Homework Sheet 6

MATH431 — Introduction to Modern Particle Theory

(Dr Thomas Teubner)

1. The Dirac wave function for the ground state of the hydrogen atom has the form (spin-up state, standard Dirac matrix representation)

$$\psi_\uparrow(r,\theta,\phi) \;=\; R(r)\begin{pmatrix} 1 \\ 0 \\ ia\cos\theta \\ iae^{i\phi}\sin\theta \end{pmatrix},$$

where $a = (1-\sqrt{1-\alpha^2})/\alpha \approx \alpha/2$ (with $\alpha \approx 1/137$ the QED coupling) and R is a function of the radial variable r only.

(a) Investigate whether $\psi_\uparrow$ is an eigenstate of L_z.
Hint: For the angular momentum operator L_z in spherical coordinates, see Homework Sheet 1, problem 2. (b).

(b) Find the expectation value of L_z and discuss the result.
Hint: Don't forget to normalise the state $\psi_\uparrow$.
What would happen in the non-relativistic limit?

(c) Show that $\psi_\uparrow$ is an eigenstate of J_z and find its eigenvalue.

2. Derive the conservation equation

$$\partial_\mu J_V^\mu = 0$$

for the four-vector current density $J_V^\mu = \bar\psi\gamma^\mu\psi$, using the covariant form of the Dirac equation and the relation $\gamma^{\mu\,\dagger} = \gamma^0\gamma^\mu\gamma^0$.

Show that the axial four-vector current density $J_A^\mu = \bar\psi\gamma^\mu\gamma^5\psi$ is not conserved but instead satisfies the covariant equation

$$\partial_\mu J_A^\mu \;=\; 2im\,\bar\psi\gamma^5\psi\,.$$

3. Derive the *Gordon decomposition* of the Dirac transition current,

$$\bar\psi_f\gamma^\mu\psi_i \;=\; \frac{1}{2m}\,\bar\psi_f\left[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu\right]\psi_i\,,$$

where $\sigma^{\mu\nu} = \frac{1}{2}i(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$.

[*Hint:* Use the Dirac equations $\bar\psi_f(\not{p}_f - m) = (\not{p}_i - m)\psi_i = 0$.]