INTERNAL STRUCTURE OF FLUX VACUA

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String sigma model:

$$\int_{\Sigma} g_{\mu\nu}(x) dx^{\mu} \wedge *dx^{\nu} + \int_{D} H$$

- $x : \Sigma \to X$ (X - target space)
- $\partial D = \Sigma$
- $H = dB_{\alpha}$ (B_{α} defined on a patch X_{α})

(SUPER)SYMMETRIES OF THE TARGET SPACE
VS.
GOLBAL CONSTRAINS

Phase space of string sigma model

- The phase space for the string sigma model on $S^1 \times \mathbb{R}$ is T^*LX — the cotangent bundle to the loop space of X (X - compact smooth manifold.)
- T^*LX is naturally a symplectic space: given a loop $x : S^1 \hookrightarrow X$, the symplectic form is
$$\omega_X = \oint_{S^1} d\sigma \left[\delta p + \imath(\partial_\sigma x) H\right]$$
(where $\oint_{S^1} d\sigma\, \imath(\partial_\sigma x) H = \frac{1}{2}\oint_{S^1} d\sigma\, \partial_\sigma x^M(\sigma) H_{MNP}(x(\sigma))\delta x^N(\sigma)\wedge \delta x^P(\sigma)$)
- Geometric quantization – a hermitian line bundle over the phase space with a connection that has curvature ω (in the patch X_α):
$$\vartheta_X = \delta z_\alpha + \oint_{S^1} d\sigma \left[p - \imath(\partial_\sigma x) B_\alpha\right]$$
with
  - ◇ $\imath(\partial_\sigma x) B = \partial_\sigma x^M B_{MN} \delta x^N(\sigma)$.
  - ◇ on the twofold intersection $X_{\alpha\beta}$: $B_\alpha - B_\beta = dA_{\alpha\beta}$
  - ◇ p is globally well defined: it is a section of $x^*(T^*X)$ ($z_\alpha - z_\beta = \oint_{S^1} x^* A_{\alpha\beta}$)

Current algebra.

- Given a section (v,ρ) of $TX \oplus T^*X$ one can construct a current:

$$J_\epsilon(v,\rho) = \oint_{S^1} d\sigma\, \epsilon(\sigma) \big[\imath(v)p + \imath(\partial_\sigma x)\,\rho\big]$$

$\epsilon(\sigma)$ is a smooth (test) function on the circle.

$\longrightarrow$ $\epsilon(\sigma)$ – smooth (test) function on the circle

$\longrightarrow$ Noether current for $\imath_v H = d\rho$

- Poisson bracket of two such currents is

$$\{J_{\epsilon_1}(v_1,\rho_1),\ J_{\epsilon_2}(v_2,\rho_2)\} = J_{\epsilon_1\epsilon_2}\big([(v_1,\rho_1),(v_2,\rho_2)]_H\big) - \frac{1}{2}\oint_{S^1} d\sigma\,(\epsilon_1\partial_\sigma\epsilon_2 - \epsilon_2\partial_\sigma\epsilon_1)\big[\imath_{v_1}\rho_2 + \imath_{v_2}\rho_1\big]$$

- $[\cdot,\cdot]_H$ is the twisted Courant bracket:

$$[(v_1,\rho_1),(v_2,\rho_2)]_H = [v_1,v_2]_{\mathrm{Lie}} + \Big\{\mathcal{L}_{v_1}\rho_2 - \mathcal{L}_{v_2}\rho_1 - \frac{1}{2} d(\imath_{v_1}\rho_2 - \imath_{v_2}\rho_1) + \imath_{v_1}\imath_{v_2}H\Big\}$$

A special case of a derived bracket : $[A,B]_D \equiv [\{A\cdot, D\}, B\cdot]$ (with $D^2 = 0$, e.g. $D = d, (d - H\wedge), \ldots$). For Lie bracket: $[\{\imath_v, d\}, \imath_w] = \imath_{[v,w]_{\mathrm{Lie}}}$

Courant and GCG

- GCG $\mathcal{J} : T \oplus T^* \longrightarrow T \oplus T^*$ $\quad (\mathcal{J}^2 = -1; \quad \mathcal{J}^\dagger \mathcal{I} \mathcal{J} = \mathcal{I})$
  - ◇ Structure group: $\Rightarrow$ U(3,3)

- GCS integrable: $\pi_+[\pi_-(v), \pi_-(w)]_{\rm Lie} = 0 \mapsto \Pi_+[\Pi_-(X), \Pi_-(Y)]_C = 0$ with Courant bracket:

$$[v + \xi,\ w + \eta] = [v, w] + \left\{ \mathcal{L}_v \eta - \mathcal{L}_w \xi - \tfrac{1}{2}\, d(\imath_v \eta - \imath_w \xi) \right\}$$

- Compatibility of two GCS's (Structure group: $\Rightarrow$ U(3)×U(3)): $[\mathcal{J}_1, \mathcal{J}_2] = 0$ $\Rightarrow$

$$G = \mathcal{J}_1 \mathcal{J}_2 = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}$$

  $g \rightarrow$ metric on T of signature $(d, 6-d)$

- Twisting: $d \mapsto d - H\wedge, \qquad [.,.]_C \mapsto [.,.]_C + \iota_v \iota_w H$

Pure spinors and integrability of GCS (GCY)

- i-eigenbundle of $\mathcal{J}$ $L_{\mathcal{J}}$ is of max dimension (6) and is null - max. isotropic.
- Spinor bundle $S = \Lambda^{\bullet} T^*$:
  - ◇ Clifford action on a spinor Φ: $(v+\zeta)\cdot\Phi = v^m \imath_{\partial_m}\Phi + \zeta_m dx^m \wedge \Phi$
  - ◇ $L_\Phi = \{v+\zeta \in T\oplus T^* \,|\, (v+\zeta)\cdot\Phi = 0\}$ is isotropic
    $(((v+\zeta)\cdot(v+\zeta)\Phi) = -(v+\zeta, v+\zeta)\Phi)$
  - ◇ If L_Φ of max. dimension – Φ - pure spinor
  - ◇ If $L_{\mathcal{J}} = L_\Phi \quad \Rightarrow \quad \mathcal{J} \leftrightarrow$ line of pure spinors
- For $A, B \in L_\Phi$, $\quad [A,B]_{\mathrm{C}}\Phi = (AB - BA)\cdot d\Phi$
  - ◇ $d\Phi = (\iota_v + \zeta\wedge)\Phi \qquad \Leftrightarrow \qquad \mathcal{J}$ -integrable
  - ◇ $d\Phi = 0$ ($[A,B]_{\mathrm{C}} \in L_\Phi = L_{\mathcal{J}}$) – GCY condition
- Global pure spinor: structure group on $TX \oplus T^*X$ SU(3,3); a pair of compatible pure spinors ($\Phi_+ \,\&\, \Phi_-$) – SU(3)×SU(3)

Pure spinors and N=1 supersymmetric vacua:

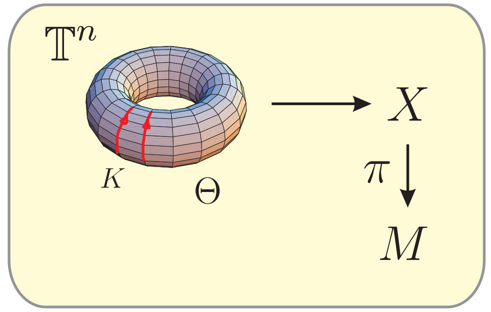
Equations of motion $\Leftrightarrow$

$$\left\{\begin{array}{l}
\diamond\ (d - H\wedge)(e^{2A-\phi}\Phi_1) = 0 \\
\diamond\ (d - H\wedge)(e^{2A-\phi}\Phi_2) = e^{2A-\phi} dA \wedge \bar{\Phi}_2 + \frac{i}{8} e^{3A} * \lambda(F) \\
\qquad\qquad \Rightarrow (d + H\wedge)(e^{4A} * F) = 0 \qquad\qquad (\lambda(F_n) = (-1)^{\mathrm{Int}[n/2]} F_n) \\
\qquad\qquad \Rightarrow d(e^{4A-2\phi} * H) = \mp e^{4A} F_n \wedge * F_{n+2} \quad (\text{for Minkowski vacua}) \\
\diamond\ dH = 0 \\
\diamond\ (d - H\wedge) F = \delta(\text{source})
\end{array}\right.$$

- With IIA $\longrightarrow$ $\Phi_1 = \Phi_+$, $\Phi_2 = \Phi_-$ IIB $\longrightarrow$ $\Phi_1 = \Phi_-$, $\Phi_2 = \Phi_+$
- Use: $\eta^1_+ \otimes \eta^{2\,\dagger}_\pm = \frac{1}{8} \sum_{k=0}^{6} \frac{1}{k!} \left(\eta^{2\,\dagger}_\pm \gamma_{m_k \ldots m_1} \eta^1_+ \right) \gamma^{m_1 \ldots m_k}$
- $\Phi_+ = \eta^1_+ \otimes \eta^{2\,\dagger}_+ \in \Lambda^{\mathrm{ev}} T^*$ and $\Phi_- = \eta^1_+ \otimes \eta^{2\,\dagger}_- \in \Lambda^{\mathrm{odd}} T^*$

$\mathbb{T}^n$ action

- Principal torus bundle $\mathbb{T}^n \hookrightarrow X \xrightarrow{\pi} M$:



- A connection on X is a globally well defined smooth 1-form Θ on X with values in $\mathfrak{t} := \mathrm{Lie}\, \mathbb{T}^n \cong \mathbb{R}^n$.
- Isometries: $\imath_K \Theta = \mathbb{I} \in \mathfrak{t}^* \otimes \mathfrak{t}$ ($\mathcal{L}_K \Theta = 0$) and $\mathcal{L}_K H = 0$
- 3-form H:

$$H = \pi^* H_3 + \langle \pi^* H_2, \Theta \rangle + \frac{1}{2} \langle \pi^* H_1, \Theta \wedge \Theta \rangle + \frac{1}{6} \langle \pi^* H_0, \Theta \wedge \Theta \wedge \Theta \rangle$$

(but $B_\alpha = B_{2\alpha} + \langle B_{1\alpha}, \Theta \rangle + \frac{1}{2} \langle B_{0\alpha}, \Theta \wedge \Theta \rangle$ (No π^*!))

$\longrightarrow$ $H_j \in \Omega^j(M; \Lambda^{3-j}\mathfrak{t})$ for $j = 0, 1, 2, 3$

$\longrightarrow$ $\langle \cdot, \cdot \rangle$: the natural pairing $\mathfrak{t}^* \otimes \mathfrak{t} \longrightarrow \mathbb{R}$

$\longrightarrow$ $dH = 0 \quad \Rightarrow \quad dH_j + \langle H_{j-1}, F \rangle = 0$

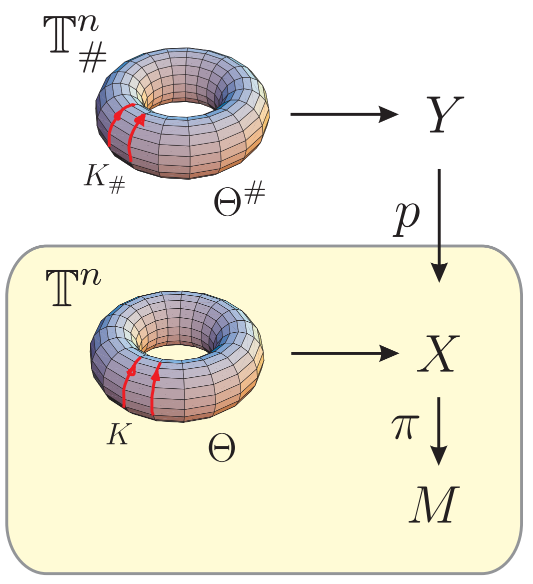
$\mathbb{T}^n$ action $\&$ Courant

- $v = v_M + \langle K, f\rangle$ and $\rho = \rho_M + \langle \phi, \Theta\rangle$
- $\mathcal{L}_K v = 0$ and $\mathcal{L}_K \rho = 0$, $\Rightarrow f \in \Omega^0(M, \mathfrak{t})$ and $\phi \in \Omega^0(M, \mathfrak{t}^*)$. ($\mathbb{T}^n$-invariant section of TX can be written as an element $(v_M, f) \in TM \oplus \mathfrak{t}$, while a $\mathbb{T}^n$-invariant section of TX^* can be written as $(\rho_M, \phi) \in T^*M \oplus \mathfrak{t}^*$.)
- $\imath_v\lambda \quad \Rightarrow \quad \imath((v_M, f))(\lambda_M, \omega) = \imath(v_M)\lambda_M + \langle \omega, f\rangle$
  $d\lambda \quad \Rightarrow \quad d(\lambda_M, \omega) = (d\lambda_M + \langle \omega, F\rangle, -d\omega)$ ....
  $\imath((v_M, f))H = \Big((\imath(v_M)H_3 + \langle H_2, f\rangle), (\imath(v_M)H_2 - \langle H_1, f\rangle), (\imath(v_M)H_1 + \langle H_0, f\rangle)\Big)$
- Courant (problems with O(n,n)):

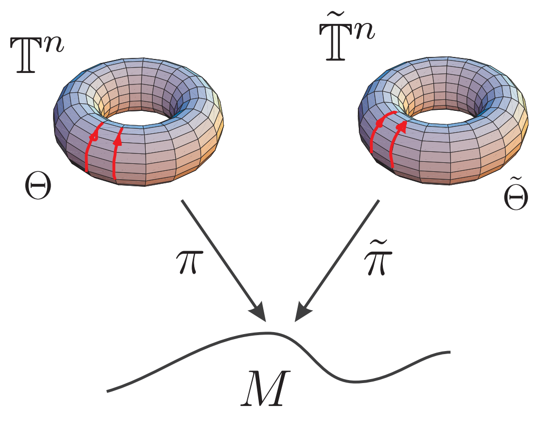
$$[(v_M, f;\rho_M, \phi),\ (w_M, g; \lambda_M, \omega)]_H = [(v_M; \rho_M),\ (w_M; \lambda_M)]_{H_3} +$$
$$\Big(0, \mathcal{L}_{v_M} g - \mathcal{L}_{w_M} f; \langle \omega, df\rangle - \langle \phi, dg\rangle + \frac{1}{2} d(\langle \omega, f\rangle - \langle \phi, g\rangle), \mathcal{L}_{v_M}\omega - \mathcal{L}_{w_M}\phi\Big) +$$
$$\Big(0, \imath_{v_M}\imath_{w_M}F; \langle \omega, \imath_{v_M}F\rangle + \langle \imath_{v_M}F_\#, g\rangle - \langle \imath_{w_M}F_\#, f\rangle - \langle \phi, \imath_{w_M}F\rangle, \imath_{v_M}\imath_{w_M}F_\#\Big) -$$
$$\Big(0, 0; \underline{\langle H_1, [f, g]\rangle}, \langle H_0, [f, g]\rangle\Big)$$

Action of O(n,n) is obstructed for $H_0 \neq 0$ & $H^1(M) \neq 0$!!!

Geometrization of H ($d(\imath_K H) = 0$) gives



instead of



Obstructed T-duality $\leftrightarrow$ "non-geometric backgrounds"

- Courant bracket as the algebra of sections of the generalised tangent bundle E:

$$0 \to T^*X \to E \to TX \to 0$$

$\longrightarrow$ Choice of $B \Rightarrow$ identification with $T \oplus T^*$

$$(v+\xi)|_{X_\alpha} = v_\alpha + (\hat{\xi}_\alpha + i_v B_\alpha).$$

$\longrightarrow$ On two-foold intersection $X_\alpha \cap X_\beta$:

$$\begin{cases} B_\alpha = B_\beta + dA_{\alpha\beta} \\ x_\alpha + \hat{\xi}_\alpha = x_\beta + \hat{\xi}_\beta - i_X dA_{\alpha\beta} \end{cases}$$

$\longrightarrow$ Courant on E $\Rightarrow$ **twisted** Courant on $T \oplus T^*$

$\longrightarrow$ Global spinor on E: $e^{B_\alpha}\Phi_\alpha = e^{B_\beta}\Phi_\beta$

Parallelizable manifolds

- A global basis of vectors and forms:

$$[\tilde{v}_m, \tilde{v}_n] = f^l_{mn}\tilde{v}_l \qquad \text{and} \qquad de^m = -\frac{1}{2} f^m_{ln} e^l \wedge e^n$$

- Passing to E:

$$E_m = \tilde{v}_m - \imath_{\tilde{v}_m} B, \qquad E^m = e^m\,.$$

- Restriction of the Courant bracket to the basis:

$$[E_m, E_n] = f^l_{mn} E_l - H_{lmn} E^l$$
$$[E_m, E^n] = -f^n_{ml} E^l$$
$$[E^m, E^n] = 0$$

- PS equations lift to E

$\rightarrow$ $$(d - H)\Phi \qquad \Leftrightarrow \qquad d\Phi_\alpha = 0$$

- T-duality preserves the form of the equations

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$$e^{B^{(2)}} \Phi^\pm_0 \mapsto e^\beta \tilde{\Phi}^\pm_0$$
$$e^B d(e^{-B}\Phi_0) = d\Phi_0 - H \wedge \Phi_0 \mapsto e^\beta d(e^{-\beta}\tilde{\Phi}_0) = d\tilde{\Phi}_0 - Q \llcorner \tilde{\Phi}_0$$

β - transform and Courant bracket

- Gluing with β transform:

$$(X+\xi)|_{X_\alpha} = (X_\alpha + \xi \lrcorner \beta_\alpha) + \hat{\xi}$$

Cannot identify with $T \oplus T^*$!!! (unless ...)

- New basis:

$$E_m = \tilde{v}_m \,, \qquad E^m = e^m + (e^m \llcorner \beta) = e^m + \beta^{mn}\tilde{v}_n$$

- New algebra:

$$[E_m, E_n] = f^l_{mn} E_l$$
$$[E_m, E^n] = -f^n_{ml} E^l - Q^{nl}_m E_l$$
$$[E^m, E^n] = Q^{mn}_l E^l$$

$\rightarrow$ Twisted tori:

$$f^m_{ln} = \begin{cases} f^a_{\mu\nu} & \text{for } a, b \text{ - along the fibre } \mathbb{T}^d, \quad ([v_a, v_b] = 0);\ \mu, \nu \text{ - along the base} \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ $\beta = \frac{1}{2}\beta^{ab}\imath_a \wedge \imath_b$ (constant β $\mathbb{T}^d$ is a symmetry of the Courant bracket)

- Can take $\beta \rightarrow const$ ($Q \rightarrow 0$) ... unless $H^1(M, \mathbb{Z}) \neq 0$.

General O(n,n)

- The spinor:

$$\Phi^{\pm} = e^{A+B+\beta}\Phi^{\pm}_{(0)}$$

- The algebra:

$$[[E_m, E_n] = f^l_{mn}E_l + H_{mnl}E^l$$
$$[E_m, E^n] = -\tilde{f}^n_{ml}E^l - \tilde{Q}^{nl}_m E_l$$
$$[E^m, E^n] = Q^{mn}_l E^l$$

The conclusion

- For $H^1(M,\mathbb{Z}) \neq 0$, O(n,n) action is obstructed, and no restriction to $T \oplus T^*$ (of anything!) is possible. For $H^1(M,\mathbb{Z}) = 0$, can think of a twisted Courant on E (i.e. choose $Q \to 0$).