Homework Sheet 5

MATH431 — Introduction to Modern Particle Theory

(Dr Thomas Teubner)

1. *Electrodynamics in covariant form:*

Let A_μ be the electromagnetic vector potential. The electromagnetic field strength tensor is then defined as

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

(a) Show that Maxwell's equations (in the absence of sources) in four-vector notation can be derived from the electromagnetic Lagrangian given by

$$L_{\text{e.m.}} \;=\; -\frac{1}{4}\, F_{\alpha\beta} F^{\alpha\beta} \,.$$

(b) Show that $L_{\text{e.m.}}$ is invariant under the transformation $A_\mu \to A_\mu - \partial_\mu \Lambda$, where Λ is a scalar function.

(c) Show that by imposing $U(1)$ symmetry, i.e. invariance of the Lagrangian under such a local $U(1)$ transformation, a mass term for the photon is forbidden.

(d) Now include the source-term $-j^\mu A_\mu$ in the Lagrangian and derive the corresponding form of Maxwell's equations with the external current j^μ. Show that this directly implies the conservation of the four-current, i.e. $\partial_\mu j^\mu = 0$.

2. *γ-gymnastics:*

(a) Defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, show that (i) $\gamma^{5\dagger} = \gamma^5$, and (ii) $\{\gamma^5, \gamma^\mu\} = 0$.

(b) By inserting $(\gamma^i)^2 = -1\!\!1$ for some $i = 1, 2, 3$ (whereas $(\gamma^0)^2 = 1\!\!1$), write both $\gamma^0\gamma^1\gamma^2$ and $\gamma^0\gamma^1\gamma^3$ as a product $\gamma^5\gamma^\nu$ for some $\nu = 0, 1, 2, 3$.

(c) Prove the following relations:

$$\text{tr}[\gamma_\mu\gamma_\nu] \;=\; 4\eta_{\mu\nu} \,,$$

$$\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \;=\; 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu \,,$$

$$\text{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] \;=\; 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}] \,.$$