Internship report

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Black Holes and the double copy

Supervised by Henning Samtleben,
Laboratoire de Physique
École Normale Supérieure de Lyon

Maxime Médevielle
ENS Lyon
M2–Magistère 3 de Physique Fondamentale d'Orsay

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Introduction

The double copy theory is a duality between gravity and Yang-Mills theories. Before introducing this duality, let us first remind some basic facts about general relativity and gauge theories. Einstein's General Relativity was published in 1915 as a generalization of his 1905 theory of Special Relativity to include the force of gravity. In Special Relativity, space and time are no longer independent and absolute but they are rather seen as forming one entity, space-time, that depends on the observer.

General Relativity goes one step further by stating that space-time is actually dynamic, and is no longer seen as a static stage on which physical processes take place. In this framework, gravity is no longer seen as a force but is rather the manifestation of the curvature of space-time. Space-time is modeled as a 4-dimensional Lorentzian manifold, with the metric $g_{\mu\nu}$ encoding the distance between two nearby points on the manifold. All of this is embodied by the celebrated Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{1}$$

and by the equation of motion called the geodesic equation:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0. \tag{2}$$

As Wheeler said about these equations "matter tells space-time how to curve and curved space-time tells matter how to move". Einstein's equations are in general highly non-linear and it is very difficult to find exact solutions. However, as soon as 1916, Karl Schwarzschild derived the first non-trivial exact solutions of Einstein's equations. This solution, known as the Schwarzschild metric, was the first relativistic description of objects called black holes.

A Black Hole is a region of space-time where the gravitational acceleration is so strong that nothing, not even light, can escape from it after getting too close. This region of space where, if crossed, you are bound to be trapped in the black hole is called the event horizon. Black Holes are of great astrophysical and theoretical importance. For our work, we will exhibit a correspondence between black holes and solutions of gauge theories. The no-hair theorem states that, in simple terms, a classical black hole can be fully described by 3 parameters:

- The mass M
- The charge Q
- The angular momentum J

This means that all in all there are 4 different types of black holes:

- The Schwarzschild black hole ($Q = J = 0$)
- The Reissner-Nordström black hole ($J = 0$)
- The Kerr black hole ($Q = 0$)
- The Kerr-Newmann black hole

Lastly, we can mention that General Relativity is still after 100 years, our best theory of gravitation, which passed every experimental test from cosmology, to gravitational waves, the perihelion of Mercury, gravitational lensing, gravitational time dilation, ...

Let us now introduce gauge theories. A gauge theory is a field theory whose Lagrangian is invariant under a Lie group of local transformations. This Lie group has an associated Lie algebra and each of its generators gives rise to a gauge field. These gauge fields are present to ensure that the symmetry is indeed local. When the symmetry group does not commute, the gauge theory is said to be non-abelian, the most famous example being Yang-Mills theories.

Yang Mills theories are gauge theories with symmetry group $SU(N)$, they are of importance because they are the theories used to describe interactions of elementary particles. Indeed, the Standard Model of particle physics is a quantum field theory whose symmetry groupe is $SU(3) \times SU(2) \times U(1)$. This successfully describes the electromagnetic, weak and strong forces thanks to the electroweak interaction and quantum chromodynamics which are part of the Standard Model. The Lagrangian of pure Yang-Mills theory (with no matter content) is therefore:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{3}$$

where $F^{\mu\nu}$ is the field-strength tensor associated to the gauge field. The associated classical equations of motions called Yang-Mills equations are then simply:

$$(\mathcal{D}_\mu F^{\mu\nu})_a = \partial_\mu F_a^{\mu\nu} + g f_{abc} A_\mu^b F^{c\mu\nu} = 0. \tag{4}$$

We can see that we immediately recover Maxwell's equations when the group is abelian. Just as General Relativity, the Standard Model has passed numerous experimental tests, some to astonishingly high degrees of accuracy and makes it the main theory used in the realm of particle physics.

Having introduced our two theories, we are ready to delve into the double copy which relates these two theories. Relations between gravity and gauge theories have been getting increasing attention. The most straightforward way is to view General Relativity as the gauge theory of Poincaré symmetry [15]. A more recent and conceptually challenging duality was the holographic principle realised in the form of the AdS/CFT correspondance conjectured by Maldacena in 1997 [16]. In this work, we will study a third kind of duality, one in which gravity can be seen as the square of Yang-Mills theory. This is the idea of the double copy. Our goal in this work is to relate classical solutions of General Relativity (namely black holes) to two copies of classical solutions of Yang-Mills theory. However, the double copy was not initally discovered in this classical context, but rather in the context of scattering amplitudes.

In 1986 Kawai-Lewellen-Tye discovered the KLT relations in string theory [10], they state that the $n-$point, tree level scattering amplitude for closed strings is related to a sum over products of $n-$point, tree level open string partial amplitudes. In field theory, a similar result was found for particles relating scatterings of gravitons to the square of the scattering amplitudes of gluons. Furthermore this was conjectured to work both at tree and loop level. This direct squaring relation is known as the BCJ double copy [12],[13]. People then started to wonder where this duality comes from, therefore in order to study the duality from a non-perturbative point of view, one way was to look at the level of solutions to the classical field equations of motion.

Having this goal in mind, our work begins with the study of the double copy procedure for a Schwarzschild black hole in flat background. Then the formalism is generalized in order to study the double copy for Schwarzschild black hole and charged black hole in Anti-de Sitter background. Lastly we use tools from supersymmetry to get insights into the double copy procedure.

Supersymmetry is a feature of some field theories where integer-valued spin fields are related to half integer-valued spin fields. The usual space-time symmetry is promoted from the Poincaré algebra to the SuperPoincaré algebra which contains fermionic generators. A crucial feature is that, when this symmetry is made local, the theory naturally contains a graviton, making the theory a supergravity [17]. The supersymmetric partner of the graviton is called the gravitino with spin 3/2. Each supergravity is labeled by its number of supercharges $\mathcal{N}$. When $\mathcal{N} > 1$ the supersymmetry is said to be extended. Lastly, we mention that a supergravity is said to be "gauged" if the gravitino is charged with respect to the gauge fields.

Chapter 1

Double copy for flat background

1.1 The formalism

In order to get familiar with the classical double copy [1], we first study a simple case, namely the Schwarzschild black hole. To this end we first look at a special kind of metrics called Kerr-Schild metrics (we will see later that the Schwarzschild metric belongs to this class). A metric is said to be Kerr-Schild if we can write it in the following way:

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu$$

Here, $\eta_{\mu\nu}$ is the usual Minkoswki metric which we will call the background metric, ϕ is a scalar field and k_μ is a null vector which is geodetic with respect to both the Minkowski and full metric. This means that it satisfies the following equations:

$$g^{\mu\nu} k_\mu k_\nu = 0 = \eta^{\mu\nu} k_\mu k_\nu \qquad \text{and} \qquad k^\mu \nabla_\mu k^\nu = 0 = k^\mu \partial_\mu k^\nu$$

Moreover, the inverse metric takes the simple form:

$$g^{\mu\nu} = \eta^{\mu\nu} - \phi k^\mu k^\nu$$

where we raised the index of k_μ using the Minkowski metric. Having defined this metric and its inverse, we move on to compute the Ricci tensor for these kind of metrics. The actual computation is rather involved so we will only present the main steps. First we compute the Christoffel symbols:

$$\begin{aligned}
\Gamma^\rho_{\mu\nu} &= \frac{1}{2} g^{\rho\delta} \left(g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta} \right) \\
&= \frac{1}{2} (\eta^{\rho\delta} - \phi k^\rho k^\delta) \left((\phi k_\mu k_\delta)_{,\nu} + (\phi k_\nu k_\delta)_{,\mu} - (\phi k_\mu k_\nu)_{,\delta} \right) \\
&= \frac{1}{2} \left((\phi k_\mu k^\rho)_{,\nu} + (\phi k_\nu k^\rho)_{,\mu} - (\phi k_\mu k_\nu)_{,}{}^{\rho} + \phi k^\rho k^\delta (\phi k_\mu k_\nu)_{,\delta} \right)
\end{aligned}$$

remembering that k_μ is null so that $k_\delta k^\delta = 0 \Rightarrow k_\delta \partial_\nu k^\delta = 0$ and is geodetic.

We define the Ricci tensor to be :

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma^\rho_{\rho\nu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\rho\nu} \tag{1.1}$$

In this tensor we have terms going from first to fourth order in ϕ. However, with some algebra using the properties of k_μ, we see that the third and fourth order terms vanish. Some second order terms do not vanish, however using a mixed convention where $\partial^\mu = \eta^{\mu\nu}\partial_\nu$ and $R^\mu{}_\nu = g^{\mu\rho}R_{\rho\nu}$, we see that these terms disappear.

It is important to understand that the Ricci tensor is first order in ϕ only with one index up and one index down. Therefore we find:

$$R^\mu{}_\nu = \frac{1}{2}\left(\partial_\rho\partial_\nu(k^\mu k^\rho \phi) + \partial_\rho \partial^\mu (k^\rho k_\nu \phi) - \partial^2(k^\mu k_\nu \phi)\right) \tag{1.2}$$

$$R = \partial_\mu \partial_\nu (k^\mu k^\nu \phi) \tag{1.3}$$

We focus now on the stationary case, this means that all time derivatives vanish. We also set $k^0 = 1$ without loss of generality. We now have the following set of equations:

$$\begin{aligned}
R^0{}_0 &= \frac{1}{2}\nabla^2\phi \\
R^i{}_0 &= -\frac{1}{2}\partial_j(\partial^i(\phi k^j) - \partial^j(\phi k^i)) \\
R^i{}_j &= \frac{1}{2}\partial_l[\partial^i(\phi k^l k_j) + \partial_j(\phi k^l k^i) - \partial^l(\phi k^i k_j)] \\
R &= \partial_i \partial_j (\phi k^i k^j)
\end{aligned}$$

We are now ready to interpret this in the spirit of the double copy. To do this, we define an abelian gauge field as $A^\mu = \phi k^\mu$, which from now on we will call the single copy. From this we can show that in the stationary case, the vaccuum Einstein equations imply that the single copy satisfies Maxwell equations.

For $\nu = 0$:

$$\begin{aligned}
\frac{1}{2}\nabla^2\phi = 0 &\Rightarrow \partial_i \partial^i \phi = 0 \\
&\Rightarrow \partial_i \partial^i (\phi k^0) - \underbrace{\partial_\mu \partial^0 (\phi k^\mu)}_{=0} = 0 \\
&\Rightarrow \partial_\mu(\partial^\mu(A^0) - \partial^0(A^\mu)) = 0 \\
&\Rightarrow \partial_\mu F^{\mu 0} = 0
\end{aligned}$$

For $\nu = i$:

$$\begin{aligned} -\partial_j[\partial^i(\phi k^j) - \partial^j(\phi k^i)] = 0 &\Rightarrow \partial_j\partial^j A^i - \partial_j\partial^i A^j = 0 \\ &\Rightarrow \partial_\mu(\partial^\mu A^i - \partial^i A^\mu) = 0 \\ &\Rightarrow \partial_\mu F^{\mu i} = 0 \end{aligned}$$

We indeed find that $R_{\mu\nu} = 0 \Rightarrow \partial_\mu F^{\mu\nu} = 0$.

Similarly we can interpret ϕ as the so called zeroth copy whose equation of motion is the Laplace equation:

$$\nabla^2\phi = 0 \tag{1.4}$$

We can therefore summarize the different fields present in the double copy in the following way: the idea is to start with the zeroth copy (scalar field) ϕ and then adding one or two copies of the vector k_μ to get respectively the single copy (gauge field) $A^\mu = \phi k^\mu$ and the double copy (graviton field) $h_{\mu\nu} = \phi k_\mu k_\nu$

It's interesting to note that the general squaring from gauge theory to gravity would come with a dilaton and 2-form field. This is similar to what we've seen in bosonic string theory, where the first excited states of the closed string correspond to massless fields: the graviton, the dilaton and the Kalb-Ramond (2-form) field. However here, the fact that k_μ is null renders the graviton traceless so there is no dilaton. Moreover here the graviton is also symmetric so there isn't any 2-form field either. This is why our mapping links the single/zeroth copy theory to Einstein's gravity.

1.2 Schwarzschild Black Hole

We are now ready to apply this formalism for the simplest case, namely the Schwarzschild black hole in a flat background. The Einstein equations are in this case:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.5}$$

with an energy-momentum tensor $T^{\mu\nu} = M v^\mu v^\nu \delta^3(\boldsymbol{x})$ and $v^\mu = (1, 0, 0, 0)$. Let us show how the Schwarzschild metric can be put in Kerr-Schild form. The usual form of the metric is

$$ds^2 = -f(r)dt'^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \tag{1.6}$$

where $f(r) = 1 - \frac{r_s}{r}$ and $r_s = 2GM$ is the Schwarzschild radius. We then perform the following change of coordinates

$$t' = t - r_s \log\left(\frac{r}{r_s} - 1\right) \tag{1.7}$$

we obtain the following line element:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + 2\frac{r_s}{r} dr dt + \left(1 + \frac{r_s}{r}\right) dr^2 + r^2 d\Omega^2 \tag{1.8}$$

which is in Kerr-Schild form $g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu$ if we define:

$$\phi = \frac{2GM}{r} \quad \text{and} \quad k^\mu = (1, x^i/r) \quad \text{with} \quad r^2 = x^i x_i, i = 1, ..., 3 \tag{1.9}$$

As we have seen previously, we can define the single copy as:

$$A^\mu = \phi k^\mu = 8\pi G \frac{M}{4\pi r} k^\mu = \frac{g c_a T^a}{4\pi r} \left(1, \frac{\boldsymbol{x}}{r}\right) \tag{1.10}$$

Where we identified the coupling constants of the two theories $8\pi G \to g$ and the charges of the two theories $M \to c_a T^a$ We can now insert this gauge field in the Maxwell equations.

For $\nu = 0$:

$$\begin{aligned} \partial_\mu F^{\mu 0} &= \partial_\mu (\partial^\mu A^0 - \underbrace{\partial^0 A^\mu}_{=0}) \\ &= \partial^2 \left(\frac{g c_a T^a}{4\pi r} \right) = \frac{g c_a T^a}{4\pi} \partial_i \partial^i \left(\frac{1}{r} \right) = -g c_a T^a \delta^3(\boldsymbol{x}) \end{aligned}$$

For $\nu = i$:

$$\partial_\mu F^{\mu i} = \partial_\mu (\partial^\mu A^i - \partial^i A^\mu) = \partial_j \partial^j A^i - \partial_j \partial^i A^j = 0$$

Therefore we obtain the following Maxwell equation:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{1.11}$$

where

$$j^\nu = -g c_a T^a v^\nu \delta^3(\boldsymbol{x}) \quad \text{with} \quad v^\nu = (1, 0, 0, 0) \tag{1.12}$$

We can physically interpret this current as a static color charge located at the origin. It is interesting to note that in the gravity side, the Schwarzschild black hole, which is sourced by a static pointlike mass M, corresponds in the single copy side to a static color charge $c_a T^a$.

In order to physically interpret the gauge field, we perform the following gauge transformation:

$$A_{\mu a} \to A_{\mu a} + \partial_\mu \left[-\frac{g c_a}{4\pi} \log \left(\frac{r}{r_0} \right) \right] \tag{1.13}$$

It's easy to show that we obtain for the gauge field:

$$A_{\mu a} = \left(\frac{g c_a}{4\pi r}, \overrightarrow{0} \right) \tag{1.14}$$

We recognize this as the Coulomb solution for the superposition of static color charges located at the origin. It's interesting to point out that, because of Birkhoff's theorem, the Schwarzschild black hole is the most general static and spherically symmetric solution of Einstein's vacuum equations. The double copy makes it correspond to a Coulomb solution which is the most general static and spherically symmetric solution of Maxwell's equations.

Chapter 2

The double copy for curved backgrounds

2.1 The Ricci tensor

We have seen a simple example of the double copy duality in which the background was flat Minkoswki metric. We now want to generalize this formalism to arbitrary backgrounds, with the aim to study the double copy in the case of maximally symmetric spacetimes, in particular the Anti-de Sitter background [2]. We therefore define generalized Kerr-Schild metrics which are metrics that can be put in the following form:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \phi k_\mu k_\nu \tag{2.1}$$

where $\bar{g}_{\mu\nu}$ is an arbitrary background metric. As before, ϕ is a scalar field and k_μ is null and geodetic with respect to both the background and full metric. We now wish to compute the Ricci tensor. Once again the computation is very involved so we will only present the main steps [8].

We set $\phi = 2H$ and $\beta_{\mu\nu} = H k_\mu k_\nu$ Let us first compute the Christoffel symbols:

$$\begin{aligned}
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}(\bar{g}^{\rho\delta} - 2\beta^{\rho\delta})((\bar{g}_{\mu\delta,\nu} + \bar{g}_{\nu\delta,\mu} + \bar{g}_{\mu\nu,\delta}) + 2(\beta_{\mu\delta,\nu} + \beta_{\nu\delta,\mu} - \beta_{\mu\nu,\delta})) \\
&= \bar{\Gamma}^\rho_{\mu\nu} - 2\beta^\rho_\sigma \bar{\Gamma}^\sigma_{\mu\nu} + \bar{g}^{\rho\delta}(\nabla_\nu \beta_{\mu\delta} + \bar{\Gamma}^\gamma_{\mu\nu}\beta_{\gamma\delta} + \bar{\Gamma}^\gamma_{\delta\nu}\beta_{\gamma\mu} + \nabla_\mu \beta_{\nu\delta} \\
&+ \bar{\Gamma}^\gamma_{\mu\nu}\beta_{\gamma\delta} + \bar{\Gamma}^\gamma_{\mu\delta}\beta_{\gamma\nu} - \nabla_\delta \beta_{\mu\nu} - \bar{\Gamma}^\gamma_{\delta\nu}\beta_{\gamma\mu} - \bar{\Gamma}^\gamma_{\mu\delta}\beta_{\gamma\nu}) - 2\beta^{\rho\delta}(\beta_{\mu\delta,\nu} + \beta_{\nu\delta,\mu} - \beta_{\mu\nu,\delta}))
\end{aligned}$$

It's easy to show that the last term is simply:

$$-2\beta^{\rho\delta}(\beta_{\mu\delta,\nu} + \beta_{\nu\delta,\mu} - \beta_{\mu\nu,\delta}) = 2H_{,\sigma}k^\sigma k^\rho \beta_{\mu\nu}$$

Therefore we have:

$$\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu} + B^\rho_{\mu\nu} \tag{2.2}$$

with $B^{\rho}_{\mu\nu} = A^{\rho}_{\mu\nu} + 2H_{,\sigma}k^{\sigma}k^{\rho}\beta_{\mu\nu}$ and $A^{\rho}_{\mu\nu} = \nabla_{\nu}\beta^{\rho}_{\mu} + \nabla_{\mu}\beta^{\rho}_{\nu} - \nabla^{\rho}\beta_{\mu\nu}$.

We now turn to the Riemann tensor:

$$\begin{aligned}
R^{\rho}{}_{\sigma\mu\nu} &= \partial_{\mu}\Gamma^{\rho}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\mu\tau}\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\rho}_{\nu\tau}\Gamma^{\tau}_{\sigma\mu} \\
&= \partial_{\mu}(\bar{\Gamma}^{\rho}_{\sigma\nu} + B^{\rho}_{\sigma\nu}) - \partial_{\nu}(\bar{\Gamma}^{\rho}_{\sigma\mu} + B^{\rho}_{\sigma\mu}) + (\bar{\Gamma}^{\rho}_{\mu\tau} + B^{\rho}_{\mu\tau})(\bar{\Gamma}^{\tau}_{\sigma\nu} + B^{\tau}_{\sigma\nu}) \\
&\quad - (\bar{\Gamma}^{\rho}_{\nu\tau} + B^{\rho}_{\nu\tau})(\bar{\Gamma}^{\tau}_{\sigma\mu} + B^{\tau}_{\sigma\mu}) \\
&= \partial_{\mu}\bar{\Gamma}^{\rho}_{\sigma\nu} + \partial_{\mu}B^{\rho}_{\sigma\nu} - \partial_{\nu}\bar{\Gamma}^{\rho}_{\sigma\mu} - \partial_{\nu}B^{\rho}_{\sigma\mu} + \bar{\Gamma}^{\rho}_{\mu\tau}\bar{\Gamma}^{\tau}_{\sigma\nu} + \bar{\Gamma}^{\rho}_{\mu\tau}B^{\tau}_{\sigma\nu} + \bar{\Gamma}^{\tau}_{\sigma\nu}B^{\rho}_{\mu\tau} \\
&\quad + B^{\rho}_{\mu\tau}B^{\tau}_{\sigma\nu} - \bar{\Gamma}^{\rho}_{\nu\tau}\bar{\Gamma}^{\tau}_{\sigma\mu} - \bar{\Gamma}^{\rho}_{\tau\nu}B^{\tau}_{\mu\sigma} - \bar{\Gamma}^{\tau}_{\sigma\mu}B^{\rho}_{\nu\tau} - B^{\rho}_{\nu\tau}B^{\tau}_{\sigma\mu} \underbrace{+\bar{\Gamma}^{\gamma}_{\mu\nu}B^{\rho}_{\sigma\gamma} - \bar{\Gamma}^{\gamma}_{\mu\nu}B^{\rho}_{\sigma\gamma}}_{=0} \\
&= \bar{R}^{\rho}{}_{\sigma\mu\nu} + \bar{\nabla}_{\mu}B^{\rho}_{\sigma\nu} - \bar{\nabla}_{\nu}B^{\rho}_{\sigma\mu} + B^{\rho}_{\mu\tau}B^{\tau}_{\sigma\nu} - B^{\rho}_{\nu\tau}B^{\tau}_{\sigma\mu}
\end{aligned}$$

The next step is to raise the second index using the full metric. With some algebra we find:

$$R^{\rho\tau}{}_{\mu\nu} = g^{\tau\sigma}R^{\rho}{}_{\sigma\mu\nu} = \bar{R}^{\rho\tau}{}_{\mu\nu} - 2\beta^{\sigma\tau}\bar{R}^{\rho}{}_{\sigma\mu\nu} + g^{\sigma\tau}(\nabla_{\mu}A^{\rho}_{\sigma\nu} - \nabla_{\nu}A^{\rho}_{\sigma\mu}) + H^{\rho\tau}{}_{\mu\nu} \quad (2.3)$$

where $H^{\rho\tau}{}_{\mu\nu}$ is a term that vanishes when its first and third indices are contracted together. To get the Ricci tensor we now contract the first and third indices of the Riemann tensor, we also reintroduce our expression in terms of ϕ and k_{μ} and since $\nabla_{\nu}A^{\rho}_{\sigma\rho} = 0$, the general formula for the Ricci tensor is :

$$R^{\mu}{}_{\nu} = \bar{R}^{\mu}{}_{\nu} - \phi k^{\mu}k^{\lambda}\bar{R}_{\lambda\nu} + \frac{1}{2}\left[\bar{\nabla}^{\lambda}\bar{\nabla}^{\mu}(\phi k_{\lambda}k_{\nu}) + \bar{\nabla}^{\lambda}\bar{\nabla}_{\nu}(\phi k_{\lambda}k^{\mu}) - \bar{\nabla}^{2}(\phi k^{\mu}k_{\nu})\right] \quad (2.4)$$

2.2 The equations of motion

We have obtained the formula of the Ricci tensor for generalized Kerr-Schild metrics. We now want to obtain a form that is more practical in the case of maximally symmetric spacetimes. A spacetime is said to be maximally symmetric if it has $\frac{1}{2}d(d+1)$ linearly independent Killing vectors (d being the dimension of the spacetime). It can be proven that this is equivalent to the Riemann tensor obeying the following relation:

$$R_{abcd} = \frac{R}{d(d-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}) \quad (2.5)$$

We focus on this class of spacetimes because they include (Anti)-de Sitter spacetimes, which will be our background metric for the following examples. Let's first rewrite equation (2.4) in the following way:

$$2(\bar{R}^{\mu}{}_{\nu} - R^{\mu}{}_{\nu}) = 2\phi k^{\mu}k^{\lambda}\bar{R}_{\lambda\nu} - \left[\bar{\nabla}^{\lambda}\bar{\nabla}^{\mu}(\phi k_{\lambda}k_{\nu}) + \bar{\nabla}^{\lambda}\bar{\nabla}_{\nu}(\phi k_{\lambda}k^{\mu}) - \bar{\nabla}^{2}(\phi k^{\mu}k_{\nu})\right] \quad (2.6)$$

We will now prove that the right hand side can be recast in this form:

$$\left[\bar{\nabla}_\lambda F^{\lambda\mu} + \frac{(d-2)}{d(d-1)}\bar{R}A^\mu\right]k_\nu + X^\mu{}_\nu + Y^\mu{}_\nu \tag{2.7}$$

where:

$$X^\mu{}_\nu \equiv -\bar{\nabla}_\nu\left[k^\mu\bar{\nabla}_\lambda A^\lambda\right] \tag{2.8}$$

$$Y^\mu_\nu \equiv F^{\rho\mu}\bar{\nabla}_\rho k_\nu - \bar{\nabla}_\rho\left(A^\rho\bar{\nabla}^\mu k_\nu - A^\mu\bar{\nabla}^\rho k_\nu\right) \tag{2.9}$$

This means that:

$$\begin{aligned}
(2.7) &= \bar{\nabla}^2 A^\mu k_\nu - \bar{\nabla}_\lambda\bar{\nabla}^\mu A^\lambda k_\nu + \frac{(d-2)}{d(d-1)}\bar{R}A^\mu k_\nu - (\bar{\nabla}_\lambda A^\lambda)\bar{\nabla}_\nu k^\mu - k^\mu\bar{\nabla}_\nu\bar{\nabla}_\lambda A^\lambda \\
&+ \bar{\nabla}_\rho A^\mu\bar{\nabla}^\rho k_\nu - \bar{\nabla}^\mu A^\rho\bar{\nabla}_\rho k_\nu - \bar{\nabla}_\rho(A^\rho\bar{\nabla}^\mu k_\nu) + \bar{\nabla}_\rho(A^\mu\bar{\nabla}^\rho k_\nu) \\
&= \bar{\nabla}^2(A^\mu k_\nu) - \bar{\nabla}^\lambda\bar{\nabla}^\mu(A^\lambda k_\nu) + \frac{(d-2)}{d(d-1)}\bar{R}A^\mu k_\nu - \bar{\nabla}^\lambda A_\lambda\bar{\nabla}_\nu k^\mu - k^\mu\bar{\nabla}_\lambda\bar{\nabla}_\nu A^\lambda \\
&+ \bar{R}^\lambda{}_{\sigma\lambda\nu}A^\sigma k^\mu \underbrace{-\bar{\nabla}^\lambda k^\mu\bar{\nabla}_\nu A_\lambda + \bar{\nabla}^\lambda k^\mu\bar{\nabla}_\nu A_\lambda}_{=0}\underbrace{-A^\lambda\bar{\nabla}_\lambda\bar{\nabla}_\nu k^\mu + A^\lambda\bar{\nabla}_\lambda\bar{\nabla}_\nu k^\mu}_{=0} \\
&= \bar{\nabla}^2(A^\mu k_\nu) - \bar{\nabla}_\lambda\bar{\nabla}^\mu(A^\lambda k_\nu) - \bar{\nabla}^\lambda\bar{\nabla}_\nu(A^\lambda k^\mu) + \frac{(d-2)}{d(d-1)}\bar{R}A^\mu k_\nu \\
&+ \bar{R}_{\sigma\nu}A^\sigma k^\mu + \bar{\nabla}_\lambda k^\mu\bar{\nabla}_\nu A^\lambda + \underbrace{A^\lambda\bar{\nabla}_\lambda\bar{\nabla}_\nu k^\mu}_{A^\lambda\bar{\nabla}_\nu\bar{\nabla}_\lambda k^\mu + R^\mu{}_{\sigma\lambda\nu}A^\lambda k^\sigma} \\
&= \bar{\nabla}^2(A^\mu k_\nu) - \bar{\nabla}_\lambda\bar{\nabla}^\mu(A^\lambda k_\nu) - \bar{\nabla}^\lambda\bar{\nabla}_\nu(A^\lambda k^\mu) + \frac{(d-2)}{d(d-1)}\bar{R}A^\mu k_\nu \\
&+ \bar{R}_{\sigma\nu}A^\sigma k^\mu + \bar{R}^\mu{}_{\sigma\lambda\nu}A^\lambda k^\sigma + \underbrace{\bar{\nabla}_\nu(A_\lambda\bar{\nabla}^\lambda k^\mu)}_{=0 \text{ since } k^\mu \text{ is geodetic}}
\end{aligned}$$

from (2.5) we know that $\bar{R}_{ab} = \frac{1}{d}\bar{R}g_{ab}$. Focusing on the terms with a Ricci tensor or scalar we find that:

$$\begin{aligned}
(2.7) &= \frac{(d-2)}{d(d-1)}\bar{R}A^\mu k_\nu + \bar{R}_{\sigma\nu}A^\sigma k^\mu + \frac{R}{d(d-1)}(\delta^\mu_\lambda g_{\sigma\nu} - \delta^\mu_\nu g_{\sigma\lambda})A^\lambda k^\sigma \\
&= \frac{(d-2)}{d(d-1)}\bar{R}A^\mu k_\nu + \bar{R}_{\sigma\nu}A^\sigma k^\mu + \frac{1}{d(d-1)}\bar{R}A^\mu k_\nu - \frac{\bar{R}}{d(d-1)}\delta^\mu_\nu\underbrace{A_\lambda k^\lambda}_{=0} \\
&= \underbrace{\frac{\bar{R}A^\mu k_\nu}{d}}_{=\bar{R}_{\sigma\nu}A^\mu k^\sigma} + \bar{R}_{\sigma\nu}A^\sigma k^\mu = 2\phi k^\mu k^\lambda\bar{R}_{\lambda\nu}
\end{aligned}$$

Collecting all the terms together we find:

$$(2.7) = 2\phi k^\mu k^\lambda\bar{R}_{\lambda\nu} - \left[\bar{\nabla}^\lambda\bar{\nabla}^\mu(\phi k_\lambda k_\nu) + \bar{\nabla}^\lambda\bar{\nabla}_\nu(\phi k_\lambda k^\mu) - \bar{\nabla}^2(\phi k^\mu k_\nu)\right] \tag{2.10}$$

which is exactly the right hand side of equation (2.6) which is what we wanted. Therefore the equation that we will exploit is:

$$2(\bar{R}^{\mu}{}_{\nu} - R^{\mu}{}_{\nu}) = \left[\bar{\nabla}_{\lambda}F^{\lambda\mu} + \frac{(d-2)}{d(d-1)}\bar{R}A^{\mu}\right]k_{\nu} + X^{\mu}{}_{\nu} + Y^{\mu}_{\nu} \tag{2.11}$$

As a consistency check, we can see if this formula gives us the correct equation (1.2) in the case of flat Minkowski background. In this case, the background Ricci tensor vanishes, and we have:

$$X^{\mu}{}_{\nu} = -(\partial_{\nu}k^{\mu})(\partial_{\lambda}(\phi k^{\lambda})) - k^{\mu}\partial_{\nu}\partial_{\lambda}(\phi k^{\lambda}) \tag{2.12}$$

$$Y^{\mu}_{\nu} = (\partial^{\rho}(\phi k^{\mu}) - \partial^{\mu}(\phi k^{\rho}))\partial_{\rho}k_{\nu} - \partial_{\rho}(\phi k^{\rho}\partial^{\mu}k_{\nu} - \phi k^{\mu}\partial_{\rho}k_{\nu}) \tag{2.13}$$

so that:

$$\begin{aligned} -2R^{\mu}{}_{\nu} &= \partial^2(\phi k^{\mu})k_{\nu} - \partial_{\lambda}\partial^{\mu}(\phi k^{\lambda})k_{\nu} - (\partial_{\nu}k^{\mu})(\partial_{\lambda}(\phi k^{\lambda})) - k^{\mu}\partial_{\nu}\partial_{\lambda}(\phi k^{\lambda}) \\ &\quad + \partial^{\lambda}(\phi k^{\mu})\partial_{\lambda}k_{\nu} - \partial^{\mu}(\phi k^{\lambda})\partial_{\lambda}k_{\nu} - \partial_{\lambda}(\phi k^{\lambda})\partial^{\mu}k_{\nu} - \phi k^{\lambda}\partial_{\lambda}\partial^{\mu}k_{\nu} \\ &\quad + \partial_{\lambda}(\phi k^{\mu})\partial^{\lambda}k_{\nu} + \phi k^{\mu}\partial^2 k_{\nu} \\ &= \partial^2(\phi k^{\mu}k_{\nu}) - \partial^{\mu}\partial_{\rho}(\phi k^{\rho}k_{\nu}) - \partial_{\nu}\partial_{\rho}(\phi k^{\rho}k^{\mu}) + \partial_{\nu}(\phi k_{\lambda})\partial^{\lambda}k^{\mu} + \phi k_{\lambda}\partial_{\nu}\partial^{\lambda}k^{\mu} \end{aligned}$$

The last two terms vanish because $\partial_{\nu}(\underbrace{\phi k^{\lambda}\partial_{\lambda}k^{\mu}}_{=0}) = 0$ so we indeed recover:

$$R^{\mu}{}_{\nu} = \frac{1}{2}\left(\partial_{\rho}\partial_{\nu}(k^{\mu}k^{\rho}\phi) + \partial_{\rho}\partial^{\mu}(k^{\rho}k_{\nu}\phi) - \partial^2(k^{\mu}k_{\nu}\phi)\right) \tag{2.14}$$

We now wish to obtain the general equations of motion of the single and zeroth copies. To get the equation of the single copy, we first use the Einstein equations with cosmological constant to rewrite the left hand side of (2.11). We will actually use the trace-reversed form of the Einstein equations:

$$R_{\mu\nu} - \frac{\Lambda g_{\mu\nu}}{\frac{d}{2}-1} = 8\pi G\left(T_{\mu\nu} - \frac{1}{d-2}Tg_{\mu\nu}\right) \tag{2.15}$$

since the background metric is a vacuum solution, the energy-momentum tensor vanishes and we immediately have:

$$\bar{R}^{\mu}{}_{\nu} = \frac{\Lambda\delta^{\mu}_{\nu}}{\frac{D}{2}-1} \tag{2.16}$$

Therefore:

$$2(\bar{R}^{\mu}{}_{\nu} - R^{\mu}{}_{\nu}) = \tfrac{2\Lambda\delta^{\mu}_{\nu}}{\frac{D}{2}-1} - 16\pi G\left(T^{\mu}{}_{\nu} - \tfrac{1}{d-2}T\delta^{\mu}{}_{\nu}\right) - \tfrac{2\Lambda\delta^{\mu}_{\nu}}{\frac{D}{2}-1} = -16\pi G\left(T^{\mu}{}_{\nu} - \tfrac{1}{d-2}T\delta^{\mu}{}_{\nu}\right)$$

We now contract the equation we obtained with a Killing vector V^ν:

$$-16\pi G\left(T^\mu{}_\nu - \frac{1}{d-2}T\delta^\mu{}_\nu\right)V^\nu = \left[\bar{\nabla}_\lambda F^{\lambda\mu} + \frac{(d-2)}{d(d-1)}\bar{R}A^\mu\right]V^\nu k_\nu + V^\nu(X^\mu{}_\nu + Y^\mu{}_\nu)$$

$$\Rightarrow \bar{\nabla}_\lambda F^{\lambda\mu} + \frac{(d-2)}{d(d-1)}\bar{R}A^\mu + \frac{V^\nu}{V_\lambda k^\lambda}(X^\mu{}_\nu + Y^\mu{}_\nu) = 8\pi G J^\mu$$

where:

$$J^\mu = \frac{-2V^\nu}{V^\lambda k^\lambda}\left(T^\mu{}_\nu - \frac{1}{d-2}T\delta^\mu{}_\nu\right) \tag{2.17}$$

This is the equation of motion for the single copy. To get the equation for the zeroth copy, we contract the single copy equation again with the Killing vector V_μ:

$$8\pi G V_\mu J^\mu = V_\mu\bar{\nabla}_\lambda(\bar{\nabla}^\lambda(\phi k^\mu) - \bar{\nabla}^\mu(\phi k^\lambda)) + \frac{(d-2)}{d(d-1)}\phi k^\mu V_\mu \bar{R} + \frac{V_\mu}{V_\lambda k^\lambda}(V^\nu X^\mu{}_\nu + V^\nu Y^\mu{}_\nu)$$

$$\Rightarrow 8\pi G\frac{V_\mu J^\mu}{V_\lambda k^\lambda} = \frac{V_\mu}{V_\sigma k^\sigma}\bar{\nabla}_\lambda\left(\phi\bar{\nabla}^\lambda k^\mu + k^\mu\bar{\nabla}^\lambda\phi - \phi\bar{\nabla}^\mu k^\lambda - k^\lambda\bar{\nabla}^\mu\phi\right) + \frac{(d-2)}{d(d-1)}\bar{R}\phi + \frac{V_\mu}{(V_\lambda k^\lambda)^2}(V^\nu X^\mu{}_\nu + V^\nu Y^\mu{}_\nu)$$

$$\Rightarrow \bar{\nabla}^2\phi = 8\pi G j - \frac{(d-2)}{d(d-1)}\bar{R}\phi - \frac{V_\mu}{V_\lambda k^\lambda}(V^\nu X^\mu{}_\nu + V^\nu Y^\mu{}_\nu + Z^\mu) - \frac{V_\mu}{V_\sigma k^\sigma}\bar{\nabla}_\lambda k^\mu \bar{\nabla}^\lambda\phi$$

where:

$$j = \frac{V_\mu J^\mu}{V_\lambda k^\lambda} \quad \text{and} \quad Z^\mu = (V_\lambda k^\lambda)\bar{\nabla}_\sigma(\phi\bar{\nabla}^\sigma k^\mu - \phi\bar{\nabla}^\mu k^\sigma - k^\sigma\bar{\nabla}^\mu\phi) \tag{2.18}$$

Chapter 3

Anti-de Sitter Black Holes

3.1 The AdS-Schwarzschild Black Hole

In the first chapter, we studied the simplest example of the double copy duality in flat space, which was the Schwarzschild black hole. In the same line of thought, the simplest example we can study in a maximally symmetric background is the Anti-de Sitter Schwarzschild black hole in $d = 4$ [2]. The first step is to find a Kerr-Schild form for the metric.

To do this we write the AdS background metric in what is called global static coordinates:

$$\bar{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\left(1-\frac{\Lambda r^2}{3}\right)\mathrm{d}t^2 + \left(1-\frac{\Lambda r^2}{3}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{3.1}$$

Moreover we take:

$$k_\mu = \left(1, \frac{1}{1-\frac{\lambda r^2}{3}}, 0, 0\right), \qquad\qquad \phi = \frac{2GM}{r}, \tag{3.2}$$

where k_μ is null and geodetic. Therefore we can write the full metric in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2GM}{r}-\frac{\Lambda r^2}{3}\right) & \frac{2GM}{r}\left(1-\frac{\lambda r^2}{3}\right)^{-1} & 0 & 0 \\ \frac{2GM}{r}\left(1-\frac{\lambda r^2}{3}\right)^{-1} & \left(1-\frac{\lambda r^2}{3}\right)^{-1}\left(1+\frac{2GM}{r}\left(1-\frac{\lambda r^2}{3}\right)^{-1}\right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin\theta \end{pmatrix} \tag{3.3}$$

The metric is already in the correct form for the θ and ϕ coordinates. We focus on the t and r coordinates:

$$\mathrm{d}s^2 = -A\,\mathrm{d}t^2 + 2B\,\mathrm{d}t\,\mathrm{d}r + C\,\mathrm{d}r^2 \tag{3.4}$$

where:

$$A = \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right) \tag{3.5}$$

$$B = \frac{2GM}{r}\left(1 - \frac{\lambda r^2}{3}\right)^{-1} \tag{3.6}$$

$$C = \left(1 - \frac{\lambda r^2}{3}\right)^{-1}\left(1 + \frac{2GM}{r}\left(1 - \frac{\lambda r^2}{3}\right)^{-1}\right) \tag{3.7}$$

We make the coordinate transformation $t = t' + F(r)$ so that $\mathrm{d}t = \mathrm{d}t' + F'(r)\,\mathrm{d}r$. To get rid of the non-diagonal terms, we choose $F'(r) = \frac{B}{A}$:

$$\begin{aligned}
\mathrm{d}s^2 &= -A(\mathrm{d}t' + \frac{B}{A}\,\mathrm{d}r)^2 + 2B(\mathrm{d}t' + \frac{B}{A}\,\mathrm{d}r) + C\,\mathrm{d}r^2 \\
&= -A\,\mathrm{d}t'^2 + (C + \frac{B^2}{A})\,\mathrm{d}r^2
\end{aligned}$$

we can check that $C + \frac{B^2}{A} = A^{-1}$ and putting back all the terms together we find:

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)\mathrm{d}t'^2 + \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{3.8}$$

which is the usual form of the AdS-Schwarzchild metric. The full metric $g_{\mu\nu}$ satisfies the Einstein equations with cosmological constant:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.9}$$

with energy momentum tensor:

$$T^{\mu}{}_{\nu} = \frac{M}{2}\mathrm{diag}(0,0,1,1)\delta^3(\mathbf{r}) \tag{3.10}$$

We follow the usual double copy procedure taking $A_\mu = \phi k_\mu$. We obtain the equations of motion by using the general formula. In the case of the single copy we obtain a Maxwell equation in curved space-time:

$$\bar{\nabla}_\mu F^{\mu\nu} = gJ^\nu. \tag{3.11}$$

we can find the source using equation (2.17) with the timelike Killing vector of the Schwarzchild metric $V^\mu = (1,0,0,0)$:

$$J^\mu = \frac{-2V^\nu}{V_\rho k^\rho}\left(T^\mu{}_\nu - \frac{T}{d-2}\delta^\mu_\nu\right) = -2\left(T^\mu{}_0 - V^\mu\frac{T}{d-2}\right) = \begin{cases} \frac{2T}{d-2} & \text{if } \mu = 0 \\ 0 & \text{otherwise} \end{cases}$$

We know that $T = M\delta^3(\mathbf{r})$ so the source is:

$$J^\mu = Q\delta^3(\mathbf{r})\delta^\mu_0 \tag{3.12}$$

where we identified the charges between the two theories $Q = M$ and the coupling constants $8\pi G = g$. We recognize a source located at the origin, which is a point-particle with charge Q, very similar to what we found in the flat case.

In the same fashion, we find the zeroth copy equation of motion:

$$\left(\bar{\nabla}^2 - \frac{\bar{R}}{6}\right)\phi = gj \tag{3.13}$$

with source $j = Q\delta^3(\mathbf{r})$.

We can see that, as opposed to the flat case where we found $\bar{\nabla}^2\phi = 0$, in an AdS spacetime the zeroth copy satisfies a Klein-Gordon-like equation. This is the equation of motion of a conformally coupled scalar field, which means equations of motion derived from an action with the following generic form:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R - 2\lambda}{16\pi G} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}R\phi^2\right] \tag{3.14}$$

After obtaining these results we can discuss a subtelty concerning the choice of the scalar function ϕ and vector k^μ. Indeed, when the full metric is in Kerr-Schild form, we see that the metric is invariant with respect to the following transformation:

$$k^\mu \to fk^\mu \qquad \phi \to \phi/f^2 \tag{3.15}$$

with f an arbitrary function. The metric being invariant, so is the Ricci tensor so the gravitational theory is not affected by this change. However we can see that the definitions for the single and zeroth copy and their equations of motion are not invariant under this transformation.

For example in the case of the AdS-Schwarzschild black hole we chose $f = 1$. And indeed this is the correct choice in order to get a sensible physical equation of motion. Suppose that we instead chose:

$$k_\mu = f(\theta)\left(1, \frac{1}{1 - \frac{\lambda r^2}{3}}, 0, 0\right), \qquad \phi = \frac{1}{f(\theta)^2}\frac{2GM}{r} \tag{3.16}$$

With this definition k_μ is still null and geodetic and the equation of motion for the single copy is:

$$\bar{\nabla}_\mu F^{\mu\nu} = Q\delta^3(\mathbf{r})\delta^\mu_0 + \tilde{j}^\mu \tag{3.17}$$

where:

$$\tilde{j}^{\mu} = -\frac{g(\theta)}{r^2 f(\theta)^3}\left(\frac{\delta^{\mu}_t}{(1-\Lambda r^2/3)} + \delta^{\mu}_r\right), g(\theta) = 2f'(\theta)^2 - f(\theta)(f''(\theta + \cot(\theta)f'(\theta)) \tag{3.18}$$

This extra source term is non-localized and changes the total charge, this is not desirable which justifies our choice of $f = 1$ because it makes that term vanish.

3.2 Charged black holes

We now turn to a new kind of black hole solution, namely the AdS Reissner-Nordström metric. This type of metric describes a non-rotating charged black hole and can as usual be put in a Kerr-Schild form:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \phi^* k_\mu k_\nu \tag{3.19}$$

where k_μ is the same as in the AdS-Schwarzschild case and:

$$\phi^* = -\frac{Q^2}{4\pi r^2} + \phi = -\frac{Q^2}{4\pi r^2} + \frac{2GM}{r} \tag{3.20}$$

Let us now pause and ponder about what we have in the gravity theory side. We are in Einstein-Maxwell theory where the field content is $(g_{\mu\nu}, A^{EM}_\mu)$. Here the gauge field is defined just like in the Ads-Schwarzschild case $A^{EM}_\mu = \phi k_\mu$ and satisfies Einstein-Maxwell equations:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\left(B_\mu{}^\psi B_{\psi\nu} + \frac{1}{4}g_{\mu\nu}B_{\psi\tau}B^{\psi\tau}\right) \tag{3.21}$$

where $B_{\mu\nu}$ is the field-strength tensor associated with A^{EM}_μ. However it is now not so clear how we define the single copy. Do we identify it with the gauge field A^{EM}_μ of the gravity side, or do we keep the usual procedure and take $A_\mu = \phi^* k_\mu$?

One tool we can use to decide between the two is to turn to supersymmetry. This is because the AdS-Reissner-Nordström black hole solution can be embedded in a $\mathcal{N} = 2$ gauged Supergravity theory [5]. Let's explain the idea in general terms first: The AdS-Reissner-Nordström blackhole preserves a supersymmetry. Therefore, if we look at the different definitions for the single and zero copies, we might pick one that preserves that supersymmetry too.

Therefore the first step is to construct the supersymmetry preserved by the Ads-R-N black hole. $\mathcal{N} = 2$ gauged supergravity is a theory with a graviton, a complex gravitino ψ_m and a gauge field A_m. The Lagrangian is:

$$e^{-1}\mathcal{L} = -\frac{1}{4}R + \frac{1}{2}\bar{\psi}_m\gamma^{mnp}D_n\psi_p + \frac{1}{4}F_{mn}F^{mn} + \frac{i}{8}(F+\hat{F})^{mn}\bar{\psi}_p\gamma_{[m}\gamma^{pq}\gamma_{n]}\psi_q$$

$$- \frac{1}{2} g \bar{\psi}_m \gamma^{mn} \psi_n - \frac{3}{2} g^2$$

The cosmological constant is fixed by supersymmetry $\Lambda = -3g^2$. The covariant derivative acting on spinorial objects is:

$$D_m = \nabla_m - igA_m \tag{3.22}$$

where $\nabla_m = \partial_m + \frac{1}{4}\omega_m{}^{ab}\gamma_{ab}$. The supercovariant field strength is:

$$\hat{F}_{mn} = F_{mn} - \mathrm{Im}(\bar{\psi}_m \psi_n) \tag{3.23}$$

The $\mathcal{N} = 2$ supersymmetry that leaves the action invariant is:

$$\delta e_m{}^a = \mathrm{Re}(\bar{\epsilon}\gamma^a \psi_m) \tag{3.24}$$

$$\delta \psi_m = \hat{\nabla}_m \epsilon \tag{3.25}$$

$$\delta A_m = \mathrm{Im}(\bar{\epsilon}\psi_m) \tag{3.26}$$

where ϵ is an infinitesimal Dirac spinor and the supercovariant derivative is given by:

$$\hat{\nabla}_m \equiv D_m + \frac{1}{2} g \gamma_m + \frac{i}{4} \hat{F}_{ab} \gamma^{ab} \gamma_m \tag{3.27}$$

For an ansatz in which the gravitino vanishes, the Lagrangian leads to the Einstein-Maxwell equations of motion with a negative cosmological-constant. Therefore we see that as we claimed, the AdS-R-N black hole is a solution of the theory. Since the gravitino field vanishes equations (3.24) and (3.26) are trivially 0. However, for the supersymmetry to be preserved, we need to find a Dirac spinor that makes equation (3.25) vanish too.

We will not show the derivation in detail (see [5] for details) but the main point is that for our case at hand (vanishing magnetic charge), we can have a vanishing Killing spinor equation if $Q = M$(sufficient but not necessary). In that case the Killing spinor equations are:

$$\bar{\nabla}_t \epsilon = \left(\partial_t - \frac{i}{2} g \right) \epsilon \tag{3.28}$$

$$\bar{\nabla}_r \epsilon = \left(\partial_r - \frac{M}{2r(r-M)} + \frac{g(r-2M)}{2U(r-M)} \gamma_1 \right) \epsilon \tag{3.29}$$

$$\bar{\nabla}_\theta \epsilon = \left(\partial_\theta - \frac{i}{2} \gamma_{012} \right) \epsilon \tag{3.30}$$

$$\bar{\nabla}_\phi \epsilon = \left(\partial_\phi - \frac{1}{2} \cos\theta \gamma_{23} - \frac{i}{2} \sin\theta \gamma_{013} \right) \epsilon \tag{3.31}$$

where:

$$U(r)=\sqrt{\left(1-\frac{M}{r}\right)^2+g^2r^2} \tag{3.32}$$

These equations vanish if we take:

$$\epsilon(t,r,\theta,\phi)=\exp\left(\frac{i}{2}gt\right)\left(\cos\frac{1}{2}\theta+i\gamma_{012}\sin\frac{1}{2}\theta\right)\left(\cos\frac{1}{2}\phi+\gamma_{23}\sin\frac{1}{2}\phi\right)\epsilon(r) \tag{3.33}$$

where:

$$\epsilon(r)=\left(\sqrt{U(r)+gr}+i\gamma_0\sqrt{U(r)-gr}\right)\frac{1}{2}(1-\gamma_1)\epsilon_0 \tag{3.34}$$

with ϵ_0 an arbitrary spinor.

We have the Killing spinor for the AdS-R-N metric. Now we turn to the single/zero copies theory and see if the supersymmetry is preserved. Since the supersymmetry is now global, the $\mathcal{N}=2$ supersymmetry theory is different than the supergravity one. First of all the multiplet we choose is a gauge multiplet which also gives us the field content of the theory:

For global $\mathcal{N}=2$:

helicity	-1	-1/2	0	1/2	1
states	1	2	1+1	2	1

So the field content of the theory is (M,N,λ,A_μ) where M and N are two scalar fields, λ is a 1/2 spin gaugino, and A_μ is the gauge field. The Lagrangian of this theory is:

$$\begin{aligned}\mathcal{L}=\mathrm{Tr}\Big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}i\bar{\lambda}_i\gamma^\mu\nabla_\mu\lambda_i+\frac{1}{2}\nabla_\mu M\nabla^\mu M+\frac{1}{2}\nabla_\mu N\nabla^\mu N\\&-i\bar{\lambda}_2[\lambda_1,M]-i\bar{\lambda}_2\gamma^5[\lambda_1,N]+\frac{1}{2}[M,N]^2\Big]\end{aligned}$$

This Lagrangian is invariant under the following supersymmetry transformations:

$$\delta A_\mu=i\bar{\xi}_i\gamma_\mu\lambda_i \tag{3.35}$$

$$\delta M=\epsilon_{ij}\bar{\xi}_i\lambda_j \tag{3.36}$$

$$\delta N=\epsilon_{ij}\bar{\xi}_i\gamma_5\lambda_j \tag{3.37}$$

$$\delta\lambda_i=-\frac{1}{2}i\sigma^{\mu\nu}\xi_iF_{\mu\nu}+i\epsilon_{ij}\gamma^\mu\nabla_\mu(M+\gamma^5N)\xi_j-i\gamma^5\xi_i[M,N] \tag{3.38}$$

where ξ_i is the supersymmetry parameter.

Now, we obviously identify our single copy with the gauge field, we don't have fermions in our theory so we take vanishing gauginos (rendering equations

(3.35),(3.36) and (3.37) trivial). Whether we identify the zeroth copy with M or N is not obvious. What our computations showed is that the expressions are much nicer if we identify the zeroth copy with the scalar field M. However neither of our choices for the single/zeroth copies seemed to preserve supersymmetry in the case of an AdS space-time. However if we take the flat limit ($\Lambda \to 0$) then we have:

$$\delta\lambda_i = -\frac{1}{2} i\sigma^{\mu\nu} F_{\mu\nu}\epsilon_i + 2i\epsilon_{ij}\gamma^\mu\partial_\mu(\phi)\epsilon_j = 0 \tag{3.39}$$

This equation is true for both choices discussed for the single/zeroth copies. This does not help us choose between the two, but it shows that for the Reissner-Nordstrom black hole in a Minkoswki background, the single/zeroth copies do preserve the supersymmetry.

We can then look at the equations of motions for the two choices to see if one of the equation we obtain does not seem physical or reasonable. It is important however to see that the analysis in the first chapter for Minkoswki backgrounds does not apply here because the gravitational theory obeys Einstein-Maxwell equations and not vacuum Einstein equations.

For $\phi = \frac{2GM}{r}$ and $A_\mu = \phi k_\mu$ the equations of motion are:

$$\partial_\mu F^{\mu\nu} = \frac{4MG}{r^3}\delta_0^\nu \tag{3.40}$$

$$\partial^2\phi = \frac{4MG}{r^3} \tag{3.41}$$

and for $\phi^* = -\frac{Q^2}{4\pi r^2} + \frac{2MG}{r}$ and $A_\mu^{EM} = \phi^{EM} k_\mu$:

$$\partial_\mu B^{\mu\nu} = \left(-\frac{3Q^2}{2\pi r^4} + \frac{4MG}{r^3}\right)\delta_0^\nu \tag{3.42}$$

$$\partial^2\phi^* = -\frac{3Q^2}{2\pi r^4} + \frac{4MG}{r^3} \tag{3.43}$$

We see that in both cases, the fields satisfy the usual Maxwell and massless Klein-Gordon equations but with source terms that are not easy to interpret in terms of the objects of the single and zeroth copies theory. We were not able to discriminate between the two choices even after looking at supersymmetry and equations of motion. Further work is needed.

Conclusion

In this work, we have studied a gauge/gravity duality at the classical level. This double copy theory allowed us to relate different Black hole solutions (ones that can be put in Kerr-Schild form) to two copies of a gauge field living on the background metric. We've seen how the vacuum Einstein equations imply Maxwell's equations when the background is Minkoswki and we applied the formalism to the simplest black hole: The Schwarzschild black hole.

We then extended this formalism to arbitrary backgrounds in order to study black Holes on an AdS background and we rederived the general equations of motions of the associated single and zeroth copies. We then went on to study the double copy procedure for the AdS-Schwarzschild black hole and for the AdS-Reissner-Nordstrom black hole. For the latter we tried to use tools from supersymmetry to help us discriminate between two possible choices for the single/zeroth copies. The procedure was not conclusive, however, we have seen that both choices preserve the supersymmetry in the flat background limit.

There are several other aspects that make the double copy interesting. For example the double copy can be used to derive exact solutions in General Relativity from gauge theories solutions. Also, the link between the classical and perturbative double copy is still not well understood. Several results have been also derived in the non-stationary limit, relating wave solutions from the two theories. One thing that would be interesting to look at would be to find a double copy correspondence with a solution in the gravitational side that is not in Kerr-Schild form. Lastly, one interesting aspect would be to study the correspondence with the two theories manifesting explicit supersymmetry. Supersymmetric black holes in supergravity have been studied extensively and finding their corresponding single copy could be interesting, for example in a $(\mathcal{N} = 4\textbf{SYM}) \times (\mathcal{N} = 4\textbf{SYM}) \sim (\mathcal{N} = 8\textbf{SUGRA})$ setting. Indeed we can see a glimpse of this relation at the level of multiplets.

For $\mathcal{N} = 4\textbf{SYM}$:

helicity	-1	-1/2	0	1/2	1
states	1	4	6	4	1

For $\mathcal{N} = 8$**SUGRA**:

helicity	-2	-3/2	-1	-1/2	0	1/2	1	3/2	2
states	1	8	28	56	70	56	28	8	1

We see that squaring the $\mathcal{N} = 4$ multiplet gives the $\mathcal{N} = 8$ one.

states	1	4	6	4	1
1	1	4	6	4	1
4	4	16	24	16	4
6	6	24	36	24	6
4	4	16	24	16	4
1	1	4	6	4	1

Indeed, summing the states of constant spin, we recover the number of states for each helicity of the $\mathcal{N} = 8$ multiplet.

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