Homework Sheet 3

MATH431 — Introduction to Modern Particle Theory

(Dr Thomas Teubner)

1. The defining equation for the Lorentz group may be written as

$$L^T \eta L = \eta \,. \tag{1}$$

Consider a 2-dimensional spacetime for which $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Show that the standard Lorentz transformation,

$$L = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix},$$

where $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$, satisfies the above condition.

2. Show that $\mathcal{L}_+^\uparrow$ is a group. This is done as follows:

(i) Use Eq. (1) to show that

$$L\eta^{-1}L^T = \eta^{-1} \,. \tag{2}$$

Hint: Recall that for matrices $(AB)^{-1} = B^{-1}A^{-1}$.

(ii) We now have from Eqs. (1), (2)

$$\eta_{\alpha\beta} L^\alpha{}_\mu L^\beta{}_\nu = \eta_{\mu\nu}, \quad \eta^{\alpha\beta} L^\mu{}_\alpha L^\nu{}_\beta = \eta^{\mu\nu} \,. \tag{3}$$

Let $\mathbf{l} = (L^1{}_0, L^2{}_0, L^3{}_0)$ and $\bar{\mathbf{l}} = (\bar{L}^0{}_1, \bar{L}^0{}_2, \bar{L}^0{}_3)$. By putting $\mu = \nu = 0$ in Eq. (3), show that $|\mathbf{l}| = \sqrt{(L^0{}_0)^2 - 1}$ and $|\bar{\mathbf{l}}| = \sqrt{(\bar{L}^0{}_0)^2 - 1}$.

(iii) By considering $(\bar{L}L)^0{}_0 = \bar{L}^0{}_\alpha L^\alpha{}_0$, show that

$$(\bar{L}L)^0{}_0 = \bar{L}^0{}_0 L^0{}_0 + \bar{\mathbf{l}} \cdot \mathbf{l} \,.$$

(iv) Use the Schwartz inequality

$$|\bar{\mathbf{l}} \cdot \mathbf{l}| \leq |\mathbf{l}||\bar{\mathbf{l}}|$$

to show that

$$(\bar{L}L)^0{}_0 \geq \bar{L}^0{}_0 L^0{}_0 - \sqrt{(\bar{L}^0{}_0)^2 - 1}\,\sqrt{(L^0{}_0)^2 - 1} \,.$$

(v) Now show that

$$\begin{aligned} (x-y)^2 \geq 0 \quad &\Rightarrow \quad x^2y^2 - 2xy + 1 \geq (x^2-1)(y^2-1) \\ &\Rightarrow \quad (xy-1)^2 \geq (x^2-1)(y^2-1) \\ &\Rightarrow \quad \text{either} \quad xy - 1 \geq \sqrt{x^2-1}\sqrt{y^2-1} \\ &\qquad\;\; \text{or} \quad xy - 1 \leq -\sqrt{x^2-1}\sqrt{y^2-1} \,. \end{aligned}$$

Deduce that if $x, y \geq 1$ then $xy - 1$ is non-negative and we must have

$$xy - \sqrt{x^2-1}\sqrt{y^2-1} \geq 1 \,.$$

Finally combine with (iv) to deduce that if $\bar{L}^0{}_0 \geq 1$ and $L^0{}_0 \geq 1$, then $(\bar{L}L)^0{}_0 \geq 1$.

(vi) Use the fact that $\det(\bar{L}L) = \det \bar{L} \det L$ to deduce that

$$\det \bar{L} = \det L = 1 \quad \Rightarrow \quad \det(\bar{L}L) = 1\,.$$

(vii) We can now deduce that $L \in \mathcal{L}_+^\uparrow$ and $\bar{L} \in \mathcal{L}_+^\uparrow \Rightarrow (\bar{L}L) \in \mathcal{L}_+^\uparrow$. Together with the obvious fact that $1 \in \mathcal{L}_+^\uparrow$, this is most of what we need to show that $\mathcal{L}_+^\uparrow$ is a group.

(viii) We still need to show that $L \in \mathcal{L}_+^\uparrow \Rightarrow L^{-1} \in \mathcal{L}_+^\uparrow$. Note that from Eq. (1) $\Rightarrow L^{-1} = \eta^{-1} L^T \eta$. So clearly $(L^{-1})^0{}_0 = L^0{}_0$. Moreover, $\det L^{-1} = \det \eta^{-1} \det L^T \det \eta = 1$. QED.

3. Let $\vec{J}$ and $\vec{K}$ be the generators of rotations and boosts, respectively.

(a) Show that

$$J^2 - K^2 \quad \text{and} \quad \vec{J} \cdot \vec{K}$$

are Lorentz invariants, i.e. that they commute with all the generators of the Lorentz group.
Hint: If you use that $\vec{J}_+$ and $\vec{J}_-$ are generating $SU(2) \otimes SU(2)$ you will not need any explicit calculation.

(b) Assume a representation (j_1, j_2) of $SU(2) \times SU(2)$. How many states are in this representation? How does this representation decompose into irreducible representations of $SU(2)_J$, where J is the total angular momentum?