Field Theory of Topological Defects

An Introduction to Solitons and Cosmology

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Abstract

Topological defects are physical structures that may have arose during phase transitions in the early universe. In this report we provide a concise mathematical background in Group Theory and Algebraic Topology, which we use to systematically study soliton-like solutions to field equations. We then provide a link between these solutions and spontaneous symmetry breaking, which we use to explain how these defects might have formed during the early stages of the universe. We explicitly analyse multiple topological defects such as Kinks and Vortices, and discuss the consequences they might have towards many aspects of modern cosmology and particle physics.

Contents

1 Introduction

In this report we will focus on soliton-like solutions to field equations and how these topological defects may have arose during phase transitions in the early universe. The existence of such topological defects is closely linked to the process of spontaneous symmetry breaking originally discovered by Nambu [13] and Glashow [2] in the early 1960s. Although the early papers were mainly aimed at condensed matter physics, it was soon realised that the idea can be used in particle physics. This advance paved the path to the Nobel prize winning discoveries of Weinberg, Salam and Glashow,who further applied the notion of symmetry breaking to unify the electromagnetic and weak interactions. Topological defects in the context of symmetry breaking were first extensively discussed by Nielsen and Olesen [15] in the 1970s, who discovered the possibility of string-like topological defects due to symmetry breaking. The line of discoveries was continued by 't Hooft [7] and Polyakov [16] who showed that the existence of monopole-like topological defects is highly possible. This area of research was then further developed by Kibble, Coleman, Hindmarsh, among others. In this report we mainly build on these ideas and explore these fascinating objects.

We will mostly discuss soliton-like defects which also exist outside of particle physics, for instance in optics and solid state physics. Since there is no one widely used definition of a soliton, we attempt to make one, which will allow us to extensively study these objects.

Definition. ***Solitons*** *are solutions to field equation with the following properties*

(i) The corresponding energy density is finite and localized in space;

(ii) They are non-dissipative meaning that they preserve their shape while propagating at constant velocity. From this it follows that they are static in time, i.e $\dot{\phi} = Const$*;*

This is quite broad and we will not always insist that our solutions satisfy all these conditions, hence the name soliton-like. In order to study these fascinating objects we first have to introduce some mathematical tools which will help us to classify and understand them. First in Section 2 we introduce the abstract concept of topological spaces and we will see how we can use such abstract ideas to gain information about the structure of some well known sets. In Section 3 we turn our attention to symmetries of physical systems which will motivate us to develop group theory in order to quantify and define these symmetries. Then in Section 4 we use these ideas to study solutions to field equations and identify any possible soliton-like solutions. Section 7 provides a link between these topological defects and cosmology, and we discuss how they may have formed in the early universe. Finally in Sections 5 and 6 we discuss specific examples of topological defects and analyse their behaviour.

2 Topology and Homotopy Theory

Topological notions are very important in the theory of solitons as they help predict the existence of soliton solutions and give a method for classifying them. We first give some basic definitions regarding general topology and then we discuss the theory of homotopy groups, which we then directly apply to our soliton system in Section 4. The following definitions and theorems are mainly based on the books [3], [5], [12], [20], [11] and [24].

2.1 Topological Spaces

First we define a topological space.

Definition. *A* ***topology*** *on set* X *is a collection* $\mathcal{O}$ *of subsets of* X *with the following properties:*

(i) $\emptyset \in \mathcal{O}$ *and* $X \in \mathcal{O}$*;*

(ii) $U_1, U_2 \in \mathcal{O} \Longrightarrow U_1 \bigcap U_2 \in \mathcal{O}$ *(Intersection property);*

(iii) $U_\lambda \in \mathcal{O}, \forall \lambda \in \Lambda \Longrightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{O}$ *(Union property).*

A pair $(X, \mathcal{O})$ *is called a* ***topological space*** *and we say the sets in* $\mathcal{O}$ *are the open sets of* X*.*

The mathematical motivation behind this definition is that $\mathcal{O}$ is the least structure we need on sets to define continuity of maps between them. Hence we make the following definition.

Definition. *Let* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *be topological spaces. A map* $f\colon X \to Y$ *is* ***continuous*** *if*

$$V \in \mathcal{O}_Y \Longrightarrow f^{-1}(V) \in \mathcal{O}_X.$$

It is important to note that in this case $f^{-1}(V)$ denotes the preimage of V as the inverse need not exist. It is natural to ask the question whether there exists structure preserving maps between topological spaces similarly to linear maps between vector spaces. Such maps do exist and they are called homeomorphisms.

Definition. *A function* $f\colon X \to Y$ *is a* ***homeomorphism*** *if it is a continuous bijection with a continuous inverse.*

Such maps preserve topological structure and hence if there is a homeomorphism between two topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ they are said to be topologically equivalent. For a simple example consider the discrete set $X = \{a, b, c\}$. We then have the following topologies:

(i) Discrete Topology: Consists of all subsets of X

$$\mathcal{O}_d = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\};$$

(ii) Indiscrete Topology:

$$\mathcal{O}_i = \{\emptyset, X\};$$

(iii) Other Topologies:

$$\mathcal{O} = \{\emptyset, X, \{a\}\}$$
$$\mathcal{O}' = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\};$$
$$\dots$$

One special case is when the topological space $X = \mathbb{R}^n$. For such Euclidian space one has an intuition of open on the basis of analysis. More precisely we can define the open sets of $\mathbb{R}^n$ to be every subset of $\mathbb{R}^n$ without a boundary. We call the collection of these open sets the standard topology on $\mathbb{R}^n$ and from now on if the topology on $\mathbb{R}^n$ is not explicitly specified, we assume it is the standard topology. For example, the standard topology on $\mathbb{R}$ consists of all open intervals (a, b) such that $a, b \in \mathbb{R}$ and $a \neq b$.
We arrive at a very important definition in algebraic topology.

Definition. *A* ***path*** *in a topological space* $(X, \mathcal{O})$ *is a continuous map* $\sigma\colon [0,1] \subset \mathbb{R} \to X$*. The point* $\sigma(0)$ *is the start and the point* $\sigma(1)$ *is the end of the path.*
A topological space is said to be ***path connected*** *if* $\forall x, x' \in X$ *there is a path from* x *to* x'*.*

It is important to note that a path need not be a homeomorphism, we only require it to be continuous, therefore paths are allowed to intersect themselves.

2.2 Homotopy Classes and Groups

We commence with one of the central definition of algebraic topology.

Definition. *Let $(X, \mathcal{O})$ be a topological space with $x_0, x_1 \in X$. We write $I = [0,1]$ with the standard topology. Two paths $\sigma_0, \sigma_1 \colon I \to X$ from x_0 to x_1 are* ***homotopic****, written $\sigma_0 \simeq \sigma_1$, if there is a continuous map $H \colon I^2 \to X$ such that*

$$\begin{aligned} H(s,0) &= \sigma_0(s) \\ H(s,1) &= \sigma_1(s) \\ H(0,t) &= x_0 \\ H(1,t) &= x_1 \end{aligned}$$

for $s, t \in I$.

Intuitively this means that two paths are homotopic if they can be continuously deformed into each other without leaving the space.
The homotopy of paths between two points in a topological space is an equivalence relation meaning it satisfies three basic properties: reflexivity, symmetry and transistivity. We denote the equivalence class of a path σ as $[\sigma] = \{\tau | \tau \simeq \sigma\}$.

Definition. *A* ***loop*** *in a topological space X based at a point $x \in X$ is a path $\sigma \colon I \to X$ such that $\sigma(0) = \sigma(1) = x$. One can equivalently write this as a map $\sigma \colon S^1 \to X$ where by construction the beginning and end point are identified.*

Such loops are of great importance because surprisingly their homotopy classes form a group.

Theorem. *The set of homotopy classes of loops based at a point $x \in X$ forms a group under the product $[\sigma][\tau] = [\sigma \star \tau]$. The identity is $[\varepsilon_x]$ and the inverse is $[\sigma]^{-1} = [\bar{\sigma}]$. This group is called the* ***fundamental group*** *of X based at $x \in X$ and is denoted $\pi_1(X, x)$.*

In the case when the topological space X is path connected, we have that $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ $\forall x_0, x_1 \in X$. Hence the fundamental group is independent of the base point and we therefore may refer to it as $\pi_1(X)$. As an example, consider the homotopy classes of loops in S^1, i.e the paths $\sigma \colon [0,1] \to S^1$. This is equivalent to the paths $\sigma \colon S^1 \to U(1)$. Since every path in S^1 either returns to the base point without going around or circles around n times where $n \in \mathbb{Z}$, we have that $\pi_1(S^1) \simeq \pi_1(U(1)) \simeq \mathbb{Z}$. One can also define higher homotopy groups analogously to the fundamental group which is also called the first homotopy group. We define the n^{th} homotopy group of a topological space X as the equivalence classes of maps $f \colon [0,1]^n \to X$ such that the boundary of the hypercube is mapped to the base point. We can again equivalently write $f \colon S^n \to X$. In general one can show that $\pi_n(X)$ for $n \geq 2$ is Abelian. Table 1 shows the higher homotopy groups of the n-sphere.

	π_1	π_2	π_3	π_4	π_5	π_6
S^1	$\mathbb{Z}$	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 1: The higher homotopy groups of spheres. Zero denotes the trivial group and $\mathbb{Z}_n$ denotes the n^{th} root of unity under multiplication.

We may also consider $\pi_0(X)$, called the zeroth homotopy group of X. This is the set of homotopy classes of images of points in X and so it is equivalent to the notion of path connectedness. This is because the statement that two points are homotopic in X is equivalent to saying that there is a path between them. Hence $\pi_0(X)$ is just the set of connected components of X. It is important to note that $\pi_0(X)$ is not generally a group, unlike $\pi_n(X)$ for $n \geq 1$.

3 Lie Groups and Geometry

Group theory plays a central role in field theory as it is the natural language in which to discuss symmetries of physical and mathematical systems. In this section we lay down some of the fundamentals of group theory, which will help us study the symmetries arising from the Lagrangian formalism in Section 4. This section is mainly based on the books [8], [4], [21] and [24]. First of all however, we have to define what we mean by a group.

3.1 Groups, Representations and Orbits

Definition. *A **group** $(G, \cdot)$ is a set of elements endowed with a composition law $(\cdot)\colon G \times G \to G$ such that it satisfies the following properties:*

(i) Closure: $\forall a, b \in G$, $c = a \cdot b \in G$;

(ii) Associativity: $\forall a, b, c \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;

(iii) Identity: $\exists \mathbb{I} \in G$ such that $\mathbb{I} \cdot a = a$ $\forall a \in G$;

(iv) Inverse: $\forall a \in G$, $\exists a^{-1} \in G$ such that $a \cdot a^{-1} = \mathbb{I} = a^{-1} \cdot a$.

If $a \cdot b = b \cdot a$, $\forall a, b \in G$ then the group is called Abelian.

We will also need to consider subgroups of G, which is a subset of $H \subset G$ which itself forms a group under the composition law on G. We call H a proper subgroup if $H \neq \{\mathbb{I}\}$ and $H \neq G$.

Definition. *Given a subgroup $H = \{h_1, h_2, \cdots, h_n\}$ of a group G, the **coset** of H given an element $g \in G$ is*

$$gH = \{gh_1, gh_2, \cdots, gh_n\} \subset G.$$

The set of all cosets of H is denoted G/H.

Definition. *A **normal subgroup** H of G is a subgroup of G that satisfies*

$$gHg^{-1} = H$$

$\forall g \in G$.

Theorem. *Given a normal subgroup H of G, the set G/H forms a group under the operation*

$$(g_1H)(g_2H) = g_1g_2H.$$

*Any group which can be written in such a way is called a **quotient group**.*

We will see that such groups are very important in physics as they will allow us to study the vacua arising from symmetry breaking processes.

Definition. *A **group homomorphism** is a map $f\colon G \to G'$ between two groups $(G, \cdot)$ and $(G', \star)$ such that $\forall g, g' \in G$,*

$$f(g \cdot g') = f(g) \star f(g').$$

Definition. *The **kernel** $Ker(f)$ of a group homomorphism $f\colon G \to G'$ is the set*

$$Ker(f) = \{g \in G | f(g) = \mathbb{I}_{G'} \in G'\},$$

where $\mathbb{I}_{G'}$ is the identity element of G'. Equivalently one can say that the kernel of a group homomorphism is the preimage of the identity element.

Theorem. *(**The isomorphism theorem**:) Given a group homomorphism $f: G \to G'$ with kernel $Ker(f) = K$, then*

$$f(G) = G/K.$$

To discuss groups in a physical context one needs to represent groups in a way that preserves their abstract structure while allowing us to easily calculate with them. This is usually done by group representations which from an important part of group theory and even started a separate area of mathematics called representation theory. Here we will give the definition of a representation. We will later see how this helps us visualise and understand the structure of many abstract groups.

Definition. *A **group representation** T,*

$$T: g \to T(g) \in GL(N, \mathbb{C}),$$

is a group homomorphism from G into the group of $N \times N$ complex invertible matrices $GL(N, \mathbb{C})$.

Definition. *Let G be a group and X be a set. The **group action** φ of G on X is a map*

$$\varphi: G \times X \to X: (g, x) \to \varphi(g, x) = g.x$$

that satisfies

(i) $ex = x$, $\forall x \in X$;

(ii) $(gg').x = g.(g'.x)$, $\forall g, g' \in G$ and $x \in X$.

Definition. *The **orbit** of an element $x \in X$ of a set X with respect to the group G is the subset of X to which x can be moved by the action of G. Explicitly, the orbit of $x \in X$ w.r.t G is the set*

$$G.x = \{g.x | g \in G\}.$$

The collection of all orbits of a set X, partitions the set into disjoint subsets. This mean that the union of all orbits of a set gives back the set, while each element of the set is only an element of a single orbit.

Definition. *The **stabilizer** subgroup G_x of G w.r.t $x \in X$, also called the isotropy group, is the set of elements in G which leave x fixed, i.e*

$$G_x = \{g \in G | g.x = x\}.$$

3.2 Lie Groups and Manifolds

Lie groups are continuous groups which describe most physical symmetries and hence are of great importance in physics.

Definition. *A **Lie group** is a group which is also a differentiable (smooth) manifold.*

We will not define a smooth manifold here, but intuitively they are sets which are locally Euclidian and hence one can do calculus on them. All groups that we will use in the later sections are Lie groups and so here we give a some examples to clarify further discussions.

(i) *SO(2)*: This group physically represents rotations in two dimensions through an arbitrary angle. This group is hence parameterised by a single parameter, say ϕ, the rotation angle. A representation of this group is the well known matrix

$$T(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

which is manifestly two dimensional.

(ii) *U(1)*: This is the group of rotations of a complex vector in complex space. It is again parameterised by single variable θ and has the simplest representation $T(\theta) = e^{i\theta}$ which is one dimensional.

These are very basic examples, but they help to understand how one might apply group theory to physical problems.

4 Field Theory and Solitons

As our best description of the fundamental interactions of particle physics are formulated in terms of fields it is natural to work with these objects. In order to do so we have to set up a framework in which we can describe topological defects in due course. This section mainly builds on books [18], [1], [11] and [25].

4.1 Lagrangian Field Theory

First, assume a Minkowski space-time $\mathbb{R} \times \mathbb{R}^d = X$ with coordinates $x = (t, \mathbf{x})$ and Minkowskian metric $\eta_{\mu\nu}$ with signature $(+, -, -, -)$. Consider a scalar field $\phi = (\phi_1, \cdots, \phi_n)$, which is a smooth map $\phi\colon X \to \mathbb{R}^n$. We will denote the set of all smooth maps from X to $\mathbb{R}^n$ as $Map(X, \mathbb{R}^n)$ and hence we have that $\phi \in Map(X, \mathbb{R}^n)$. Take the general simplest Lorentz invariant Lagrangian for ϕ

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - U(\phi), \tag{4.1}$$

where $U(\phi)$ is a non negative scalar potential. The corresponding Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + U(\phi). \tag{4.2}$$

For a physical system the potential $U(\phi)$ has to be bounded from below, and hence we may always rescale it such that $U_{min} = 0$. The Lagrangian (4.1) satisfies the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0, \tag{4.3}$$

and after substitution we get the field equation

$$\partial_\mu \partial^\mu \phi + \frac{dU(\phi)}{d\phi} = 0,$$

which is non-linear and Lorentz invariant. It is important to note that the Lagrangian (4.1) possesses some internal symmetries, apart from Lorentz invariance. An example is $SO(n)$ symmetry, since under the rotation

$$\phi_i \to R_{ij}\phi_j,$$

with $R \in SO(n)$, the Lagrangian remains unchanged.

4.2 Gauge Theory

So far we have seen that the Lagrangian can possesses some internal symmetries. We have shown that the Lagrangian (4.1), is invariant under global $SO(n)$ transformation. The term global means that the transformation is of the form $\phi \to \Lambda\phi$ where Λ is an appropriate representation of the Lie group and is independent of space-time x. Now, one might be interested in whether we can impose local symmetries of our Lagrangian. A local symmetry is one where the Lagrangian is invariant under transformations of the form $\phi \to \Lambda(x)\phi$, where $\Lambda(x)$ is again a representation of the Lie group, but now it is dependent on space-time x. Construction of such locally invariant Lagrangians are possible and we will now consider a specific example of how one might proceed.

Consider the Lagrangian (4.1) with two real scalar fields, or equivalently one complex field $\phi = \phi_1 + i\phi_2$, given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi^*) - U(\phi\phi^*). \tag{4.4}$$

It is globally $U(1)$ invariant since under the transformation

$$\phi \to e^{i\alpha}\phi$$

the Lagrangian transforms as

$$\mathcal{L} \to \frac{1}{2}(\partial_\mu e^{i\alpha}\phi)(\partial^\mu e^{-i\alpha}\phi^*) - U(e^{i\alpha}e^{-i\alpha}\phi\phi^*) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi^*) - U(\phi\phi^*) = \mathcal{L}.$$

On the other hand, under the local $U(1)$ transformation

$$\phi \to e^{i\alpha(x)}\phi \tag{4.5}$$

the Lagrangian transforms to

$$\mathcal{L} \to \frac{1}{2}(\partial_\mu e^{i\alpha(x)}\phi)(\partial^\mu e^{-i\alpha(x)}\phi^*) - U(e^{i\alpha(x)}e^{-i\alpha(x)}\phi\phi^*) = \frac{1}{2}(\partial_\mu e^{i\alpha(x)}\phi)(\partial^\mu e^{-i\alpha(x)}\phi^*) - U(\phi\phi^*) \neq \mathcal{L}$$

and so even though the potential $U(\phi\phi^*)$ is locally $U(1)$ invariant the derivatives do not allow such symmetry of the Lagrangian. We can make the Lagrangian locally $U(1)$ invariant by introducing an electromagnetic field to which our fields couple to. Such a theory is called scalar electrodynamics since the EM fields couple to a scalar field instead of vector fields as in QED. Consider the gauge potential $A_\mu = (\Phi, \mathbf{A})$ where Φ is the electric scalar potential and $\mathbf{A}$ is the magnetic vector potential. From classical electrodynamics we know that the resulting physics is invariant under transformations $A_\mu \to A_\mu + \partial\chi(x)$ with $\chi(x)$ scalar function. This motivates us to postulate that the gauge potential transforms under (4.5) as

$$A_\mu \to A_\mu + \partial_\mu \alpha(x).$$

Moreover, the electromagnetic field tensor $F_{\mu\nu}$, defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

is also invariant under such transformations. We now attempt to fix the non-symmetric terms in the Lagrangian (4.4), by introducing a covariant derivative defined by

$$D_\mu \phi = \partial_\mu \phi - ieA_\mu \phi,$$

and which transforms as

$$D_\mu \phi \to \partial_\mu (e^{i\alpha(x)}\phi) - ieA_\mu e^{i\alpha(x)}\phi = e^{i\alpha(x)} D_\mu \phi,$$

i.e as the field itself. By now we have constructed a dynamical variable of the fields and gauge fields that is gauge invariant in the form $(D_\mu\phi)^*(D^\mu\phi)$ and $F_{\mu\nu}F^{\mu\nu}$. Using these results we can now write down a gauge invariant Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - U(\phi\phi^*). \tag{4.6}$$

We can get the corresponding Euler-Lagrange equations by the variation of ϕ and A, which gives

$$D_\mu D^\mu \phi = -2\phi \frac{dU}{d(\phi\phi^*)} \tag{4.7}$$

$$\partial_\mu F^{\mu\nu} = \frac{-ie}{2}\left(\phi^* D^\nu \phi - \phi D^\nu \phi^*\right). \tag{4.8}$$

4.3 The Vacuum and Spotaneous Symmetry Breaking

Let the potential $U(\phi)$ in (4.1) have a minimal value $U_{min} = 0$. We denote by $\mathcal{V}$ the manifold where U attains its minimal value and $\dot{\phi} = \nabla\phi = 0$, i.e

$$\mathcal{V} = \{\phi_0 \in \text{Map}(X, \mathbb{R}^n) | \dot{\phi} = \nabla\phi = U(\phi) = 0\}.$$

We call this the vacuum manifold of the potential $U(\phi)$ as, if $\phi \in \mathcal{V}$ then the total energy is zero. It is a submanifold of the space $\text{Map}(X, \mathbb{R}^n)$, i.e all possible ϕ. Since the fields in $\mathcal{V}$ are constant, we can think of this manifold as being a subset of $\mathbb{R}^n$.

Given a Lagrangian invariant under the action of a group G, we require the physical observables like energy be also invariant under G. Mathematically, this means that the orbit of field $\phi \in \text{Map}(X, \mathbb{R}^n)$ w.r.t G should have the same energy. Assume that the minima of $U(\phi)$ are attained on a single orbit of $\text{Map}(X, \mathbb{R}^n)$, i.e

$$\mathcal{V} = G.\phi_0,$$

with ϕ_0 being some fixed element in $\text{Map}(X, \mathbb{R}^n)$. We now have to distinguish two possibilities:

(1) *Unbroken Symmetry:* If the vacuum orbit consists of a single point $\mathcal{V} = \{\phi_0\}$, then the action of G will be trivial and hence the stabilizer will be the group itself, i.e

$$\mathcal{V} = G.\phi_0 = \{\phi_0\} \Rightarrow \varphi_G(\mathcal{V}) = \mathcal{V} \Rightarrow G_{\phi_0} = G.$$

In this case $\mathcal{V}$ possesses the same symmetry as $U(\phi)$ and we say the symmetry is unbroken in the vacuum.

(2) *Spontaneous Symmetry Breaking:* If the vacuum orbit, on the other hand, consists of multiple points, then the vacuum manifold will no longer be invariant under G, i.e

$$\mathcal{V} = G.\phi_0 = \{\phi_0, \phi_1, \cdots, \phi_m\} \Rightarrow \varphi_G(\mathcal{V}) \neq \mathcal{V} \Rightarrow G_{\phi_0} = H \neq G.$$

In such a case we say the symmetry G of the Lagrangian has been spontaneously broken down to H in the vacuum.

It follows from above that the true vacuum configuration of our system is given by the quotient group $\mathcal{V} = G/G_x$ which in the above cases gives

$$\mathcal{V} = \begin{cases} G/G \simeq \{\mathbb{I}\} & \text{(Unbroken Symmetry)} \\ G/H & \text{(Broken Symmetry).} \end{cases}$$

As an example, consider our Lagrangian (4.1) which we have seen posesses $SO(n)$ symmetry. If we take the potential to be

$$U(\phi) = (\phi_i\phi_i - F)^2,$$

then for $F \leq 0$ the vacuum is a single point at $\phi = 0$, and the symmetry is unbroken. Conversely, if $F > 0$, then the vacuum manifold consists of points $|\phi| = \sqrt{F}$. This means that vacuum solutions ϕ_0 have the form

$$\phi_0 = (0, 0, \cdots, \sqrt{F}, \cdots, 0, 0)$$

which manifestly has $SO(n-1)$ symmetry. Hence, the symmetry has been broken in the vacuum and the vacuum manifold is given by

$$\mathcal{V} = SO(n)/SO(n-1) \simeq S^{n-1}.$$

4.4 Vacuum Topology and Solitons

Consider a general Lagrangian (4.1) for a scalar field $\phi = (\phi_1, \cdots, \phi_n)$ with scalar potential $U(\phi)$. The total energy of such lagrangian is given by an integral over the Hamiltonian density (4.2)

$$E(\phi) = \int_{-\infty}^{\infty} \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + U(\phi)d^dx. \tag{4.9}$$

To gain soliton solutions we require the boundary condition that the total energy be finite. Imposing this condition on (4.9) gives that in the limit, as $|\mathbf{x}| \to \infty$, the following have to be satisfied:

$$|\mathbf{x}| \to \infty \;\Rightarrow \begin{cases} \dot{\phi} = 0 \\ \nabla\phi = 0 \\ U(\phi) = 0. \end{cases} \tag{4.10}$$

This is equivalent to the requirement

$$\lim_{|\mathbf{x}|\to\infty} \phi(x) \in \mathcal{V},$$

which means that to get a finite energy solution the field has to take its values in the vacuum manifold $\mathcal{V}$ at spatial infinity. Therefore the possible finite energy solutions are classified by a map φ^∞ which maps the (d-1)-sphere at infinity to $\mathcal{V}$

$$\varphi^\infty \colon S^{d-1}_\infty \to \mathcal{V},$$

which is a (d-1)-loop in $\mathcal{V}$. Each loop in the vacuum manifold has a specific topological winding number, also called topological charge) associated with it, namely the image of the homotopy equivalence class $[\varphi^\infty]$ in $\pi_{d-1}(\mathcal{V})$. Hence the solutions ϕ carry a topological charge, one of the elements of $\pi_{d-1}(\mathcal{V})$. Solutions with the same topological charge can be continuously deformed into each other and so any solution which carries a trivial charge can be deformed to the vacuum. The non-trivial fields interpolate between different vacua. We will now see how much information we can gain about soliton solutions in d-dimensions without further specifying the form of $U(\phi)$.

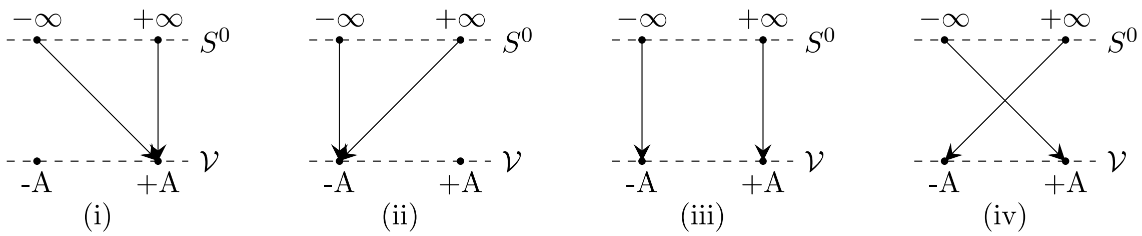

Figure 1: The possible mappings from S^0 to $\mathcal{V}$. The maps $(i) - (ii)$ are contractible to a point so they are trivial, while $(iii) - (iv)$ carry a non-trivial topological charge.

(1) *d=1:* In one spatial dimension the map φ^∞ is of the form $\varphi^\infty \colon S^0_\infty = (-\infty, +\infty) \subset \mathbb{R} \to \mathcal{V}$. So the solutions are classified by the elements of $\pi_0(\mathcal{V})$. In the special case when the vacuum manifold is given by $\mathcal{M} = S^0_A = (-A, +A) \subset \mathbb{R}$, the possible mappings φ^∞ are topologically classified into two groups as shown in Figure 1. We see that out of the four possible mapping $(i) - (iv)$, only two $(iii) - (iv)$ are non-trivial. These correspond to solutions that interpolate between the two vacua, meaning that they satisfy the specific boundary conditions

$$\mathbf{x} \to \pm\infty \;\Rightarrow \begin{cases} \phi(x) \to \pm A & \text{(iii)} \\ \phi(x) \to \mp A & \text{(iv)}. \end{cases}$$

These topologically non trivial solutions in $1+1$ dimensions are called kink-like.

(2) *d=2:* In two spatial dimensions the map φ^{∞} is of the form $\varphi^{\infty}\colon S^1_{\infty} \subset \mathbb{R}^2 \to \mathcal{V}$. So the solutions are classified by the elements of the fundamental group $\pi_1(\mathcal{V})$. For each possible field ϕ there corresponds an image in $\pi_1(\mathcal{V})$ given by the boundary conditions it satisfies. Again, each solution carries a topological charge and those solutions, which correspond to the identity element in $\pi_1(\mathcal{V})$, can be deformed to the vacuum. In the special case when $\mathcal{V} = S^n_A$ we only have a non-trivial fundamental group if $n = 1$. In that case $\pi_1(S^1_A) = \mathbb{Z}$ and so each solution ϕ carries an integer topological charge called the vortex number. Such non-trivial solutions are called vortex-like.

In Table 2 we summarise these topological defects and give the conditions for their existence.

Name	d	*Condition*
Kink	1	$\pi_0(\mathcal{V}) \neq \mathbb{I}$
Domain Wall	2 or 3	
Vortex	2	$\pi_1(\mathcal{V}) \neq \mathbb{I}$
String	3	

Table 2: The classification of topological defects, with the conditions for their existence. Here $\pi_i(\mathcal{V}) \neq \mathbb{I}$ means that it is non-trivial, i.e contains more than one element.

4.5 Bogomolny Bounds

Consider again a Lagrangian of the form (4.1), from which we get the field equation

$$\partial_\mu \partial^\mu \phi + \frac{dU(\phi)}{d\phi} = 0,$$

which is a second order differential equation. As we are interested in finding static solitonic solutions, $\dot{\phi} = 0$, and the field equation reduces to

$$\nabla^2 \phi - \frac{dU(\phi)}{d\phi} = 0. \tag{4.11}$$

Bogomolny developed a method which allows us to reduce (4.11) to a first order differential equation, by finding a lower bound on the energy and requiring that the physical solutions attain this bound [22]. In our case, this can be done by a much more direct method making use of the boundary conditions (4.10). We start by noticing that after multiplying through (4.11) by $\nabla\phi$ we can rewrite it as

$$\nabla \left(\frac{1}{2}(\nabla\phi)^2 - U(\phi) \right) = \nabla^2\phi \nabla\phi - \frac{dU(\phi)}{d\phi}\nabla\phi = 0.$$

Hence the term in the brackets

$$\frac{1}{2}(\nabla\phi)^2 - U(\phi) = \text{Const}$$

is a constant of motion. The constant can be determined by evaluating the above equation at spatial infinity and imposing the boundary conditions (4.10). This implies that the constant must be equal to zero and thus we get the first order equation

$$\frac{1}{2}(\nabla\phi)^2 - U(\phi) = 0 \tag{4.12}$$

which is often referred to as the Bogomolny equation.

5 Kinks

In this section we consider the simplest types of solitons which arise in $1+1$-dimensions. Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\partial_x\phi)^2 - U(\phi), \tag{5.1}$$

discussed in Section 4. The general theory in Section 4.4 suggests that solitonic solutions only exist if the potential $U(\phi)$ has degenerate minima. This motivates us to consider the following two scalar potentials

$$U_\phi(\phi) = \frac{\lambda}{4}(\phi^2 - F^2)^2$$
$$U_S(\phi) = A\left(1 - \cos\frac{2\pi\phi}{F}\right),$$

with $F \in \mathbb{R}$ and $A, \lambda \in \mathbb{R}^+$, called the ϕ^4 and sine-Gordon potentials respectively.

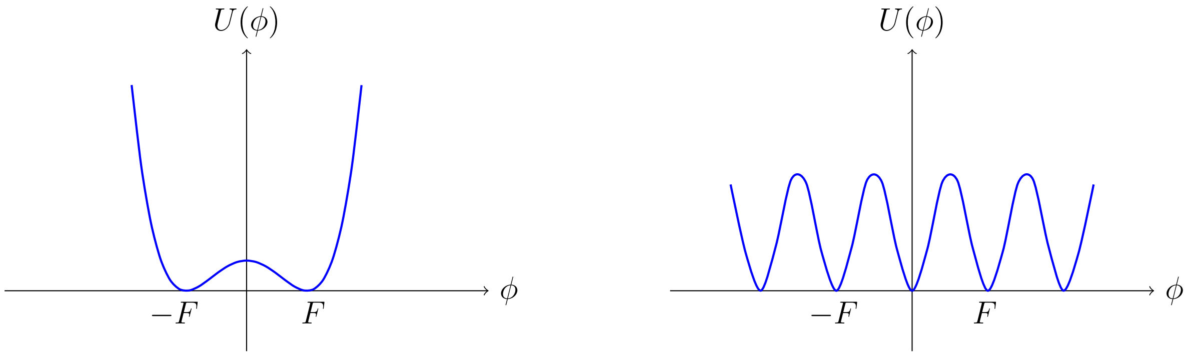

Figure 2: The ϕ^4 [left] and sine-Gordon [right] potentials.

Figure 2 shows these potentials and we see that the ϕ^4 potential has two minima at $\pm F$, while the sine-Gordon potential has infinite degenerate minima at nF with $n \in \mathbb{Z}$. These minima occur at constant field values nF in both cases and so the vacuum manifold for the two systems takes the form

$$\mathcal{V}_\phi = \{+F, -F\} = S^0_F \subset \mathbb{R}$$
$$\mathcal{V}_S = \{nF | n \in \mathbb{Z}\} \subset \mathbb{R}.$$

We will now consider these two cases separately.

5.1 The ϕ^4 Kink

We have already met this form of the vacuum manifold $\mathcal{V}_\phi$ in Section 4.4 and so we know that the non-trivial solutions carry a topological charge: one of the elements of $\pi_0(\mathcal{V}) = \pi_0(S^0) = \mathbb{Z}_2$. Hence, we expect two solitonic solutions, which interpolate between the two vacua at $\pm F$ and carry different topological charges. The full Lagrangian density for the ϕ^4 potential is

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{\lambda}{4}(\phi^2 - F^2)^2, \tag{5.2}$$

where $\lambda \in \mathbb{R}^+$ and $F \in \mathbb{R}$ are real parameters. The corresponding Bogomolny equation is given by

$$\frac{1}{2}(\partial_x\phi)^2 = \frac{\lambda}{4}(\phi^2 - F^2)^2.$$

Taking the square root, we get two separate equations

$$\partial_x\phi = \pm\sqrt{\frac{\lambda}{2}}(\phi^2 - F^2).$$

This has a straightforward solution

$$\phi(x) = \pm F \tanh\left(\frac{F\sqrt{\lambda}}{\sqrt{2}}(x - x_0)\right) \tag{5.3}$$

shown in Figure 3.

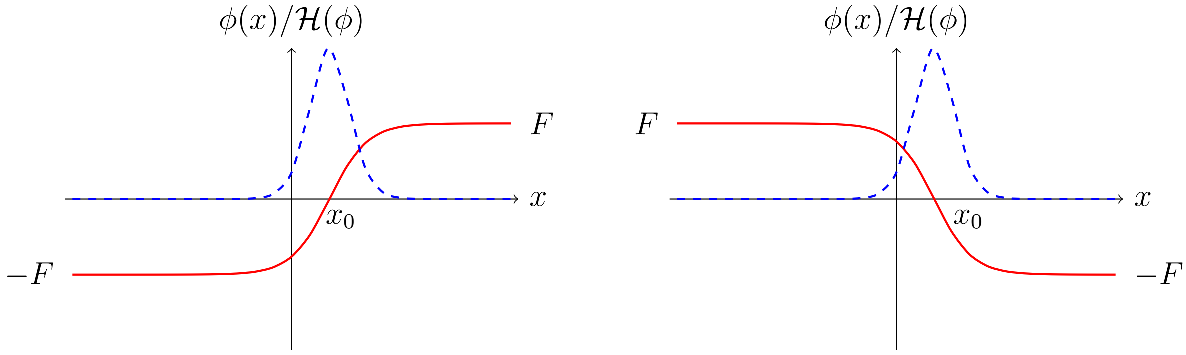

Figure 3: The kink [left] and anti-kink [right] solutions with the corresponding energy densities as dashed lines.

We see that as expected we have two non-trivial solutions, which interpolate between the two elements of $\mathcal{V}$. These two solutions correspond to cases (iii) and (iv) in Figure 1 and thus carry different topological charges. To get the corresponding energy density we need to substitute the solutions (5.3) into the Hamiltonian (4.2), which gives

$$\mathcal{H} = \frac{F^4\lambda}{2}\operatorname{sech}^4\left(\frac{F\sqrt{\lambda}}{\sqrt{2}}(x - x_0)\right)$$

for both solutions. This energy density is also plotted in Figure 3 along with the solutions. We see that we get a localised energy density about x_0 which we therefore call the centre of the kink.

5.2 The Sine-Gordon Kink

The vacuum manifold in this case is infinitely degenerate. It consists of points at nF, with $n \in \mathbb{Z}$, on the real line and so $\pi_0(\mathcal{M}_S) = \mathbb{Z}$. This means each solution will carry an integer topological charge, but unlike in the ϕ^4 model, now we have infinitely many possible charges. The full Lagrangian density for the sine-Gordon potential is

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\partial_x\phi)^2 - A\left(1 - \cos\frac{2\pi\phi}{F}\right),$$

where A and F are nonzero real parameters. The corresponding Bogomolny equation is given by

$$\frac{1}{2}(\partial_x\phi)^2 - A\left(1 - \cos\left(\frac{2\pi}{F}\phi\right)\right) = 0. \tag{5.4}$$

Using the double angle formula $\cos(2\theta) = 1 - 2\sin^2(\theta)$ we may rewrite equation (5.4) as

$$\frac{1}{2}(\partial_x\phi)^2 = 2A\sin^2\left(\frac{\pi}{F}\phi\right),$$

which has analytic solutions

$$\phi(x) = \frac{2F}{\pi}\arctan\left(e^{\pm 2\pi\sqrt{A}/F(x - x_0)}\right).$$

This is shown in Figure 4 along with the corresponding energy density which is again localised [17].

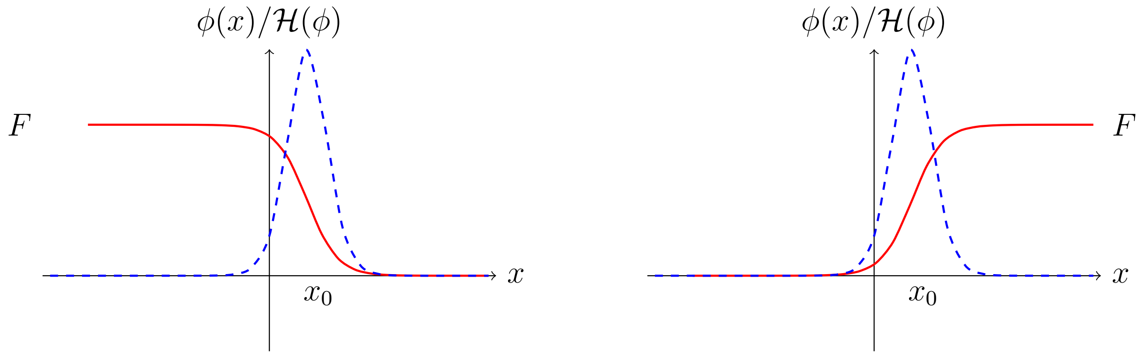

Figure 4: The kink [right] and anti-kink [left] solutions for the sine-Gordon model with the corresponding energy densities as dashed lines.

6 Vortices

In this section we discuss the simples types of vortex solutions that arise due to the breaking of a U(1) symmetry of the Lagrangian. We attempt to find solitons in 2+1 dimensions, but can also then be trivially extended two three dimensions. Consider a complex field $\phi = \phi_1 + i\phi_2$ which can equivalently be described by ϕ and ϕ^*. The simplest Lagrangian for such fields takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi^*) - U(\phi), \tag{6.1}$$

where now $\phi \in C^\infty(X, \mathbb{R}^2)$. It possesses $U(1)$ internal symmetry as under the global transformation

$$\phi \to e^{i\theta}\phi,$$

the Lagrangian (6.1) remains unchanged. Motivated by Section 4.4, we consider the potential

$$U(\phi) = \frac{\lambda}{2}(\phi^*\phi - F^2)^2 \tag{6.2}$$

plotted in Figure 5, which has degenerate minima at $\phi^*\phi = F^2$. This means the vacuum manifold in this case is given by

$$\mathcal{V} = \{\phi \in \mathrm{Map}(X, \mathbb{R}^n) | \dot{\phi} = \nabla\phi = 0 \,\mathrm{and}\, |\phi| = F\}$$

which equivalent to the vectors of length F in $\mathbb{R}^2$ and hence equivalent to S^1_F. This implies that each vacuum solution can be written as $\phi_0 = Fe^{i\chi}$ where χ is an arbitrary angle. Such a solution is shown in Figure 5. For the gradient term to vanish, the parameter χ has to be fixed, which breaks the $U(1)$ symmetry. We have already met the vacuum manifold S^1 in Section 4.4, so we expect to gain soliton solutions classified by integer topological charges. We will now check whether this is actually the case. The Hamiltonian density corresponding to the Lagrangian density (6.1) is

$$\mathcal{H} = \frac{1}{2}\dot{\phi}\dot{\phi}^* + \frac{1}{2}\nabla\phi\nabla\phi^* + \frac{\lambda}{2}(\phi^*\phi - F^2)^2. \tag{6.3}$$

Imposing the boundary condition (4.10) gives

$$\lim_{|\mathbf{x}|\to\infty} \phi(x) = Fe^{i\chi}, \tag{6.4}$$

meaning at spatial infinity the fields must take their value in the vacuum manifold $\mathcal{V} = S^1$.

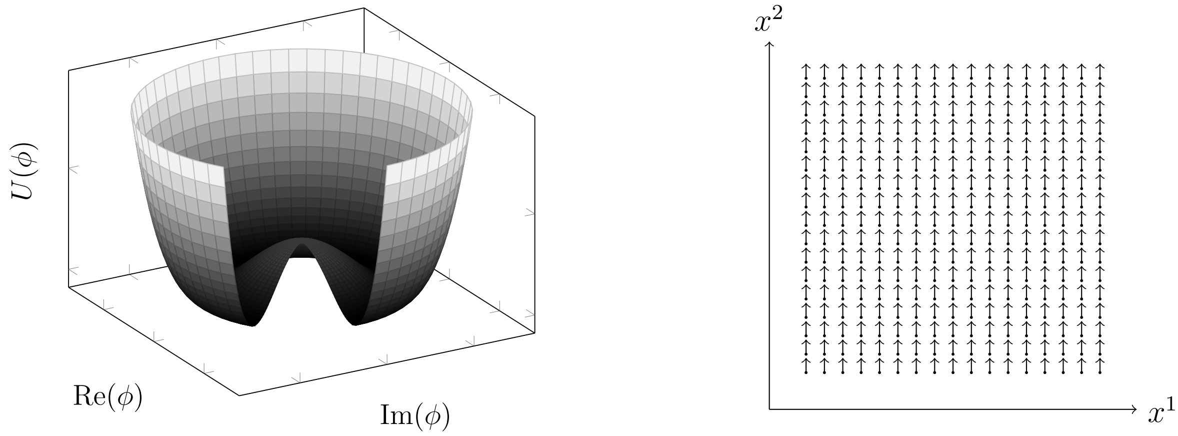

Figure 5: The ϕ^4 potential (6.2) [left] plotted against the real and imaginary parts of the field. A vacuum solution [right] depicted as a plot of the vector field $\phi = (\phi_1, \phi_2)$ in two dimensional space (x^1, x^2).

For a general solution, the boundary conditions mean that the field at infinity must take the form

$$\phi^\infty(r, \theta) = F e^{i\chi(\theta)},$$

where we have denoted by (r, θ) the standard polar coordinates on $\mathbb{R}^2$. We see that compared to the vacuum solutions, the parameter $\chi(\theta)$ is now a function of θ. This precisely meas that a solution will be classified by a map $\varphi^\infty \colon S^1 \to S^1$ and so equivalently by $\pi_1(S^1) = \mathbb{Z}$. Moreover, since φ^∞ is single valued, χ must be periodic in 2π giving $\chi(2\pi) = \chi(0) + 2\pi N$ with $n \in \mathbb{Z}$. As expected, the vortex solutions carry a non-trivial integer topological charge, which is precisely the number N above. Solutions satisfying the above boundary conditions with different values for N are shown in Figure 6.

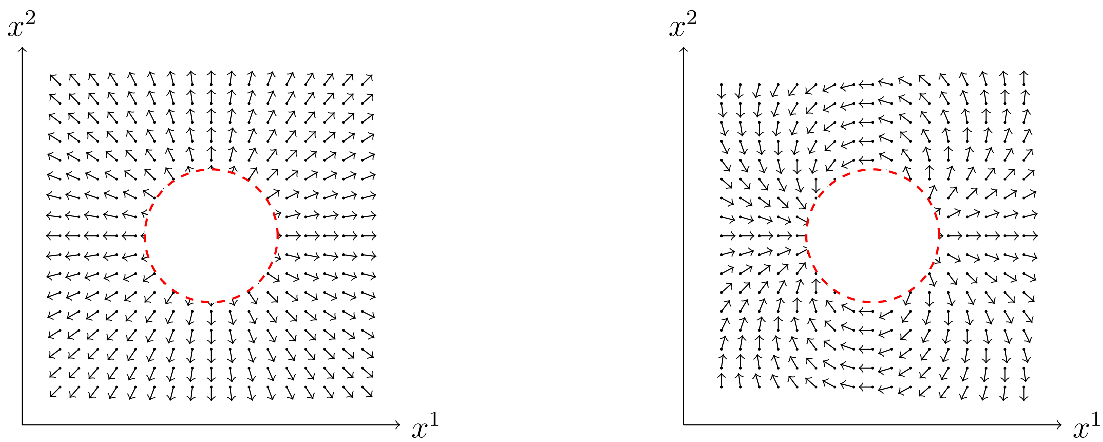

Figure 6: General solutions satisfying the boundary conditions (6.4) with $N = 1$ [left] and $N = 2$ [right], depicted as a plot of the vector field $\phi = (\phi_1, \phi_2)$ in two dimensional space (x^1, x^2). The boundary conditions only specify the behaviour near spatial infinity, so at this point the field inside the dashed circle is arbitrary.

6.1 Global Vortices

Substituting our Lagrangian (6.1) into the Euler-Lagrange equation (4.3) gives

$$\nabla^2 \phi + \frac{dU}{d\phi^*} = 0 \tag{6.5}$$

$$\nabla^2 \phi^* + \frac{dU}{d\phi} = 0. \tag{6.6}$$

These two equations are each others complex conjugate and so they contain the same information. In other words it is sufficient to continue with only one of the above equations. Substituting the specific form of our potential (6.2) gives

$$\nabla^2\phi + \lambda(\phi^*\phi - F^2)\phi = 0,$$

called the Ginzburg-Landau equation [11]. The $SU(1)$ symmetry of the Lagrangian implies that under transformations $\theta \to \theta+\alpha$ the field behaves as $\phi(r,\theta+\alpha) = e^{ik\alpha}\phi(r,\theta)$, where k is an integer. This condition is only satisfied if the untransformed field takes the form $\phi(r,\theta) = e^{ik\theta}\phi(r)$. The topological charge of such field is given by $N = k$ and so our solutions take the general form

$$\phi(r,\theta) = e^{iN\theta}\phi(r),$$

with $\phi(r)$ being real. This means that equation (6.5) reduces to a trivial angular and a radial part

$$\partial_r^2\phi(r) + \frac{1}{r}\partial_r\phi(r) - \frac{N^2}{r^2}\phi(r) + \lambda(\phi^2(r) - F^2)\phi(r) = 0, \tag{6.7}$$

which again is a non-linear second order field equation. Unlike the equations we have met before, this is not analitically solvable. We can find the asymptotic solutions by expansion, which gives [14]

$$\phi(r) \simeq \begin{cases} r^N + \mathcal{O}(r^{n+2}) & \text{as} \quad r \to 0 \\ 1 - \frac{N^2}{2r^2} + \mathcal{O}(1/r^4) & \text{as} \quad r \to \infty. \end{cases}$$

One can also numerically solve (6.7), which gives an energy density localised about the vortex core, as expected for solitons [11, 14].

It is key to realise that the total energy written as an integral over (6.3)

$$E(\phi) = \int_{-\infty}^{\infty} \left(\frac{1}{2}\nabla\phi\nabla\phi^* + \frac{\lambda}{2}(\phi^*\phi - F^2)^2\right) d^2x$$

does not converge. To justify this statement, we rewrite the Hamiltonian in terms of polar coordinates to give

$$E(\phi) = \int_0^{\infty}\int_0^{2\pi} \left(\frac{1}{2}\partial_r\phi\partial_r\phi^* + \frac{1}{2r^2}\partial_\theta\phi\partial_\theta\phi^* + \frac{\lambda}{2}(\phi^*\phi - F^2)^2\right) rdrd\theta. \tag{6.8}$$

In the limit at spatial infinity, i.e as $r \to \infty$, the field must take the form $\phi^\infty(r,\theta) = Fe^{i\chi(\theta)}$. Substituting this form into the energy density (6.8) gives that as

$$r \to \infty \Rightarrow \mathcal{H}(\phi) \to \frac{1}{2r^2}F^2 e^{2i\chi(\theta)}\left(\frac{d\chi}{d\theta}\right)^2.$$

This means that outside a shell of sufficiently large radius ρ, the total energy is given by

$$\int_\rho^{\infty}\int_0^{2\pi}\left(\frac{1}{2r^2}F^2 e^{2i\chi(\theta)}\left(\frac{d\chi}{d\theta}\right)^2\right) rdrd\theta \propto [\log(r)]_\rho^\infty$$

and hence the energy is logarithmically divergent. Despite the divergent energy, this global vortex still plays a role in condensed matter systems as the divergence is sufficiently small [11]. As we will now show, this divergence can be fixed by introducing a gauge field and making the Lagrangian locally $U(1)$ invariant. Such a gauged Lagrangian gives rise to what we call gauged or Nielsen-Olesen vortices.

6.2 The Nielsen-Olesen Vortex

In Section 4.2 we derived the Lagrangian for a gauged complex scalar field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - U(\phi\phi^*). \tag{6.9}$$

and the corresponding Euler-Lagrange equations (4.8)

$$D_\mu D^\mu \phi = -2\phi \frac{dU}{d(\phi\phi^*)} \tag{6.10}$$

$$\partial_\mu F^{\mu\nu} = \frac{-i}{2}\left(\phi^* D^\nu \phi - \phi D^\nu \phi^*\right). \tag{6.11}$$

Separating the time and space components in (6.9), the Lagrangian naturally splits into kinetic and potential terms

$$\mathcal{L} = \left[\frac{1}{2}e_i e_i + \frac{1}{2}(D_0\phi)^*(D_0\phi)\right] - \left[\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}(D_i\phi)^*(D_i\phi) + U(\phi\phi^*)\right].$$

We again require that the solutions be static in time and so the kinetic term vanishes which gives that the total energy is

$$E = \int \left(\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}(D_i\phi)^*(D_i\phi) + U(\phi\phi^*)\right) d^2x. \tag{6.12}$$

To minimise the total energy, the following conditions are required

$$\begin{cases} \phi^*\phi = F^2 \\ F_{ij}F_{ij} = 0 \\ D_i\phi = 0. \end{cases}$$

The first condition is satisfied if the field takes the form $\phi = Fe^{i\chi(x)}$. The second condition is equivalent to the magnetic field being zero as

$$F_{ij}F_{ij} = 4B^2 = 0.$$

This means that A_i has to be a pure gage i.e $A_i = \partial_i\alpha(x)$, where $\alpha(x)$ is an arbitrary function. The covariant derivative then vanishes if

$$iF(\partial_i\chi(x) - eA_i)e^{i\chi(x)} = iF(\partial_i\chi(x) - e\partial_i\alpha(x))e^{i\chi(x)} = 0.$$

Hence we have $\partial_i(\chi - e\alpha) = 0$, which gives $\alpha = \chi + \text{Const.}$, where we have absorbed the factor of e into the constant and the arbitrary functions. We can thus conclude that the vacuum solutions take the form

$$\phi_0(x) = Fe^{i\chi(x)} \quad \text{and} \quad A_i = \partial_i\chi(x).$$

Hence the solutions look very similar to those of the global vortices. It is important to note that the introduction of the gauge potential allows us to absorb the divergence of the global energy (6.8) into the gauge potential, by choosing it appropriately [19]. Thus local vortices do exist with finite energy, and we will now proceed to discuss how such topological defect could have formed within a physical context.

7 Cosmology and Phase Transitions

In this section we discuss how topological defects could have formed during the early universe. We will not have an in depth discussion, rather we lay down some of the fundamental notions that allow us to view the solitonic solutions from a cosmological viewpoint. As a motivating example we turn our attention to phase transitions in condensed matter physics. Consider water in its liquid phase. Such a system possesses a very large symmetry as all directions look the same. When this system goes through freezing, i.e a phase transition, the water molecules arrange in a crystal structure. In crystals there are preferred directions and so not every direction looks the same, meaning that the original large symmetry of the system has been broken down to a much smaller symmetry group.
Now turning our attention back to cosmology, one might ask the question whether such processes took place during the evolution of the universe, as similarly to the condensed matter example, the universe has been cooling ever since the big bang. It was first realised by Kirzhnits [9, 10], that phase transitions in the early universe could have had a similar effect on the symmetries of elementary particles. In this section is mainly based on ideas presented in [6] and [23].

7.1 Phase Transitions

In the hot big bang model, the universe started at a very high temperature and gradually cooled to today's temperature of around 2.7K. During the expansion, the universe went through multiple phase transitions where the symmetry of the elementary particles were broken down to a smaller symmetry group. This theory suggest that close to the big bang all fundamental interaction were unified into one large gauge symmetry group G, and during the evolution of the universe each interaction broke out of this symmetry group during phase transitions at different temperatures. Hence the evolution of the universe can be described by a sequence of symmetry breakings

$$G \to H \to \cdots \to SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1) \to U(1) \to \mathbb{I}$$

which we call a symmetry breaking pattern. Here each group corresponds to the symmetry of a fundamental interaction, i.e. $U(1)$ for EM.

7.2 Defect Formation

If such phase transitions occurred, then at each symmetry breaking, the fields had to pick a new vacuum state from the new degenerate vacuum states. In regions largely separated, this vacuum is chosen randomly and so the universe developed a domain structure as shown in Figure 7.

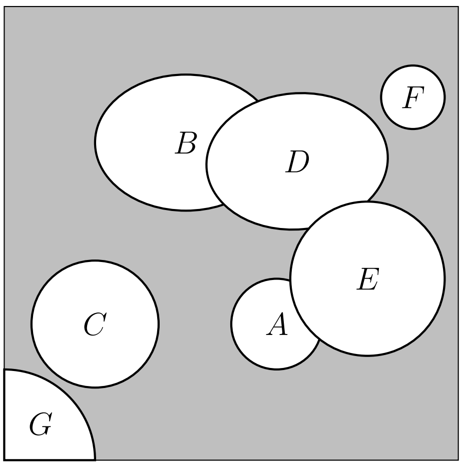

Figure 7: The formation of domains during symmetry breaking phase transitions in the early universe. Each letter corresponds to a different vacuum and hence the borders between these domain give rise to topological defects.

Hence in different regions of space, the vacuum state is different and this gives rise to solitonic structures on the borders between these domains. This is due to our previous discussion where we concluded that solitons interpolate between different vacua. In this scenario, one would expect to find multiple types of topological defects scattered around space, each originating from one of the phase transitions. As we know that these solitonic solutions are determined by the structure of the vacuum manifold, which in turn is defined by the symmetry breaking pattern, knowing these patterns allows us to determine the types of topological defects one would expect.
Conversely, the detection of such structures would provide evidence for these patterns and could even give us implications of what the Grand Unified group G might be. This is very significant as so far these theories could only be tested in particle accelerators. Despite the expectations these defects have not been observed to this day, but this is possibly due to many factors, such as their rarity.

8 Conclusion

We have seen that the formation of topological defects during the early universe is possible and that the form of these defects depend on the symmetry structure of the fundamental interactions. These solutions appear to be stable, static and possess finite energy. The observation of such topological defects would open up a new era in cosmology and would allow us to further understand not only large scale cosmology, but the fundamental interactions themselves. This highly motivates the study of such objects. Moreover their existence is unavoidable in many current cosmological models. We have seen that topological defects possess a localised energy density, which could have triggered large scale structure formation in the early universe. By modelling the formation of such topological defects one could compare the results to the current CMB spectrum and hence we would be able to see whether such processes are possible. This is not so simple because it would require developing a theory to describe the evolution and interaction of these defects in order to make a feasible prediction of their current day properties. As this is one of the first possibilities where one could compare the results from the theory of topological defects with actual observational data, this path of research is worth pursuing.

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