fianl part in which we show that the on-shell action indeed gives the area law of the RT proposal.

5.3.1 Kinematics

Let us first review the geometric replica setup for $q \in \mathbb{Z}_+$. In this case the replica geometry $\mathcal{B}_q$ is well understood as the q-fold branched cover of $\mathcal{B}$, with branching over $\partial\mathcal{A}$. This provides a proper boundary condition for the variational problem of finding $\mathcal{M}_q$. On the other hand this viewpoint does not seem to allow continuation to $q \in \mathbb{R}$.

At this point one may ask whether we have any more information about $\mathcal{B}_q$ to exploit. The answer is yes, there is the $\mathbb{Z}_q$ symmetry of the replica construction. Indeed one of the assumptions of the LM proof is that the $\mathbb{Z}_q$ symmetry naturally extends to the smooth bulk replica manifold $\mathcal{M}_q$. It is though important to realise that this symmetry may not act smoothly everywhere in $\mathcal{M}_q$, there may be $\mathbb{Z}_q$ fixed points. We can then consider the quotient space $\hat{\mathcal{M}}_q \equiv \mathcal{M}_q/\mathbb{Z}_q$, which will have singular fixed points. We call the collection of these singular points $\mathbf{e}_q$. The key assumption made by LM is that $\mathbf{e}_q$ is a bulk codimension-2 surface. This seems natural as we can think of $\mathbf{e}_q$ as the extension of $\partial\mathcal{A}$ into the bulk.

To find out more about this singular locus, we adopt local coordinates adjusted to the the codimension-2 surface $\mathbf{e}_q$. This means that we pick a coordinates such that: (y^i), with $i \in \{1, 2, \cdots, d-1\}$, are tangential and (x, t_E) are normal to $\mathbf{e}_q$. The surface in these coordinates is located at $x = 0$, $t_E = 0$. Expanding the metric around $\mathbf{e}_q$ gives

$$ds_E^2 = dt_E^2 + dx^2 + (\gamma_{ij} + 2K_{ij}^x x + 2K_{ij}^t t_E)dy^i dy^j + \cdots ,$$

where K_{ij}^a is the extrinsic curvature of $\mathbf{e}_q$, γ_{ij} is the induced metric on $\mathbf{e}_q$ and the dots represent higher derivative corrections. It is instructive to change to polar coordinated in the normal direction by writing $x \pm it_E = re^{\pm i\tau}$, where $\tau \sim \tau + 2\pi$ for regularity. These coordinates are shown in Figure 5.3. The replica symmetry in the bulk implies that near $\mathbf{e}_q$

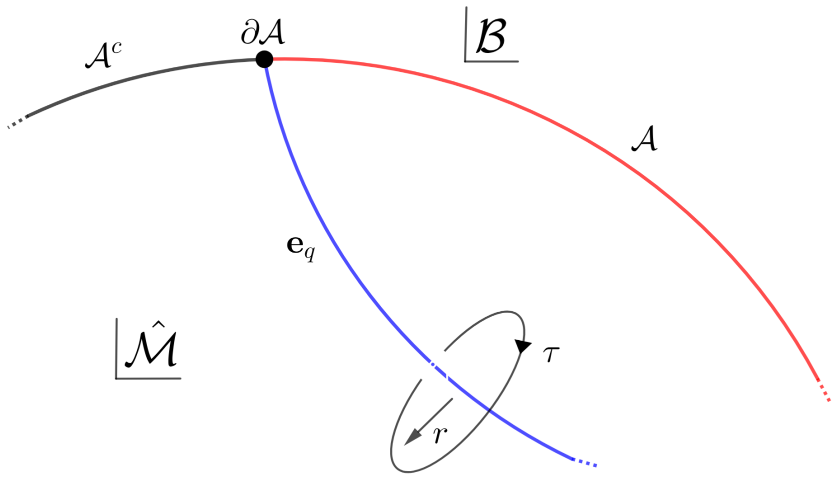

FIGURE 5.3: A pictorial representation of the bulk normal coordonates. Here τ and r are the radial normal coordinates defined with respect to the codimension-2 locus $\mathbf{e}_q$.

the coordinate τ has to instead be identified as $\tau \sim \tau + 2\pi q$. Then, using the smoothness of the covering bulk space $\mathcal{M}_q$ this implies that the metric in $\hat{\mathcal{M}}_q$ near $\mathbf{e}_q$ takes the form

$$ds^2 = q^2 dr^2 + r^2 d\tau^2 + (\gamma_{ij} + 2K^x_{ij} r^q \cos\tau + 2K^t_{ij} r^q \sin\tau) dy^i dy^j + \cdots . \tag{5.5}$$

The importance in this formula comes from the non-trivial q-dependence. This means that the identification $\tau \sim \tau + 2\pi q$ together with the smoothness of the covering space causes a backreaction on the geometry. The quotient orbifold geometry $\hat{\mathcal{M}}_q$ allows us think differently about the $q \to 1$ limit. The identification $\tau \sim \tau + 2\pi q$ can be equivalently thought of as introducing a conical singularity in the quotient space with defect angle $2\pi/q$.

This orbifolded viewpoint is very useful as we know what is means to have a conical defect angle of $2\pi/q$ for any $q \in \mathbb{R}_+$, that is the analytic continuation is well understood in this context. In the orbifolded construction we can also further exploit the $\mathbb{Z}_q$ symmetry in the bulk to concentrate on just one domain of its action, i.e. we may write

$$S_{\rm Cl}[\mathcal{M}_q] = q\, S_{\rm Cl}[\hat{\mathcal{M}}_q],$$

which will prove useful later. This, together with (5.4), implies that the Renyi entropy is given by

$$S^{(q)}_{\mathcal{A}} = \frac{q}{1-q}\big(S_{\rm Cl}[\mathcal{M}] - S_{\rm Cl}[\hat{\mathcal{M}}_q] \big), \tag{5.6}$$

where $\mathcal{M} = \hat{\mathcal{M}}_1$.

5.3.2 Dynamics

What remains now is to check what condition the dynamical field equations impose, and then calculate the on shell action for the ansatz geometry. Calculating the Ricci scalar of the geometry (5.5) we find contributions $\propto \frac{(q-1)}{r} K^a$, where $K^a \equiv \gamma^{ij} K^a_{ij}$ is the trace of the extrinsic curvature [21]. Since this is divergent, it leads us to conclude that the EH dynamics constrains the geometry of $\mathbf{e}_q$ to ones with $K^a = 0$, i.e. to those with vanishing extrinsic curvature. The fact that we are working with states with a time reflection symmetry $t \to -t$ already implies that $K^t = 0$. Thus the above constraint of the vanishing of the extrinsic curvature precisely reproduces the minimal area constraint of the RT proposal, i.e.

$$\lim_{q\to 1} \mathbf{e}_q = \mathcal{E}_{\mathcal{A}}, \qquad \mathcal{E}_{\mathcal{A}} \subset \mathcal{M} \text{ s.t. } K^x = 0.$$

5.3.3 On-Shell Action

We have now identified the minimal surface and have a good understanding of its geometry. Hence all that remains is to check wether the on shell action of our ansatz gemetry (5.5) indeed reproduces the area law of the RT proposal. LM in [20] noticed that instead of working with the Renyi entropy, considering a closely related object

$$\tilde{S}^{(q)}_{\mathcal{A}} \equiv q^2 \partial_q \left(\frac{q-1}{q} S^{(q)}_{\mathcal{A}} \right),$$

called the *modular entropy*, simplifes the calculation. This is because the modular entropy then satisfies the relation

$$\tilde{S}^{(q)}_{\mathcal{A}} = q^2 \partial_q S_{\text{Cl}}[\hat{\mathcal{M}}],$$

which follows directly from (5.6). Also, similarly to the Renyi entropy, the $q \to 1$ limit of the modular entropy still gives the entanglement entropy, that is $\lim_{q\to 1} \tilde{S}^{(q)}_{\mathcal{A}} = S_{\mathcal{A}}$. This allows us to instead only consider the variation of the on-shell action with respect to q, which we now recognise as a continuous parameter. The variation in q is essentially the variation of buandary conditions near the fixed point set $\mathbf{e}_q$. We can see this by considering $\delta g_{(q)} = \partial_q g_{(q)}$ which satisfies $\partial_q g_{(q)}|_{\mathcal{B}_q} = 0$ and $\partial_q g_{(q)}|_{\mathbf{e}_q} \neq 0$, where $g_{(q)}$ is the metric on $\hat{\mathcal{M}}_q$ [5]. This implies that this q variation of the action will reduce to a boundary term localised on this singular locus and thus will have no contribution from the conformal boundary. We use a tubular regulator around $\mathbf{e}_q$, i.e. restricting to $r > \varepsilon$ in Figure 5.3, we denote this bulk codimension-1 regulating surface as $\mathbf{e}_q(\varepsilon)$. The result is then given in the limit as $\varepsilon \to 0$.

We know that the variation of a gravitational action can always be decomposed into its equations of motion and some other boudary terms, i.e.

$$\delta S_{\text{Cl}}[\hat{\mathcal{M}}_q] = \int_{\hat{\mathcal{M}}_q} \Big(\text{EoM} \times \delta g_{(q)} + d\Theta\big(g_{(q)}, \delta g_{(q)}\big) \Big),$$

where Θ represents a general boundary term. For our gravity action, this is the standard setup for a variatinal problem and the boundary term is given by the so called *Gibbons-Hawking boundary term,* that is we have

$$S_{\text{EH}} \to S_{\text{EH}} + S_{\text{GH}} \qquad \text{with} \qquad S_{\text{GH}} = \frac{1}{8\pi G_N} \int_{\mathbf{e}_q(\varepsilon)} \sqrt{-\gamma}\, K,$$

where the integral is over the boundary, K is a boudary term and γ is the induced metric [22]. In our case the boundary term will be localised on the codimension-1 regulating surface $\mathbf{e}_q(\varepsilon)$.

For the orbifolded geometry (5.5), this boundary term is given by

$$\partial_q S_{\rm Cl} = \frac{1}{16\pi G_N} \int \sqrt{-\gamma}\, \mathbf{n}^N \big(\nabla^M \partial_q\, g_{NM} - g^{MP} \nabla_N\, \partial_q\, g_{MP}\big),$$

as shown in [23]. In the $q \to 1$ limit this indeed reproduces the RT proposal [20], i.e.

$$\lim_{q\to 1} \partial_q S_{\rm Cl}[\hat{\mathcal{M}}_q] = \lim_{q\to 1} \frac{\text{Area}(\mathbf{e}_q)}{4q^2 G_N} = \frac{\text{Area}(\mathbf{e}_q)}{4G_N} = \lim_{q\to 1} \tilde{S}^{(q)}_{\mathcal{A}} = S_{\mathcal{A}}.$$

This is remarkable, as we have only used one basic entry of the AdS/CFT dictionary.

Thus we see that the use of the orbifolded geometry $\hat{\mathcal{M}}$ allowed us not only to make sense of general q, but it also allows to easily calculate the on-shell action via the modular entropy. This derivation, nevertheless, does not address all the issues, for example there is no mention of the homology constraint, which forms a major part of the RT proposal. The problem is tackled in the recent paper [24].

5.4 Extremal Surfaces in AdS

We now show some examples of etremal surfaces in AdS and develop methods to find them. We adopt local coordinates to the extremal surface y^i and denote its induced metric γ_{ij}, which is defined as the pull-back on the AdS metric, i.e.

$$\gamma_{ij} = \frac{\partial X^M}{\partial y^i} g_{MN} \frac{\partial X^N}{\partial y^j},$$

where $X^M(y)$ is the embedding of the extremal surface $\mathcal{E}_{\mathcal{A}}$. The area fuctional of the surface is then given by

$$S[\mathcal{E}_{\mathcal{A}}] = \int d^{d-1}y \sqrt{-\gamma}.$$

Assuming that the metric takes the form [5]

$$ds^2_{\mathcal{M}} = \frac{1}{z^2}(dz^2 + \mathfrak{g}_{\mu\nu} dx^\mu dx^\nu),$$

implies in turn that the area functional must be

$$S = \int d^{d-1}y \sqrt{\det\left(\frac{\partial z}{\partial y^i}\frac{\partial z}{\partial y^j} + \mathfrak{g}_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}\right)}.$$

This choice is called the *Fefferman-Graham coordinates*, and the existance and consequences of such coordinates is discussed in [25].

The above discussion allows us to nicely set up the variational problem, as we are required to extremise S with the boundary condition $\mathcal{E}_{\mathcal{A}}|_{z=0} = \partial\mathcal{A}$. Once we have found the extremal surface, it is only left to evaluate the on-shell action of the geometry.

We now give a simple example of an extremal surface. The above extremization procedure in general is done numerically as they can only be done analytically in the most simple of theories, although in some cases the symmetries can be exploited.

For simplicity we consider the case of the vacuum state of CFT_2 which will have a dual geometry in AdS_3, this is a nice low dimensional case to consider first. Let the CFT_2 have coordinates (x,t) and take $\mathcal{A} = \{x = [-a,+a], t = 0\}$. Due to the static nature of the vacuum, we can concetrate on $t = 0$, that is we are required to extremise the action

$$S = \int dy \frac{\sqrt{x'^2(y) + z'^2(y)}}{z}$$

with[4] the boundary condition $x(z=0) = \pm a$. This is solved by the semicircle $x(y) = a \cos y$ and $z(y) = a \sin y$ precisely as shown, for example, in Figure 5.2. We can then evaluate the on-shell action of this solution to get the entanglement entropy.

[4]Note that the x inside the integral is the AdS coordinate while the x above is the CFT coordinate.

6 Conclusion

Now, let us take a step back and appreciate what we have discussed. To start, we introduced the basic building blocks of the AdS/CFT correspondence, namely Anti-de Sitter space and conformal field theory. We then stated and explored many aspects of the AdS/CFT correspondence including the calculation of boundary observables. Next, we gave an introduction to quantum information theory, which we then used to formulate entanglement entropy in QFT. We did this by representing QFT states by path integrals over different geometries. This allowed for a precise definition of the reduced density matrix of a field theory and lead to a replica construction to evaluate the powers of it. All this work with path integrals allowed us to find an expression for the Renyi entropy in terms of partition functions over some branched cover geometry of the original field theory spacetime. This turned out to be very useful as it readily connects to the AdS/CFT correspondence through the equivalence of the CFT and AdS partition functions.

After having developed all the ingredients, we proceeded to formulate the RT and HRT prescriptions of holographic entanglement entropy. We saw how it connects field theory states on one side to purely geometric objects on the other and discussed its implications. We then gave a holographic proof of the entropy inequality formulas which provided a nice check on the validity of the proposals. It also demonstrated the power and simplicity of the geometric viewpoint. We then attempted to give a holographic derivation of the time-independent RT proposal, which higlighted many of the subtle aspects of the relation. In the end, we had a brief discussion on how to set up the problem of calculating extremal AdS surfaces.

All in all, we found that the entanglement entropy of a field theory is directly proportional to the area of a specific codimension-2 bulk surface $\mathcal{E}_{\mathcal{A}}$, that is

$$S_{\mathcal{A}} \propto \text{Area}(\mathcal{E}_{\mathcal{A}}).$$

This formula allows for a lot more fundamental study of the AdS/CFT correspondence and may help understand the fabric of spacetime itself. We say this as what we have studied proves a deep connection between quantum dynamics and pure geometry. This is very exciting as we may hope to find out more about the quantum nature of spacetime, i.e. quantum gravity.

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