Homework Sheet 2

MATH431 — Introduction to Modern Particle Theory

(Dr Thomas Teubner)

1. Prove

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\mathbf{j} \;=\; 0\,,$$

where

$$\mathbf{j} \;=\; -\frac{i\hbar}{2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]\,,$$

using the standard non-relativistic Schrödinger equation.

2. (a) The Lagrange density for a real scalar field is given by

$$\mathcal{L}_0 \;=\; \frac{1}{2}\,(\partial\phi)^2 - \frac{1}{2}\,m^2\phi^2\,.$$

Show that the Hamiltonian H_0 is given by

$$H_0 \;=\; \frac{1}{2}\int \left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right) \mathrm{d}^3x\,.$$

(b) The canonical commutation relations are given by

$$\begin{aligned} [\phi(\mathbf{x},t), \pi(\mathbf{x}',t)] &= i\hbar\delta(\mathbf{x}-\mathbf{x}')\,, \\ [\phi(\mathbf{x},t), \phi(\mathbf{x}',t)] &= 0\,, \\ [\pi(\mathbf{x},t), \pi(\mathbf{x}',t)] &= 0\,. \end{aligned}$$

Show that with these commutation relations the (Heisenberg) equations of motion for the time evolution of the operators ϕ and π,

$$i\hbar\dot{\phi} \;=\; [\phi, H_0] \qquad \text{and} \qquad i\hbar\dot{\pi} \;=\; [\pi, H_0]\,,$$

imply the Klein-Gordon equation for the field ϕ.

(c) Now consider the Lagrange density

$$\mathcal{L} \;=\; \mathcal{L}_0 + \mathcal{L}_I\,,$$

with the interaction Lagrangian

$$\mathcal{L}_I \;=\; -\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4\,.$$

Derive the equation of motion from the general form of the Euler-Lagrange equations

$$\partial^\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} \;=\; 0\,.$$

p.t.o.

3. *Creation and annihilation operators:*

(a) Two-particle states are defined by $|\mathbf{p}_1, \mathbf{p}_2\rangle \;=\; a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0\rangle$. Show that

$$\langle \mathbf{p}_1', \mathbf{p}_2' | \mathbf{p}_1, \mathbf{p}_2\rangle \;=\; (2\pi)^6 (2p_1^0)(2p_2^0)\left[\delta(\mathbf{p}_1 - \mathbf{p}_1')\delta(\mathbf{p}_2 - \mathbf{p}_2') + \delta(\mathbf{p}_1 - \mathbf{p}_2')\delta(\mathbf{p}_2 - \mathbf{p}_1')\right] .$$

(b) Show that the number operator, defined by

$$N \;:=\; \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}}{2p^0}\, a^\dagger(\mathbf{p})a(\mathbf{p})\,,$$

satisfies

$$[N, a^\dagger(\mathbf{p})] \;=\; a^\dagger(\mathbf{p})\,,$$

and hence

$$N\,|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle \;=\; n\,|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle\,.$$