Internship report

18 June 2018 – 02 August 2018

Derivation of the Poincaré Superalgebra

Supervised by Benjamin Fuks,
LPTHE
Université Pierre et Marie Curie

Maxime Médevielle
Université Paris XI
M1–Magistère 2 de Physique Fondamentale d'Orsay

Remerciements

Je tiens à remercier très chaleureusement Benjamin Fuks d'avoir accepté de superviser ce stage. Je le remercie pour tout le temps qu'il m'a accordé, que je sais précieux, pour sa pédagogie, sa patience à mon égard et surtout pour sa grande sympathie.

Laboratoire

Le Laboratoire de Physique Théorique et Hautes Énergies (LPTHE) est une Unité Mixte de Recherche de Sorbonne Université et du CNRS. Il est rattaché à l'Institut de Physique (INP) du CNRS.
Il comprend 19 chercheurs CNRS et 10 enseignants chercheurs.

L'activité scientifique du LPTHE, est centrée sur le principe unificateur qu'est la théorie quantique des champs aussi bien dans ses aspects très théoriques que ses applications.
Les domaines suivants sont représentés au LPTHE :
-Physique Mathématique
-Cordes, Branes et Champs
-Matière Condensée et Physique Statistique
-Physique des Particules et Cosmologie

Abstract

Supersymmetry (SUSY) is a postulated symmetry of nature that relates elementary integer-valued spin particles (bosons) with half-integer spin particles (fermions). If SUSY is realised in nature, it could potentially resolve a number of problems faced in physics, for example the hierarchy problem; the gauge coupling unification; it might contain a dark matter candidate, etc...
This report focuses on how the spin-statistics and Noether's theorem lead to the Poincaré superalgebra, which is the supersymmetric extension of the Poincaré algebra (the algebra of the symmetry group of Special Relativity). We will also rederive its (anti)-commutation relations which completely specify this superalgebra.

Résumé

La supersymétrie (SUSY) est une symétrie postulée de la nature qui associe les particules élémentaires de spin demi-entier (fermions) avec les particules de spin entier (bosons). Si la supersymétrie est réalisée dans la nature, elle permettrait de potentiellement résoudre de nombreux problèmes en physique notamment le problème de la hierarchie, l'unification des constantes de couplage, elle pourrait contenir un candidat pour la matière noire, etc...
Dans ce rapport, nous allons redémontrer comment, à partir des théorèmes de Noether et spin-statistique, on obtient naturellement la superalgèbre de Poincaré, qui est l'extension supersymetrique de l'algèbre de Poincaré (l'algèbre du groupe de symétrie de la relativité restreinte). Nous allons aussi redémontrer les relations d'(anti)-commutations qui caractérisent complètement cette algèbre.

Contents

1 Lie Superalgebras

In this section we will show that, in a quantum field theory setting, spin-statistics and Noether's theorems naturally lead to a Lie superalgebra structure.
Let $\mathcal{L}(\Phi^a, \Psi^i)$ be the Lagrangian of the theory and Φ^a and Ψ^i be two sets of bosonic and fermionic fields. Then, we consider a symmetry generated by B_A^1 and B_A^2 leaving the lagrangian invariant, such that:

$$\delta_A \Phi^a = (B_A^1)^a{}_b \Phi^b \text{ and } \delta_A \Psi^i = (B_A^2)^i{}_j \Psi^j$$

We introduce the conjugate momenta associated to the fields:

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi^a)} \quad \text{and} \quad \rho_i = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi^i)}$$

1.1 Lie algebra for the bosonic charges

Let's find an expression for the conserved charge B_A. We start by varying the lagrangian:

$$\delta_A \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \Phi^a} \delta_A \Phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \delta_A(\partial_\mu \Phi^a) + \frac{\partial \mathcal{L}}{\partial \Psi^i} \delta_A \Psi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)} \delta_A(\partial_\mu \Psi^i) \qquad (1)$$

We then use the following Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \Psi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)} = 0 \qquad (3)$$

δ_A and ∂_μ commute, such that:

$$\begin{aligned}\delta_A \mathcal{L} &= \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)}\right) \delta_A \Phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \partial_\mu \delta_A \Phi^a + \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)}\right) \delta_A \Psi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)} \partial_\mu \delta_A \Psi^i \\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \delta_A \Phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)} \delta_A \Psi^i\right)\end{aligned}$$

Since this transformation is a symmetry, the lagrangian must remain invariant: $\delta_A \mathcal{L} = 0$. We therefore find a conserved current that satisfies $\partial_\mu J^\mu = 0$:

$$\begin{aligned}J^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \delta_A \Phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)} \delta_A \Psi^i \\ &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} (B_A^1)^a{}_b \Phi^b + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)} (B_A^2)^i{}_j \Psi^j\end{aligned}$$

We can therefore express the conserved charge as $B_A = -i\int \mathrm{d}^3x J^0$ since:

$$\frac{\partial}{\partial t}B_A = -i\frac{\partial}{\partial t}\int \mathrm{d}^3x J^0 = -i\int \mathrm{d}^3x \partial_0 J^0 - i\int \mathrm{d}^3x \partial_i J^i = -i\int \mathrm{d}^3x \partial_\mu J^\mu = 0$$

The $-i$ factor is just a matter of normalisation convention. Finally:

$$\begin{aligned} B_A &= -i\int \mathrm{d}^3x \left(\frac{\partial\mathcal{L}}{\partial(\partial_0\Phi^a)}(B_A^1)^a{}_b\Phi^b + \frac{\partial\mathcal{L}}{\partial(\partial_0\Psi^i)}(B_A^2)^i{}_j\Psi^j\right) \\ &= -i\int \mathrm{d}^3x \left(\Pi_a(B_A^1)^a{}_b\Phi^b + \rho_i(B_A^2)^i{}_j\Psi^j\right) \end{aligned}$$

We will now determine the action of the conserved charge on the fields and momenta:

$$\begin{aligned} [\Phi^a, B_A] &= \left[\Phi^a, -i\int \mathrm{d}^3x \left(\Pi_b(B_A^1)^b{}_c\Phi^c + \rho_i(B_A^2)^i{}_j\Psi^j\right)\right] \\ &= -i\left[\Phi^a, \int \mathrm{d}^3x \Pi_b(B_A^1)^b{}_c\Phi^c\right] \\ &= -i\int \mathrm{d}^3x \left[\Phi^a, \Pi_b\right](B_A^1)^b{}_c\Phi^c \\ &= -i\int \mathrm{d}^3x\, i\delta^{(3)}(x-y)\delta^a_b(B_A^1)^b{}_c\Phi^c \\ &= (B_A^1)^a{}_b\Phi^b \end{aligned}$$

where we used the equal time commutation relations from QFT.

$$\begin{aligned} [\Pi_a, B_A] &= \left[\Pi_a, -i\int \mathrm{d}^3x \left(\Pi_b(B_A^1)^b{}_c\Phi^c + \rho_i(B_A^2)^i{}_j\Psi^j\right)\right] \\ &= -i\int \mathrm{d}^3x \Pi_b(B_A^1)^b{}_c\left[\Pi_a, \Phi^c\right] \\ &= -\Pi_b(B_A^1)^b{}_a \end{aligned}$$

$$\begin{aligned} [\Psi^i, B_A] &= \left[\Psi^i, -i\int \mathrm{d}^3x \left(\Pi_b(B_A^1)^b{}_c\Phi^c + \rho_j(B_A^2)^j{}_k\Psi^k\right)\right] \\ &= -i\int \mathrm{d}^3x \left[\Psi^i, \rho_j(B_A^2)^j{}_k\Psi^k\right] \\ &= -i\int \mathrm{d}^3x \left([\Psi^i,\rho_j](B_A^2)^j{}_k\Psi^k + \rho_k[\Psi^i,(B_A^2)^j{}_k\Psi^k]\right) \\ &= -i\int \mathrm{d}^3x \left(\{\Psi^i,\rho_j\} - 2\rho_j\Psi^i\right)(B_A^2)^j{}_k\Psi^k + \rho_j\left(\{\Psi^i,(B_A^2)^j{}_k\Psi^k\} - 2(B_A^2)^j{}_k\Psi^k\Psi^i\right) \end{aligned}$$

$$= -i \int \mathrm{d}^3x \left(i\delta^{(3)}(x-y)\delta^i_j {(B^2_A)^j}_k \Psi^k - 2\rho_j \Psi^i {(B^2_A)^j}_k \Psi^k + 0 - 2\rho_j {(B^2_A)^j}_k \Psi^k \Psi^i \right)$$

We notice that in the last term, anticommuting the two fermionic fields will make a $-$ appear which will cancel out with the second term.

$$= {(B^2_A)^i}_k \Psi^k$$

$$\begin{aligned}
[\rho_i, B_A] &= \left[\rho_i, -i \int \mathrm{d}^3x \left(\Pi_b {(B^1_A)^b}_c \Phi^c + \rho_j {(B^2_A)^j}_k \Psi^k \right) \right] \\
&= -i \int \mathrm{d}^3x [\rho_i, \rho_j {(B^2_A)^j}_k \Psi^k] \\
&= -i \int \mathrm{d}^3x \left([\rho_i, \rho_j] {(B^2_A)^j}_k \Psi^k + \rho_j [\rho_i, {(B^2_A)^j}_k \Psi^k] \right) \\
&= -i \int \mathrm{d}^3x \left((\{\rho_i, \rho_j\} - 2\rho_j\rho_i) {(B^2_A)^j}_k \Psi^k + \rho_j (\{\rho_i, {(B^2_A)^j}_k \Psi^k\} - 2{(B^2_A)^j}_k \Psi^k \rho_i) \right) \\
&= -i \int \mathrm{d}^3x \left(0 - 2\rho_j\rho_i {(B^2_A)^j}_k \Psi^k + i\rho_j \delta^{(3)}(x-y)\delta^k_i {(B^2_A)^j}_k - 2\rho_j {(B^2_A)^j}_k \Psi^k \rho_i \right) \\
&= \rho_j {(B^2_A)^j}_i - i \int \mathrm{d}^3x \left(2\rho_j\rho_i {(B^2_A)^j}_k \Psi^k - 2\rho_j {(B^2_A)^j}_k (i\delta^{(3)}(x-y)\delta^k_i - \rho_i \Psi^k) \right) \\
&= \rho_j {(B^2_A)^j}_i - i \int \mathrm{d}^3x (-2\rho_j {(B^2_A)^j}_k i\delta^{(3)}(x-y)\delta^k_i) \\
&= \rho_j {(B^2_A)^j}_i - 2\rho_j {(B^2_A)^j}_i = -\rho_j {(B^2_A)^j}_i
\end{aligned}$$

Knowing these relations, we now show that the bosonic charges give rise to a Lie algebra. We therefore need to verify the Jacobi identity $[B_A, [B_C, B_D]] + [B_C, [B_D, B_A]] + [B_D, [B_A, B_C]] = 0$ and that the algebra is closed $[B_A, B_C] = {f_{AC}}^D B_D$. Let's first check the closure:

$$\begin{aligned}
[B_A, B_C] &= \left[-i \int \mathrm{d}^3x \left(\Pi_a {(B^1_A)^a}_b \Phi^b + \rho_i {(B^2_A)^i}_j \Psi^j \right), B_C \right] \\
&= -i \int \mathrm{d}^3x \left([\Pi_a {(B^1_A)^a}_b \Phi^b, B_C] + [\rho_i {(B^2_A)^i}_j \Psi^j, B_C] \right)
\end{aligned}$$

We compute the first commutator:

$$\begin{aligned}
[\Pi_a {(B^1_A)^a}_b \Phi^b, B_C] &= \Pi_a [{(B^1_A)^a}_b \Phi^b, B_C] + [\Pi_a, B_C] {(B^1_A)^a}_b \Phi^b \\
&= \Pi_a {(B^1_A)^a}_b {(B^1_C)^b}_d \Phi^d - \Pi_d {(B^1_C)^d}_a {(B^1_A)^a}_b \Phi^b \\
&= \Pi_a {(B^1_A)^a}_b {(B^1_C)^b}_d \Phi^d - \Pi_a {(B^1_C)^a}_b {(B^1_A)^b}_d \Phi^d \\
&= \Pi_a {[B^1_A, B^1_C]^a}_d \Phi^d
\end{aligned}$$

and the second commutator:

$$\begin{aligned}[\rho_i(B^2_A)^i{}_j\Psi^j, B_C] &= \rho_i[(B^2_A)^i{}_j\Psi^j, B_C] + [\rho_i, B_C](B^2_A)^i{}_j\Psi^j \\ &= \rho_i(B^2_A)^i{}_j(B^2_C)^j{}_k\Psi^k - \rho_k(B^2_C)^k{}_i(B^2_A)^i{}_j\Psi^j \\ &= \rho_i(B^2_A)^i{}_j(B^2_C)^j{}_k\Psi^k - \rho_i(B^2_C)^i{}_j(B^2_A)^j{}_k\Psi^k \\ &= \rho_i[B^2_A, B^2_C]^i{}_k\Psi^k\end{aligned}$$

We find:

$$[B_A, B_C] = -i\int \mathrm{d}^3x\left(\Pi_a[B^1_A, B^1_C]^a{}_d\Phi^d + \rho_i[B^2_A, B^2_C]^i{}_k\Psi^k\right) \tag{4}$$

According to this relation, for the algebra to close we need to impose:

$$[B^1_A, B^1_C] = f_{AC}{}^D B^1_D \text{ et } [B^2_A, B^2_C] = f_{AC}{}^D B^2_D$$

And we find as wanted:

$$\begin{aligned}[B_A, B_C] &= f_{AC}{}^D\left(-i\int \mathrm{d}^3x(\Pi_a(B^1_D)^a{}_d\Phi^d + \rho_i(B^2_D)^i{}_k\Psi^k)\right) \\ &= f_{AC}{}^D B_D\end{aligned}$$

For the Jacobi identity, we notice that by bilinearity of the commutator, the Jacobi identity will apply to B_1 and B_2 which are matrices. Therefore, by associativity of the matrix product, Jacobi is trivially verified:

$$[B_A, [B_C, B_D]] + [B_C, [B_D, B_A]] + [B_D, [B_A, B_C]] = 0$$

Conclusion: The bosonic charges B_A give rise to a Lie algebra.

1.2 A new symmetry

We now consider a new symmetry generated by F^1_I and F^2_I that transforms bosonic fields into fermionic fields and vice-versa:

$$\delta_I\Phi^a = (F^1_I)^a{}_i\Psi^i \text{ and } \delta_I\Psi^i = (F^2_I)^i{}_a\Phi^a$$

A computation similar to the one performed in the first part shows that the conserved charge has the form:

$$F_I = -i\int \mathrm{d}^3x\left(\Pi_a(F^1_I)^a{}_i\Psi^i + \rho_i(F^2_I)^i{}_a\Phi^a\right) \tag{5}$$

We determine like before the action of the conserved charge upon the fields and momenta:

$$\begin{aligned}
[\Phi^a, F_I] &= [\Phi^a, -i\int \mathrm{d}^3x \left(\Pi_b (F_I^1)^b{}_i \Psi^i + \rho_i (F_I^2)^i{}_b \Phi^b\right)] \\
&= -i\int \mathrm{d}^3x [\Phi^a, \Pi_b](F_I^1)^b{}_i \Psi^i \\
&= (F_I^1)^a{}_i \Psi^i
\end{aligned}$$

$$\begin{aligned}
[\Pi_a, F_I] &= [\Pi_a, -i\int \mathrm{d}^3x \left(\Pi_b (F_I^1)^b{}_i \Psi^i + \rho_i (F_I^2)^i{}_b \Phi^b\right)] \\
&= -i\int \mathrm{d}^3x \rho_i (F_I^2)^i{}_b [\Pi_a, \Phi^b] \\
&= -\rho_i (F_I^2)^i{}_a
\end{aligned}$$

$$\begin{aligned}
\{\Psi^i, F_I\} &= \{\Psi^i, -i\int \mathrm{d}^3x \left(\Pi_a (F_I^1)^a{}_j \Psi^j + \rho_i (F_I^2)^i{}_a \Phi^a\right)\} \\
&= -i\int \mathrm{d}^3x \left(\{\Psi^i, \Pi_a (F_I^1)^a{}_j \Psi^j\} + \{\Psi^i, \rho_j (F_I^2)^j{}_a \Phi^a\}\right)
\end{aligned}$$

First anticommutator:

$$\begin{aligned}
\{\Psi^i, \Pi_a (F_I^1)^a{}_j \Psi^j\} &= \Pi_a (F_I^1)^a{}_j \underbrace{\{\Psi^i, \Psi^j\}}_{=0} + \underbrace{[\Psi^i, \Pi_a (F_I^1)^a{}_j]}_{=0} \Psi^j \\
&= 0
\end{aligned}$$

Second anticommutator:

$$\begin{aligned}
\{\Psi^i, \rho_j (F_I^2)^j{}_a \Phi^a\} &= \rho_j (F_I^2)^j{}_a \{\Psi^i, \Phi^a\} + [\Psi^i, \rho_j (F_I^2)^j{}_a]\Phi^a \\
&= 2\rho_j (F_I^2)^j{}_a \Phi^a \Psi^i + (\{\Psi^i, \rho_j\} - 2\rho_j \Psi^i)(F_I^2)^j{}_a \Phi^a \\
&= 2\rho_j (F_I^2)^j{}_a \Phi^a \Psi^i + \{\Psi^i, \rho_j\}(F_I^2)^j{}_a \Phi^a - 2\rho_j \Psi^i (F_I^2)^j{}_a \Phi^a \\
&= \{\Psi^i, \rho_j\}(F_I^2)^j{}_a \Phi^a
\end{aligned}$$

Finally:

$$\begin{aligned}
\{\Psi^i, F_I\} &= -i\int \mathrm{d}^3x \{\Psi^i, \rho_j\}(F_I^2)^j{}_a \Phi^a \\
&= (F_I^2)^i{}_a \Phi^a
\end{aligned}$$

Similarly:

$$\{\rho_i, F_I\} = -i \int \mathrm{d}^3x \left(\{\rho_i, \Pi_a (F_I^1)^a{}_j \Psi^j\} + \{\rho_i, \rho_j (F_I^2)^j{}_a \Phi^a\} \right)$$

First anticommutator:

$$\{\rho_i, \Pi_a (F_I^1)^a{}_j \Psi^j\} = \Pi_a (F_I^1)^a{}_j \{\rho_i, \Psi^j\} + \underbrace{[\rho_i, \Pi_a]}_{=0} (F_I^1)^a{}_j \Psi^j$$

Second anticommutator:

$$\begin{aligned}
\{\rho_i, \rho_j (F_I^2)^j{}_a \Phi^a\} &= \rho_j (F_I^2)^j{}_a \{\rho_i, \Phi^a\} + [\rho_i, \rho_j] (F_I^2)^j{}_a \Phi^a \\
&= 2\rho_j (F_I^2)^j{}_a \rho_i \Phi^a + (\{\rho_i, \rho_j\} - 2\rho_j \rho_i)(F_I^2)^j{}_a \Phi^a \\
&= 2\rho_j (F_I^2)^j{}_a \rho_i \Phi^a - 2\rho_j \rho_i (F_I^2)^j{}_a \Phi^a \\
&= 0
\end{aligned}$$

Finally:

$$\{\rho_i, F_I\} = \Pi_a (F_I^1)^a{}_i$$

We will now show that the composition of two fermionic operators F_I and F_J is a bosonic operator:

$$\{F_I, F_J\} = -i \int \mathrm{d}^3x \left(\{\Pi_a (F_I^1)^a{}_i \Psi^i, F_J\} + \{\rho_i (F_I^2)^i{}_a \Phi^a, F_J\} \right)$$

First anticommutator:

$$\begin{aligned}
\{\Pi_a (F_I^1)^a{}_i \Psi^i, F_J\} &= \Pi_a (F_I^1)^a{}_i \{\Psi^i, F_J\} - [\Pi_a (F_I^1)^a{}_i, F_J] \Psi^i \\
&= \Pi_a (F_I^1)^a{}_i (F_J^2)^i{}_b \Phi^b + \rho_j (F_J^2)^j{}_a (F_I^1)^a{}_i \Psi^i
\end{aligned}$$

Second anticommutator:

$$\begin{aligned}
\{\rho_i (F_I^2)^i{}_a \Phi^a, F_J\} &= \rho_i (F_I^2)^i{}_a [\Phi^a, F_J] + \{\rho_i (F_I^2)^i{}_a, F_J\} \Phi^a \\
&= \rho_i (F_I^2)^i{}_a (F_I^1)^a{}_j \Psi^j + \Pi_b (F_J^1)^b{}_i (F_I^2)^i{}_a \Phi^a
\end{aligned}$$

We get:

$$\{F_I, F_J\} = -i \int \mathrm{d}^3x \left(\Pi_a (F_I^1 F_J^2 - F_J^1 F_I^2)^a{}_b \Phi^b + \rho_i (F_I^2 F_J^1 + F_J^2 F_I^1)^i{}_j \Psi^j \right)$$

Once again, we impose the algebra closure:

$$F_I^1F_J^2 + F_J^1F_I^2 = g_{IJ}{}^A B_A^1 \text{ et } F_I^2F_J^1 + F_J^2F_I^1 = g_{IJ}{}^A B_A^2$$

We finally get:

$$\begin{aligned}\{F_I, F_J\} &= -ig_{IJ}{}^A \int \mathrm{d}^3x(\Pi_a(B_A^1)^a{}_b\Phi^b + \rho_i(B_A^2)^i{}_j\Psi^j)\\ &= g_{IJ}{}^A B_A\end{aligned}$$

1.3 Conserved charges as a Lie superalgebra

To show that the conserved charges form a Lie superalgebra we need to check the Lie super-identities and $[B_A, B_C] = f_{AC}{}^D B_D$, $\{F_I, F_J\} = g_{IJ}{}^A B_A$ and $[B_A, F_I] = h_{AI}{}^J F_J$.
The first two equalities have been checked previously, let's prove the third one:

$$[B_A, F_I] = -i \int \mathrm{d}^3x \left([\Pi_a\delta_A\Phi^a, F_I] + [\rho_i\delta_A\Psi^i, F_I]\right)$$

First anticommutator:

$$\begin{aligned}[\Pi_a\delta_A\Phi^a, F_I] &= \Pi_a[\delta_A\Phi^a, F_I] + [\Pi_a, F_I]\delta_A\Phi^a\\ &= \Pi_A\delta_I\delta_A\Phi^a + \delta_I\Pi_a\delta_A\Phi^a\end{aligned}$$

Second anticommutator:

$$\begin{aligned}[\rho_i\delta_A\Psi^i, F_I] &= \rho_i[\delta_A\Psi^i, F_I] + [\rho_i, F_I]\delta_A\Psi^i\\ &= \rho_i(\{\delta_A\Psi^i, F_I\} - 2F_I\delta_A\Psi^i) + (\{\rho_i, F_I\} - 2F_I\rho_i)\delta_A\Psi^i\\ &= \rho_i\delta_I\delta_A\Psi^i - 2\rho_iF_I\delta_A\Psi^i + \delta_I\rho_i\delta_A\Psi^i - 2F_I\rho_i\delta_A\Psi^i\\ &= \rho_i\delta_I\delta_A\Psi^i - 2\delta_I\rho_i\delta_A\Psi^i + 2F_I\rho_i\delta_A\Psi^i + \delta_I\rho_i\delta_A\Psi^i - 2F_I\rho_i\delta_A\Psi^i\\ &= \rho_i\delta_I\delta_A\Psi^i - \delta_I\rho_i\delta_A\Psi^i\end{aligned}$$

The sum of the two commutators gives:

$$\begin{aligned}[\Pi_a\delta_A\Phi^a, F_I] + [\rho_i\delta_A\Psi^i, F_I] &= \Pi_a\delta_I\delta_A\Phi^a + \delta_I\Pi_a\delta_A\Phi^a + \rho_i\delta_I\delta_A\Psi^i - \delta_I\rho_i\delta_A\Psi^i\\ &= \Pi_a(B_A^1)^a{}_b(F_I^1)^b{}_i\Psi^i - \rho_i(F_I^2)^i{}_a(B_A^1)^a{}_b\Phi^b + \rho_i(B_A^2)^i{}_j(F_I^2)^j{}_b\Phi^b\\ -\Pi_a(F_I^1)^a{}_i(B_A^2)^i{}_j\Psi^j\\ &= \Pi_a\left(B_A^1F_I^1 - F_I^1B_A^2\right)^a{}_i\Psi^i + \rho_i\left(B_A^2F_I^2 - F_I^2B_A^1\right)^i{}_a\Phi^a\end{aligned}$$

As usual, we want the algebra to close:

$$B^1_A F^1_I - F^1_I B^2_A = h_{AI}{}^J F^1_J \text{ and } B^2_A F^2_I - F^2_I B^1_A = h_{AI}{}^J F^2_J$$

Finally:

$$\begin{aligned}[B_A, F_I] &= h_{AI}{}^J \left(-i \int \mathrm{d}^3x (\Pi_a (F^1_J)^a{}_i \Psi^i + \rho_i (F^2_J)^i{}_a \Phi^a) \right) \\ &= h_{AI}{}^J F_J\end{aligned}$$

The Superalgebra is indeed closed and like before the associativity of the matrix product ensures that the Jacobi super-identities are verified:

$$\begin{aligned}[B_A, [B_C, B_D]] + [B_C, [B_D, B_A]] + [B_D, [B_A, B_C]] = 0 \\ [B_A, [B_C, F_I]] + [B_C, [F_I, B_A]] + [F_I, [B_A, B_C]] = 0 \\ [B_A, \{F_I, F_J\}] + \{F_I, [F_J, B_A]\} - \{F_J, [B_A, F_I]\} = 0 \\ [F_I, \{F_J, F_K\}] + [F_J, \{F_K, F_I\}] + [F_K, \{F_I, F_J\}] = 0\end{aligned}$$

Conclusion: The conserved charges form a Lie superalgebra.

1.4 Compatibility

In this section we show that the transformation laws considered thus far are compatible with the structure of a Lie superalgebra.
To do so let's consider first:

$$\begin{aligned}[\delta_A, \delta_B]\Phi^a &= (\delta_A \delta_B - \delta_B \delta_A)\Phi^a \\ &= [[\Phi^a, B_B], B_A] - [[\Phi^a, B_A], B_B] \\ &= [[B_A, B_B], \Phi^a]\end{aligned}$$

We recover as expected the Jacobi identity. Moreover, a composition of symmetries is still a symmetry so:

$$\begin{aligned}[\delta_A, \delta_B]\Phi^a &= X_{AB}{}^C \delta_C \Phi^a \\ &= [\Phi^a, X_{AB}{}^C B_C] \\ &= [-X_{AB}{}^C B_C, \Phi^a]\end{aligned}$$

By equating both expressions we recover the fact that the algebra of the bosonic charges closes with structure constants $f_{AB}{}^C = -X_{AB}{}^C$. We showed that the transformation laws are compatible with a Lie algebra structure but the generalisation to the case of Lie super-algebras follows the same path of reasoning.

1.5 Beyond Lie superalgebras?

We can wonder if in a relativistic quantum field theory setting, we can obtain other structures than Lie superalgebras. There are two possible ways to generalize:
- The first one is to consider a new kind of charge other than bosonic and fermionic, which would be bold but not necessarily excluded (see Plektons).
- The second one would be to use multiple (anti)-commutators to get new algebraic structures (see ternary algebras for example)

2 The Super-Poincaré algebra

Let $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the Poincaré superalgebra, for which,

$$\mathfrak{g}_0 = \mathfrak{iso}(1,3) = \{M_{\mu\nu}, P_\mu, \mu, \nu = 0, ..., 3\}$$

is the Poincaré algebra and,

$$\mathfrak{g}_1 = (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2}) = \{Q_\alpha, \alpha = 1,2\} \oplus \{\bar{Q}^{\dot{\alpha}}, \dot{\alpha} = 1,2\}$$

To review the Poincaré algebra and the representations of the Lorentz group, see appendix.

2.1 The (anti)-commutation relations

We start by computing the commutators $[M, Q]$ and $[P, Q]$. For this purpose we introduce $\sigma^\mu = (\mathbb{1}, \overrightarrow{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\overrightarrow{\sigma})$ with $\overrightarrow{\sigma}$ the Pauli matrices.
We then define:

$$(\sigma^{\mu\nu})_\alpha{}^\beta \doteq \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta$$
$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \doteq \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}$$

which satisfy the Lorentz algebra. Given that Q_α is a spinor, it transforms according to the following law $Q'_\alpha = exp(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$.
It also serves as an operator that transofrms under Lorentz as $Q'_\alpha = U^\dagger Q_\alpha U$ with $U = exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$. By equating these two relations to first order we get:

$$(\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta = (\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})Q_\alpha(\mathbf{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$$
$$\Rightarrow Q_\alpha - \frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta = Q_\alpha - \frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu}Q_\alpha - Q_\alpha M^{\mu\nu})$$
$$\Rightarrow [M^{\mu\nu}, Q_\alpha] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$$

We show in the same way that:

$$[M^{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}$$

We know that Q_α lies in the $(\frac{1}{2}, 0)$ representation and P^μ is in the $(\frac{1}{2}, \frac{1}{2})$ one, therefore the commutator must either be in the $(1, \frac{1}{2})$ or in the $(0, \frac{1}{2})$ representation. The first one is not possible because there are no generators of the superalgebra that lie in this representation. This means that we can write in full generality:

$$[P^\mu, Q_\alpha] = c(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$$

By taking the hermitian conjugate, and taking care of indices, we have:

$$[P^\mu, \bar{Q}^{\dot{\alpha}}] = c^*(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}Q_\alpha$$

By applying Jacobi to P^μ, P^ν and Q_α we find:

$$\begin{aligned}
0 &= [P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] + [Q_\alpha, \underbrace{[P^\mu, P^\nu]}_{=0}] \\
&= c(\sigma^\nu)_{\alpha\dot{\alpha}}[P^\mu, \bar{Q}^{\dot{\alpha}}] - c(\sigma^\mu)_{\alpha\dot{\alpha}}[P^\nu, \bar{Q}^{\dot{\alpha}}] \\
&= c(\sigma^\nu)_{\alpha\dot{\alpha}}c^*(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}Q_\beta - c(\sigma^\mu)_{\alpha\dot{\alpha}}c^*(\bar{\sigma}^\nu)^{\dot{\alpha}\beta}Q_\beta \\
&= |c|^2(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta Q_\beta \\
&= -4i|c|^2\underbrace{(\sigma^{\nu\mu})_\alpha{}^\beta}_{\neq 0} Q_\beta
\end{aligned}$$

This relation must be verified for a general Q_β this means that we necessarily have $|c|^2 = 0$.
Conclusion:

$$[P^\mu, Q_\alpha] = [P^\mu, \bar{Q}^{\dot{\alpha}}] = 0$$

We are now looking for anticommutation relations between fermionic generators. To do so we first notice that $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$ and the only vector in our algebra is the four-momentum. We have :

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -iC^{te}(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu \tag{6}$$

Let's show that the constant is positive:
Let $|\Psi\rangle$ be a state in our Hilbert space. We can write:

$$\begin{aligned}
-iC^{te}(\sigma^\mu)_{\alpha\dot{\alpha}}\langle\Psi|P_\mu|\Psi\rangle &= \langle\Psi|\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}|\Psi\rangle \\
&= \langle\Psi|Q_\alpha Q_\alpha^\dagger|\Psi\rangle + \langle\Psi|Q_\alpha^\dagger Q_\alpha|\Psi\rangle \\
&= |Q_\alpha^\dagger|\Psi\rangle|^2 + |Q_\alpha|\Psi\rangle|^2 > 0
\end{aligned}$$

by summing over α and $\dot{\alpha}$ we find

$$\begin{aligned} -iC^{te}tr(\sigma^{\mu})\langle\Psi|P_{\mu}|\Psi\rangle &= -i2C^{te}\delta^{\mu 0}\langle\Psi|P_{\mu}|\Psi\rangle \\ &= -i2C^{te}\langle\Psi|\underbrace{P^{0}}_{=iH}|\Psi\rangle > 0 \end{aligned}$$

By positivity of the energy we have $C^{te} > 0$ and by convention we set the constant to 2 and we get:

$$\{Q_{\alpha}, \bar{Q}_{\dot{\alpha}}\} = -2i(\sigma^{\mu})_{\alpha\dot{\alpha}}P_{\mu} \tag{7}$$

For the anticommutator $\{Q_{\alpha}, Q_{\beta}\}$ we see that the result will be a linear combination of operators in the $(0,0)$ and $(1,0)$ representations. The only element in the algebra in the $(1,0)$ representation is the self dual of $M^{\mu\nu}$ because it is the only rank 2 anti-symmetric tensor in the algebra.

Indeed we us the self duality propretry of the $\sigma^{\mu\nu}$ matrix to get:

$$\tfrac{i}{2}\epsilon^{\mu\nu\rho\lambda}\sigma_{\rho\lambda} = \sigma^{\mu\nu} \Rightarrow \tfrac{i}{2}\epsilon^{\mu\nu\rho\lambda}\sigma_{\rho\lambda}M_{\mu\nu} = \sigma^{\mu\nu}M_{\mu\nu}$$

We can write :

$$\{Q_{\alpha}, Q_{\beta}\} = K\epsilon_{\alpha\beta} + K'(\sigma^{\mu\nu})_{\alpha\beta}M_{\mu\nu} \tag{8}$$

The left hand-side is symmetrical with respect to α and β whereas the first term of the right hand side is anti-symmetric with respect to these indices so $K = 0$. Moreover, the left handside commutes with P^{μ}, we can see that by using the Jacobi super-identity : $[P, \{Q, Q\}] - \{[P, Q], Q\} - \{Q, [P, Q]\} = 0$, we've shown that P commutes with Q so the last two terms vanish like expected. However, the second term of the right hand-side of equation (8) does not commute with P so $K' = 0$. We reason similarly for right-handed spinors and we get:

$$\{Q_{\alpha}, Q_{\beta}\} = \{\bar{Q}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} = 0 \tag{9}$$

2.2 An equivalent formulation of the Poincaré superalgebra

Let's give an equivalent formulation of the Poincaré superalgebra by introducing 4-component spinors:

$$Q_a = \begin{pmatrix} Q_{\alpha} \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix}$$

First off we trivially have:

$$[Q_a, P^{\mu}] = 0 \tag{10}$$

Then we also find easily:

$$[M^{\mu\nu}, Q_a] = (\Sigma^{\mu\nu})_a{}^b Q_b \tag{11}$$

with $\Sigma^{\mu\nu}$ being the Lorentz algebra generators in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation. Finally for the fermionic sector:

$$
\begin{aligned}
\{Q_a, Q_b\} &= \left\{ \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix}, \begin{pmatrix} Q_\beta & \bar{Q}^{\dot\beta} \end{pmatrix} \right\} \\
&= \begin{pmatrix} \{Q_\alpha, Q_\beta\} & \{Q_\alpha, \bar{Q}^{\dot\beta}\} \\ \{Q_\beta, \bar{Q}^{\dot\alpha}\} & \{\bar{Q}^{\dot\alpha}, \bar{Q}^{\dot\beta}\} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \{Q_\alpha, \bar{Q}_{\dot\gamma}\}\epsilon^{\dot\beta\dot\gamma} \\ \{Q^\delta, \bar{Q}^{\dot\alpha}\}\epsilon_{\beta\delta} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2i(\sigma^\mu)_{\alpha\dot\gamma}\epsilon^{\dot\beta\dot\gamma}P_\mu \\ -2i(\bar\sigma^\mu)^{\delta\dot\alpha}\epsilon_{\beta\delta}P_\mu & 0 \end{pmatrix} \\
&= -2i \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\gamma} \\ (\bar\sigma^\mu)^{\delta\dot\alpha} & 0 \end{pmatrix} \begin{pmatrix} \epsilon_{\beta\delta} & 0 \\ 0 & \epsilon^{\dot\beta\dot\gamma} \end{pmatrix} P_\mu \\
&= -2i \left(\gamma^\mu C^{-1}\right) P_\mu
\end{aligned}
$$

with C being the charge conjugation matrix. We have:

$$\{Q_a, Q_b\} = -2i \left(\gamma^\mu C^{-1}\right)_{ab} P_\mu \tag{12}$$

We can make one final comment by rewriting the supersymmetry algebra in the Majorana representation where the Dirac matrices are imaginary. In this case we see that the constant structure are all imaginary.
We can then absorb the i factor by redefining our generators and this shows us that the algebra is real, which was not clear in the beginning.

3 Conclusion

We have seen how considering a relativistic quantum field theory naturally lead to the strucuture of Lie superalgebras. We then went on to study the basic properties of the supersymmetry algebra, namely the Poincaré superalgebra.The next steps would be to focus on the representations of the extended supersymmetry (meaning with N supercharges).

For reference, we can mention that since its discovery, supersymmetry has found applications in other areas of physics such as condensed matter, optics and dynamical systems to name a few.

On a personal level, considering that I have been interested in learning supersymmetry, I was very pleased to complete my internship this summer. Through the length of this internship, I was able to apply what I learned this year in field theory and Lie algebras. The completion of this internship left me even more eager to pursue my higher learning in theoretical physics.

A Review on the Poincaré Group and algebra

The Poincaré group corresponds to the isometry group of Minkowski space-time of Special Relativity. It's a 10-dimensional non-compact Lie group composed of the Lorentz group (3 rotations and 3 boosts) to which we add the 4 translations (formally the semi-direct product of the Lorentz group with the translations).

Let's develop the main results.

We define the Lorentz group as the matrix group that preserves the following minkoswkian scalar product:

$$\eta_{\mu\nu}x^\mu y^\nu$$

The Lorentz transformation takes the form $x' = \Lambda x$ so we have:

$$\eta_{\mu\nu}x^\mu y^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma y^\sigma \Rightarrow \Lambda^T\eta\Lambda = \eta \tag{13}$$

The set of matrices Λ that verify this relation is the Lorentz group $SO(1,3)$. The Lorentz group has 4 connected components. The part connected to the identity $SO(1,3)^\uparrow_+$ is called the proper orthochronous Lorentz group (because it preserves space orientation and the arrow of time). It's this group that will be considered from now on. This group contains the group of rotations $SO(3)$ as a subgroup and the boosts (that do not form a group on their own).

To obtain the Poincaré group, we add the translations, then 4-vectors will transform in the following way:

$$x' = T(\Lambda, a) = \Lambda x + a \tag{14}$$

We write $ISO(1,3)$ the Poincaré group (I for inhomogenous). Let's study the assiocated algebra. $U(\Lambda, a)$ are the representations of the Poincaré group acting on a vector space. The Poincaré Group being a Lie group, we can write its elements as:

$$U(\Lambda, a) = e^{\frac{i}{2}\epsilon_{\mu\nu}M^{\mu\nu}}e^{ia_\mu P^\mu} \tag{15}$$

where the explicit form of the generators M and P depends on the representation. We then obtain the commutations relations by considering the following equality:

$$U(\Lambda, a)U(\Lambda', a')U^{-1}(\Lambda, a) = U(\Lambda\Lambda'\Lambda^{-1}, a + \Lambda a' - \Lambda\Lambda'\Lambda^{-1}a)$$

After some involved calculations we find that the commutations relations of the Poincaré algebra are:

$$i[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} \tag{16}$$

$$i[P^\mu, M^{\rho\sigma}] = \eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho \tag{17}$$

$$[P^\mu, P^\nu] = 0 \tag{18}$$

We can then define the SO(3) generators **J** (angular momentum) and the boosts generators **K** as:

$$J^i \doteq -\tfrac{1}{2}\epsilon_{ijk}M^{jk}, \qquad K^i \doteq M^{0i}$$

The Poincaré algebra commutation relations then take the form:

$$\begin{array}{lll} [J^i, J^j] = i\epsilon_{ijk}J^k & [J^i, P^j] = i\epsilon_{ijk}P^k & [P^i, P^j] = 0 \\ [J^i, K^j] = i\epsilon_{ijk}K^k & [K^i, P^j] = i\delta_{ij}P^0 & [J^i, P^0] = 0 \\ [K^i, K^j] = -i\epsilon_{ijk}J^k & [K^i, P^0] = iP^i & [P^i, P^0] = 0 \end{array}$$

B Review on the irreducible representations of the Lorentz group

To review the Lorentz group we will complexify the Lorentz algebra which means that we consider : $\mathfrak{so}(1,3)_{\mathbb{C}} = \mathfrak{so}(1,3) \otimes \mathbb{C}$.
In this algebra we can define the following elements:

$$N^i \doteq \frac{1}{2}(J^i + iK^i), \qquad \bar{N}^i \doteq \frac{1}{2}(J^i - iK^i)$$

for these generators of $\mathfrak{so}(1,3)_{\mathbb{C}}$ we get the following commutation relations:

$$\begin{aligned}
[N^i, N^j] &= i\epsilon_{ijk} N^k \\
[\bar{N}^i, \bar{N}^j] &= i\epsilon_{ijk} \bar{N}^k \\
[N^i, \bar{N}^j] &= 0
\end{aligned}$$

Therefore, the N^i and $\bar{N}^i$ form two independant sub-algebras $\mathfrak{sl}(2,\mathbb{C})$ that commute with each other. We have:

$$\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \tag{19}$$

We obtain the representations of $\mathfrak{so}(1,3)_{\mathbb{C}}$ by tensoring two representations of $\mathfrak{sl}(2,\mathbb{C})$. Those are labeled by an integer $n \in \mathbb{N}$ that is twice the "spin". Knowing that the operator $\bar{N}^i$ is the conjugate of N^i we have:

$$\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \overline{\mathfrak{sl}(2,\mathbb{R})} \tag{20}$$

The representations of $\mathfrak{so}(1,3)$ will be labeled by two integers and we will write them as $\left(\frac{n}{2}, \frac{m}{2}\right)$.
A few examples :
- The $(0,0)$ representation is the 1-dimensional trivial representation, it corresponds to scalars.
- The $(\frac{1}{2}, 0)$ representation is 2-dimensional and the vectors on which it acts are called left-handed spinors (it's the case of Q in the Super-Poincaré algebra).
- The $(0, \frac{1}{2})$ is also 2-dimensional and it acts on right-handed spinors ($\bar{Q}$ in our superalgebra).
- The $(\frac{1}{2}, \frac{1}{2})$ is 4-dimensional and we identify it with the defining vector representation of $\mathfrak{so}(1,3)$ which acts on 4-vectors.

Références

[1] Adel Bilal, *Introduction to Supersymmetry*, (2001).

[2] Fernando Quevedo, Sven Krippendorf and Oliver Schlotterer, *Cambridge Lectures on Supersymmetry and Extra Dimensions*, (2010).

[3] A. Wipf, *Introduction to Supersymmetry*, (2016).

[4] Gernot Eichmann, *Introduction to hadron physics : Appendix B Poincaré group*, (2015)

[5] Ave Khamseh, *Representations of the Poincaré Group*, (2013)

[6] D. Bernard, Y. Laszlo and D. Renard, *éléments de théorie des groupes et symétries quantiques*, (2012)

[7] M. Sohnius, *Introducing Supersymmetry*, (1985)

[8] V. Démery, *Groupe de Poincaré et théorie relativiste des champs*, (2007)

[9] L. Frappat, *Introduction à la supersymétrie*, (2015)