Homework Sheet 7

MATH431 — Introduction to Modern Particle Theory

(Dr Thomas Teubner)

1. Starting from the solutions $u^s(p)$ and $v^s(p)$ $(s_1, s_2 \in \{\uparrow, \downarrow\})$ of the free Dirac equation as discussed in the lectures, prove the following relations:

(i) Normalisation: $u^{(s_1)\dagger}(p)\, u^{(s_2)}(p) \;=\; v^{(s_1)\dagger}(p)\, v^{(s_2)}(p) \;=\; 2\,E\,\delta_{s_1 s_2}$.

(ii) Orthogonality: $u^{(s_1)\dagger}(p)\, v^{(s_2)}(\bar{p}) \;=\; v^{(s_1)\dagger}(\bar{p})\, u^{(s_2)}(p) \;=\; 0$, with $\bar{p} := (p^0, -\vec{p})$.

(iii) Completeness:

$$\sum_{s=\uparrow,\downarrow} u^{(s)}(p)\, \bar{u}^{(s)}(p) \;=\; \not{p} + m\,,$$
$$\sum_{s=\uparrow,\downarrow} v^{(s)}(p)\, \bar{v}^{(s)}(p) \;=\; \not{p} - m\,.$$

2. Show that a unitary matrix U can be written as e^{iH}. What are the conditions that the matrix H must satisfy?

3.* Prove that for the group $SU(2)$ the fundamental representation (the representation by complex 2×2 matrices) is equivalent to the complex conjugate representation. This means that there exists a unitary 2×2 matrix W such that $U^\star = W^\dagger\, U\, W$ (unitary equivalence).

Hints: The complex conjugate representation can be written as

$$U^\star \;=\; \exp(-\frac{i}{2}\,\theta_a\,\sigma_a^\star)\,,$$

with the three Pauli-matrices in the usual representation,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

First show that $i\sigma_2$ is unitary and that $\sigma_2\,\sigma_a^\star\,\sigma_2 = -\sigma_a$.

Explain the consequences this has for the construction of mass terms in the Standard Model. [This last part can be done only towards the end of the course.]

* *This problem is for real aficionados only.*