Homework Sheet 9

MATH431 — Introduction to Modern Particle Theory

(Dr Thomas Teubner)

1. Suppose we have two complex scalar fields, ϕ_1 and ϕ_2, which form a doublet of $SU(2)_I$. Write down the Lagrangian and show that $m_1 \neq m_2$ entails explicit breaking of the symmetry. Can we write a Higgs potential which preserves the symmetry? Describe how to break the symmetry in this case.

2. Let the Lagrangian for three interacting real scalar fields ϕ_1, ϕ_2, ϕ_3 be given by

$$L = \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\mu^2 \phi_i^2 - \frac{1}{4}\lambda(\phi_i^2)^2 ,$$

with $\mu^2 < 0$ and $\lambda > 0$, and where the summation over i is implied.

(a) Explain the meaning of the different terms and to which elements of the corresponding Feynman rules they lead. Why is μ^2 chosen negative, but λ positive?

(b) Show that after spontaneous symmetry breaking this Lagrangian describes one massive field with mass $\sqrt{-2\mu^2}$ and two massless 'Goldstone bosons'.

3. In perturbative QCD the scale (energy) dependence of the 'running' coupling $\alpha_s = g_s^2/(4\pi)$ is described by the differential equation

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\ln E} = -b_0 \alpha_s^2 - b_1 \alpha_s^3 + \mathcal{O}(\alpha_s^4) ,$$

with b_0, $b_1 > 0$ the first two coefficients of the QCD beta-function.

(a) Neglect the term $\sim \alpha_s^3$ and verify that the solution is given by

$$\alpha_s(E) = \frac{\alpha_s(\mu)}{1 + b_0 \alpha_s(\mu) \ln(E/\mu)} ,$$

where the initial condition is fixed by the coupling $\alpha_s(\mu)$ at a reference scale μ.

(b) Show that by defining a mass scale

$$\Lambda_{\mathrm{QCD}} = \mu \exp\left[-\frac{1}{b_0 \alpha_s(\mu)} \right]$$

the solution reads

$$\alpha_s(E) = \frac{1}{b_0 \ln(E/\Lambda_{\mathrm{QCD}})} .$$

Experimentally, $\Lambda_{\mathrm{QCD}} \simeq 200$ MeV. Sketch the behaviour of $\alpha_s(E)$ and discuss the physical consequences in the two cases $E \to \Lambda_{\mathrm{QCD}}$ and at asymptotically large energies.

(c) Derive a solution similar to the one in part (b), but taking into account the term $-b_1\alpha_s^3$ in the differential equation and chosing a suitable redefinition of Λ_{QCD} (formula for Λ_{QCD} at next order not required).

4. The decomposition of the fundamental representation of $SU(5)$ under $SU(3) \times SU(2) \times U(1)$ is given by:

$$5 \;=\; \left(3, 1, -\frac{2}{3}\right) + (1, 2, 1)\,.$$

(i) Derive the charge assignment of the leptoquark vector bosons in the adjoint (the 24) representation.

(ii) Draw the Feynman diagrams which mediate proton decay via leptoquark exchange.