Homework Sheet 4

MATH431 — Introduction to Modern Particle Theory

(Dr Thomas Teubner)

1. Consider the Poincaré group in 1+2 dimensions, with the infinitesimal line element

$$\mathrm{d}s^2 \;=\; \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2\,.$$

(a) Write down in matrix form the metric for this line element and its inverse.

(b) What are the transformations under which this line element is invariant?
Give the generators associated with each transformation.

(c) The Pauli-Lubanski vector in four dimensions is given by

$$W^\mu \;=\; -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho} J_{\nu\sigma} P_\rho\,,$$

where $K_i = J_{i0}$ and $J_i = \frac{1}{2}\epsilon_{ijk}\, J_{jk}$ are the generators of boosts and rotations, respectively, and P_ρ is the momentum four-vector. Give W^μ in terms of K_i, J_i, P_0, P_i. Write down the four components of the Pauli-Lubanski vector in the case where the line element $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2$ is viewed as embedded in four space-time dimensions, for both massless and massive particle states.

2. Suppose that we live on a two dimensional surface with a line element on it given by

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\,.$$

(A convenient notation is $\theta^\mu \equiv (\theta, \phi)\,,\ \mu = 1, 2.$)

(a) Write the corresponding metric $g_{\mu\nu}$ in explicit matrix form. Give $g^{\mu\nu}$ in matrix form.

(b) Find the set of infinitesimal transformations of the form

$$\theta^\mu \;\rightarrow\; \theta^\mu + \epsilon\, \zeta^\mu(\theta, \phi)$$

for which the line element $\mathrm{d}s^2$ is invariant.

(c) Define the operators

$$J = \zeta^\mu \frac{\partial}{\partial\theta^\mu}\,.$$

(Einstein's summation convention is assumed: The summation over the index μ is implicit.) Show that all *three* independent operators satisfy the commutation relation

$$[J_i, J_j] \;=\; i\epsilon_{ijk} J_k\,.$$

(d) What is the geometric meaning of this example?