UV to IR Evolution of Quasi-Scale-Invariant Gauge Theories and Comparisons with Lattice Measurements

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Univ. of Colorado Workshop: Lattice Meets Experiment: Beyond the SM, Oct. 27, 2012

Outline

- Renormalization-group flow from UV to IR; types of IR behavior; role of an exact or approximate IR fixed point; conditions for approximately scale-invariant behavior

- Higher-loop calculations of UV to IR evolution, including IR zero of β and anomalous dimension γ_m of fermion bilinear

- Some comparisons with lattice measurements of γ_m

- Study of scheme-dependence in calculation of IR fixed point

- Application to models of dynamical electroweak symmetry breaking

- Conclusions

Some new results covered in this talk are from the following recent papers by T. A. Ryttov and R. Shrock, which will also be covered in Thomas Ryttov's talk:

- Phys. Rev. D 83, 056011 (2011), arXiv:1011.4542
- Phys. Rev. D 85, 076009 (2012), arXiv:1202.1297
- Phys. Rev. D 86, 065032 (2012), arXiv:1206.2366
- Phys. Rev. D 86, 085005 (2012), arXiv:1206.6895

as well as earlier related papers.

Renormalization-group Flow from UV to IR; Types of IR Behavior and Role of IR Fixed Point

Consider an asymptotically free, vectorial gauge theory with gauge group G and N_f massless fermions in representation R of G.

The asymptotic freedom property means theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale μ in deep ultraviolet (UV).

The question of how this theory behaves in the infrared (IR) is of fundamental field-theoretic significance. This motivates a detailed study of the UV to IR evolution.

The results are relevant to models of dynamical electroweak symmetry breaking (discussed further below).

Denote running gauge coupling at scale μ as $g = g(\mu)$, and let $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $a(\mu) = g(\mu)^2/(16\pi^2) = \alpha(\mu)/(4\pi)$.

As theory evolves from the UV to the IR, $\alpha(\mu)$ increases, as governed by the beta function

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ ,$$

where $t = \ln \mu$, $\ell =$ loop order of the coefficient, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$.

Coefficients b_1 and b_2 in β are independent of the regularization/renormalization scheme, while b_ℓ for $\ell \geq 3$ are scheme-dependent.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$.

As the scale μ decreases from large values, $\alpha(\mu)$ increases. Denote α_{cr} (dependent on R) as minimum value for formation of bilinear fermion condensates and resultant spontaneous chiral symmetry breaking (SχSB).

There are two possibilities for the β function and resultant UV to IR evolution:

- There may not be any IR zero in β, so that as μ decreases, $\alpha(\mu)$ increases, eventually beyond the region where one can calculate it perturbatively in a self-consistent manner. This is the case for QCD.
- β may have a zero at a certain value (closest to the origin) denoted α_{IR}, so that as μ decreases, $\alpha \longrightarrow \alpha_{IR}$ (Caswell, Banks+Zaks). In this class of theories, there are two further generic possibilities: $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$.

If $\alpha_{IR} < \alpha_{cr}$, the zero of β at α_{IR} is an exact IR fixed point (IRFP) of the ren. group; as $\mu \longrightarrow 0$ and $\alpha \longrightarrow \alpha_{IR}$, $\beta \longrightarrow \beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant.

If β has no zero, or if β has an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as μ decreases through a scale denoted Λ, $\alpha(\mu)$ exceeds α_{cr} and SχSB occurs. The fermions then gain dynamical masses $\sim \Lambda$ (e.g., light quarks gain constituent quark masses $\sim \Lambda_{QCD} \simeq 300$ MeV in QCD).

If SχSB occurs, then in low-energy effective field theory applicable for $\mu < \Lambda$, one integrates these fermions out, and β function becomes that of a pure gauge theory, which has no IR zero. Hence, in the case where β has a zero at $\alpha_{IR} > \alpha_{cr}$, this is only an approximate IRFP.

If $\alpha_{IR} > \alpha_{cr}$, the effect of the approximate IRFP at α_{IR} on the behavior of the theory depends on how close it is to α_{cr}.

If α_{IR} is only slightly greater than α_{cr}, then, as $\alpha(\mu)$ approaches α_{IR}, since $\beta = d\alpha/dt \to 0$, $\alpha(\mu)$ varies very slowly as a function of the scale μ, i.e., there is approximately scale-invariant, i.e. dilatation-invariant or slow-running ("walking") behavior. For these theories, this is equivalent to quasiconformal behavior.

Denote Λ_* as scale μ where $\alpha(\mu)$ grows to O(1) (with Λ the scale where SχSB occurs). In the slow-running case, $\Lambda << \Lambda_*$. The approximate dilatation symmetry applies in this interval $\Lambda << \mu << \Lambda_*$.

The SχSB and attendant fermion mass generation at Λ spontaneously break the approximate dilatation symmetry, plausibly leading to a resultant light Nambu-Goldstone boson, the dilaton (dilaton mass estimates vary). The dilaton is not massless, because β is not exactly zero for $\alpha(\mu) \neq \alpha_{IR}$.

At the two-loop (2ℓ) level, $\beta = -[\alpha^2/(2\pi)](b_1 + b_2 a)$, so the condition for an IR zero in β is $b_1 + b_2 a = 0$, i.e.,

$$\alpha_{IR,2\ell} = -\frac{4\pi b_1}{b_2}$$

which is physical for $b_2 < 0$. One-loop coefficient b_1 is

$$b_1 = \frac{1}{3}(11 C_A - 4 N_f T_f)$$

(Gross, Wilczek; Politzer), where $C_A \equiv C_2(G)$ is the quadratic Casimir invariant, and $T_f \equiv T(R)$ is the trace invariant. We focus here on $G = \mathrm{SU}(N)$; more general groups discussed in T. Ryttov's talk.

As N_f increases, b_1 decreases and vanishes at

$$N_{f,b1z} = \frac{11 C_A}{4 T_f}$$

Hence, for asymptotic freedom, require $N_f < N_{f,b1z}$; for fund. rep., this is $N_f < (11/2)N$.

Two-loop coefficient b_2 is

$$b_2 = \frac{1}{3}\left[34C_A^2 - 4(5C_A + 3C_f)N_f\,T_f\right]$$

(Caswell, Jones). For small N_f, $b_2 > 0$; b_2 decreases with increasing N_f and vanishes with sign reversal at $N_f = N_{f,b2z}$, where

$$N_{f,b2z} = \frac{34C_A^2}{4T_f(5C_A + 3C_f)} \ .$$

For arbitrary G and R, $N_{f,b2z} < N_{f,b1z}$, so there is always an interval in N_f for which β has an IR zero, namely

$$I: \quad N_{f,b2z} < N_f < N_{f,b1z}$$

If $R =$ fund. rep., then

$$I: \quad \frac{34N^3}{13N^2-3} < N_f < \frac{11N}{2}$$

For example, for $N = 2$, this is $5.55 < N_f < 11$, and for $N = 3$, $8.05 < N_f < 16.5$. (Here, we evaluate these expressions as real numbers, but understand that the physical values of N_f are nonnegative integers.)

As $N \to \infty$, interval I is $2.62N < N_f < 4.5N$.

For N_f near the lower end of I, $b_2 \to 0$ and $\alpha_{IR,2\ell}$ is too large for the calculation to be reliable.

In the interval I, α_{IR} is a decreasing function of N_f. As N_f decreases below $N_{f,b1z}$ where $b_1 = 0$, α_{IR} increases from 0. As N_f decreases to a value denoted $N_{f,cr}$, α_{IR} increases to α_{cr}, so

$$N_f = N_{f,cr} \quad \text{at} \quad \alpha_{IR} = \alpha_{cr}$$

The value of $N_{f,cr}$ is of fundamental importance in the study of a non-Abelian gauge theory, since it separates two different regimes of IR behavior, viz., an IR conformal phase with no SχSB and an IR phase with SχSB.

With a given G, regarding N_f as a variable, this is thus a (zero-temperature) chiral transition (Appelquist, Wijewardhana).

$N_{f,cr}$ is not exactly known. To obtain $N_{f,cr}$ for a given gauge group, we need, as inputs, calculations of α_{IR} as function of N_f and an estimate of α_{cr}.

To estimate α_{cr}, analyze Dyson-Schwinger (DS) equation for the fermion propagator. For $\alpha > \alpha_{cr}$, this yields a nonzero sol. for a dynamically generated fermion mass.

Ladder approx. to DS eq. yields $3\alpha_{cr}C_2(R)/\pi = 1$. Given the strong-coupling nature of the physics, this is only a rough estimate. Corrections to ladder approx. studied by several groups (Appelquist, Lane, Mahanta..)

Although DS eq. ignores confinement and instantons, the corrections from including these tend to cancel each other in estimate for α_{cr}; DS eq. takes Euclidean loop integration interval as $0 \le k \le \infty$, but confinement produces minimum bound state momentum $k \sim \pi/\Lambda$, decreases loop integration interval, while instantons enhance SχSB (Brodsky and Shrock, Phys. Lett. B 666, 95 (2008), arXiv:0806.1535).

Combining estimate of α_{cr} from ladder approx. to DS eq. with 2-loop calculation of $\alpha_{IR} \equiv \alpha_{IR,2\ell}$ yields $N_{f,cr} \simeq 4N$.

Lattice gauge simulations provide promising way to determine $N_{f,cr}$ and measurement of anomalous dimension $\gamma \equiv \gamma_m$ describing running of m and bilinear operator, $\bar{F}F$ as a function of $\ln\mu$. Intensive current work on this.

Higher-Loop Corrections to UV $\longrightarrow$ IR Evolution of Gauge Theories

Because of the strong-coupling nature of the physics at an approximate IRFP, with $\alpha \sim O(1)$, there are significant higher-order corrections to results obtained from the two-loop β function.

This motivates calculation of location of IR zero in β, α_{IR}, and resultant value of γ evaluated at α_{IR} to higher-loop order. We have done this to 3-loop and 4-loop order in Ryttov and Shrock, PRD 83, 056011 (2011), arXiv:1011.4542; see also Pica and Sannino, PRD 83,035013 (2011), arXiv:1011.5917.

Although coeffs. in β at $\ell \geq 3$ loop order are scheme-dependent, results give a measure of accuracy of the 2-loop calc. of the IR zero, and similarly with the value of γ evaluated at this IR zero.

We use $\overline{MS}$ scheme, for which coeffs. of β and γ have been calculated to 4-loop order by Vermaseren, Larin, and van Ritbergen. The value of this sort of higher-loop calcululation using $\overline{MS}$ scheme is demonstrated by the excellent fit of the four-loop $\alpha_s(\mu)$ to data as function of $\mu^2 = Q^2$ in QCD (cf. Bethke).

For 3-loop analysis, we need

$$b_3 = \frac{2857}{54} C_A^3 + T_f N_f \left[2C_f^2 - \frac{205}{9} C_A C_f - \frac{1415}{27} C_A^2 \right] + (T_f N_f)^2 \left[\frac{44}{9} C_f + \frac{158}{27} C_A \right]$$

Coefficient b_3 is quadratic function of N_f and vanishes, with sign reversal, at two values of N_f, denoted $N_{f,b3z,1}$ and $N_{f,b3z,2}$. $b_3 > 0$ for small N_f and vanishes first at $N_{f,b3z,1}$, which is smaller than $N_{f,b2z}$, the left endpoint of interval I. Furthermore, $N_{f,b3z,2} > N_{f,b1z}$, the right endpoint of interval I. For example,

$$\text{for} \quad N = 2, \quad N_{f,b3z,1} = 3.99 < N_{f,b2z} = 5.55$$

$$N_{f,b3z,2} = 27.6 > N_{f,b1z} = 11$$

$$\text{for} \quad N = 3, \quad N_{f,b3z,1} = 5.84 < N_{f,b2z} = 8.05$$

$$N_{f,b3z,2} = 40.6 > N_{f,b1z} = 16.5$$

Hence, $b_3 < 0$ in interval I of interest for IR zero of β.

At this 3-loop level,

$$\beta = -\frac{\alpha^2}{2\pi}(b_1 + b_2 a + b_3 a^2)$$

so $\beta = 0$ away from $\alpha = 0$ at two values,

$$\alpha = \frac{2\pi}{b_3}\Big(- b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}\ \Big)$$

Since $b_2 < 0$ and $b_3 < 0$, this is

$$\alpha = \frac{2\pi}{|b_3|}\Big(- |b_2| \mp \sqrt{b_2^2 + 4b_1 |b_3|}\ \Big)$$

One of these solutions is negative and hence unphysical; the other is manifestly positive, and is $\alpha_{IR,3\ell}$

We find that for any fermion rep. R for which β has a 2-loop IR zero, the value of the IR zero decreases when calculated at the 3-loop level, i.e.,

$$\alpha_{IR,2\ell} > \alpha_{IR,3\ell}$$

Proof:

$$\alpha_{IR,2\ell} - \alpha_{IR,3\ell} = \frac{4\pi b_1}{|b_2|} - \frac{2\pi}{|b_3|}(-|b_2| + \sqrt{b_2^2 + 4b_1|b_3|}\ )$$
$$= \frac{2\pi}{|b_2 b_3|}\Big[2b_1|b_3| + b_2^2 - |b_2|\sqrt{b_2^2 + 4b_1|b_3|}\ \Big]$$

The expression in square brackets is positive if and only if

$$(2b_1|b_3| + b_2^2)^2 - b_2^2(b_2^2 + 4b_1|b_3|) > 0$$

This difference is equal to the positive-definite quantity $4b_1^2 b_3^2$, which proves the inequality.

For the 4-loop analysis, we use b_4, which is a cubic polynomial in N_f. It is positive for $N_f \in I$ for $N = 2, 3$ but is negative in part of I for higher N.

The 4-loop β function is $\beta = -[\alpha^2/(2\pi)](b_1 + b_2 a + b_3 a^2 + b_4 a^3)$, so β has three zeros away from the origin. We determine the smallest positive real zero as $\alpha_{IR,4\ell}$.

We find

- As noted, when one goes from 2-loop level to 3-loop level, there is a decrease in the value of the IR zero of β

- As one goes from 3-loop to 4-loop level, there is a slight change in the value of the IR zero, but the change is smaller than the decrease from 2-loops to 3-loops, so $\alpha_{IR,4\ell} < \alpha_{IR,2\ell}$.

- The fractional changes in the value of the IR zero of β decrease in magnitude as N_f increases toward its maximum, $N_{f,b1z}$, and all of the values of $\alpha_{IR,n\ell} \to 0$.

Our finding that the fractional change in the location of the IR zero of β is reduced at higher-loop order agrees with the general expectation that calculating a quantity to higher order in perturbation theorydy. should give a more stable and accurate result.

Given $\alpha_{cr} \sim O(1)$ for SχSB, the decrease in α_{IR} at higher-loop order, together with the property that α_{IR} increases as N_f decreases, suggests that the actual lower boundary of the IR-conformal phase could lie somewhat below the estimate that $N_{f,cr} \simeq 4N$ from the 2-loop $\alpha_{IR,2\ell}$ plus DS eq. This is also suggested by our SUSY study (T. Ryttov's talk).

Several lattice simulations of various cases also find this; e.g., for $N = 3$, lower boundary of IR-conformal phase is somewhat below $4N = 12$ (although other groups argue that $N_f = 12$ is in SχSB phase).

Some numerical values of $\alpha_{IR,n\ell}$ at the 2-loop, 3-loop, and 4-loop level for fermions in fund. rep., $N_f \in I$, and illustrative groups $G = \mathrm{SU}(2)$ and $G = \mathrm{SU}(3)$:

N	N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$
2	7	2.83	1.05	1.21
2	8	1.26	0.688	0.760
2	9	0.595	0.418	0.444
2	10	0.231	0.196	0.200
3	10	2.21	0.764	0.815
3	11	1.23	0.578	0.626
3	12	0.754	0.435	0.470
3	13	0.468	0.317	0.337
3	14	0.278	0.215	0.224
3	15	0.143	0.123	0.126
3	16	0.0416	0.0397	0.0398

(For N_f values sufficiently close to $N_{f,b1z}$, $\alpha_{IR,n\ell}$ is so large that the perturbative calculation is not reliable; these are omitted.)

We have performed the corresponding higher-loop calculations for SU(N) gauge theories with N_f fermions in the adjoint, symmetric and antisymmetric rank-2 tensor representations. The general result $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ applies. The difference $\alpha_{IR,4\ell} - \alpha_{IR,3\ell}$ tends to be relatively small, but can have either sign.

For example, for $R =$ adjoint, $N_{f,b1z} = 11/4$ and $N_{f,b2z} = 17/16$ (indep. of N), so interval I where β has an IR zero, viz., $N_{f,b2z} < N_f < N_{f,b1z}$, is $1.06 < N_f < 2.75$, which includes only one physical, integral value, $N_f = 2$. For this value of N_f and some illustrative values of N, the results are:

N	$\alpha_{IR,2\ell,adj}$	$\alpha_{IR,3\ell,adj}$	$\alpha_{IR,4\ell,adj}$
2	0.628	0.459	0.450
3	0.419	0.306	0.308
4	0.314	0.2295	0.234

The anomalous dimension $\gamma_m \equiv \gamma$ for the fermion bilinear operator is

$$\gamma = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \Big(\frac{\alpha}{\pi}\Big)^\ell$$

where $\bar{c}_\ell = c_\ell/4^\ell$ is the ℓ-loop coeff. The one-loop coeff. c_1 is scheme-independent, the c_ℓ with $\ell \geq 2$ are scheme-dependent, and the c_ℓ have been calculated up to 4-loop level (Vermaseren, Larin, van Ritbergen):

$$c_1 = 6C_f$$

$$c_2 = 2C_f\Big[\frac{3}{2}C_f + \frac{97}{6}C_A - \frac{10}{3}T_f N_f\Big]$$

$$c_3 = 2C_f\Big[\frac{129}{2}C_f^2 - \frac{129}{4}C_f C_A + \frac{11413}{108}C_A^2 + C_f T_f N_f(-46 + 48\zeta(3)) - C_A T_f N_f(\frac{556}{27} + 48\zeta(3)) - \frac{140}{27}(T_f N_f)^2\Big]$$

and similarly for c_4.

It is of interest to calculate γ at the exact IRFP in the IR conformal phase and the approximate IRFP in the phase with SχSB.

We denote γ calculated to n-loop ($n\ell$) level as $\gamma_{n\ell}$ and, evaluated at the n-loop value of the IR zero of β, as

$$\gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha = \alpha_{IR,n\ell})$$

N.B.: In the IR conformal phase, an all-order calc. of γ evaluated at an all-order calc. of α_{IR} would be an exact property of the theory, but in the broken phase, just as the IR zero of β is only an approximate IRFP, so also, the γ is only approx., describing the running of $\bar{\psi}\psi$ and the dynamically generated fermion mass near the zero of β:

$$\Sigma(k) \sim \Lambda\Big(\frac{\Lambda}{k}\Big)^{2-\gamma}$$

In both phases, γ is bounded above as $\gamma < 2$. At the 2-loop level we calculate $\gamma_{IR,2\ell} =$

$$\frac{C_f(11C_A - 4T_fN_f)[455C_A^2 + 99C_AC_f + (180C_f - 248C_A)T_fN_f + 80(T_fN_f)^2]}{12[-17C_A^2 + 2(5C_A + 3C_f)T_fN_f]^2}$$

Our analytic expressions for $\gamma_{IR,n\ell}$ at the 3-loop and 4-loop level are too complicated to list here. Illustrative numerical values of $\gamma_{IR,n\ell}$ at the 2-, 3-, and 4-loop level are given below for fermions in the fund. rep. and for the illustrative values $N = 2,\ 3$.

N	N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
2	7	(2.67)	0.457	0.0325
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	10	(4.19)	0.647	0.156
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259

(Two-loop values in parentheses for N_f in lower part of interval I are unphysically large, reflect inadequacy of lowest-order perturbative calculation in this region.)

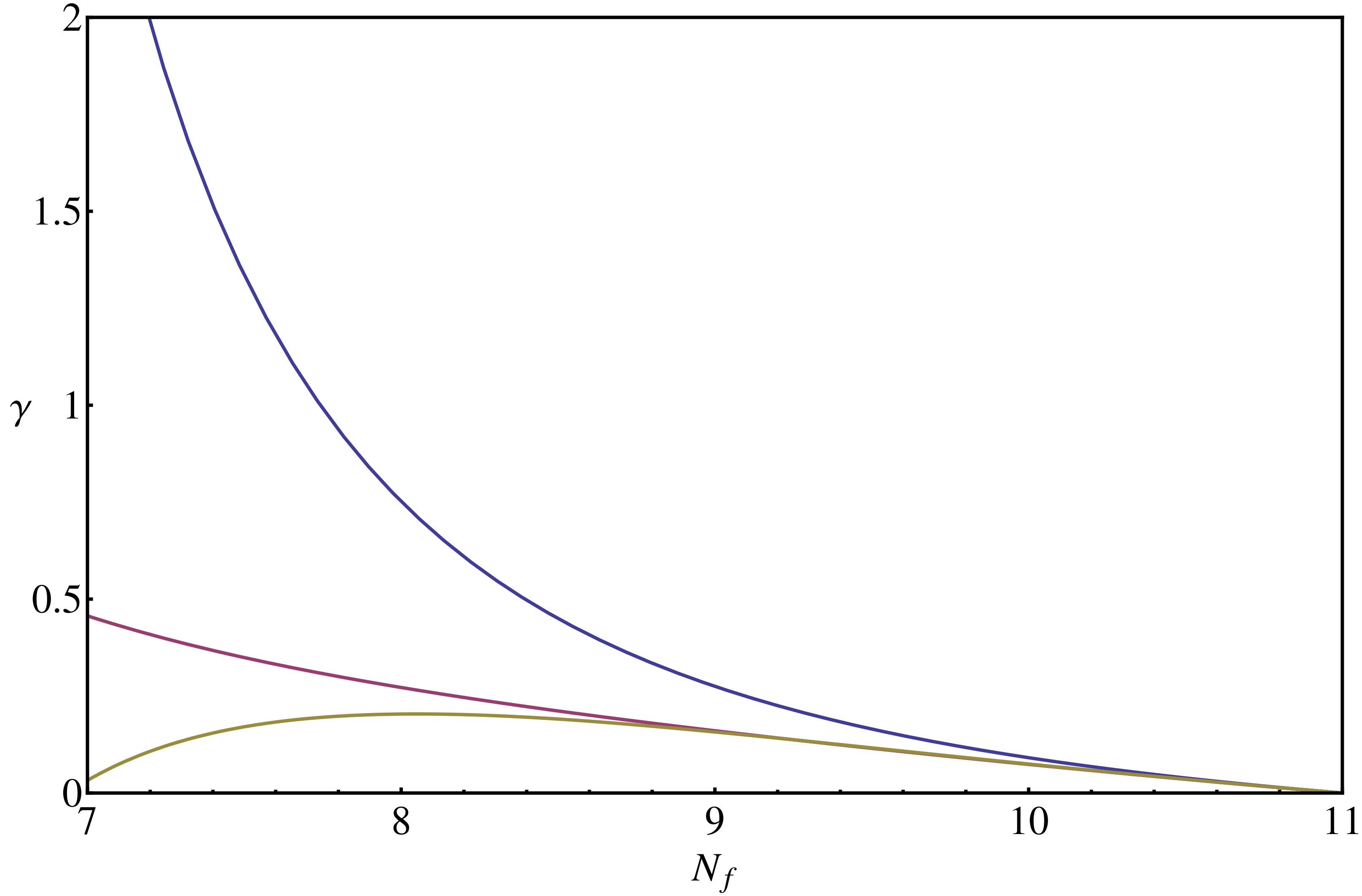

Figure 1: Anomalous dimension $\gamma_m \equiv \gamma$ for SU(2) for N_f fermions in the fundamental representation; (i) blue: $\gamma_{IR,2\ell}$; (ii) red: $\gamma_{IR,3\ell}$; (iii) brown: $\gamma_{IR,4\ell}$.

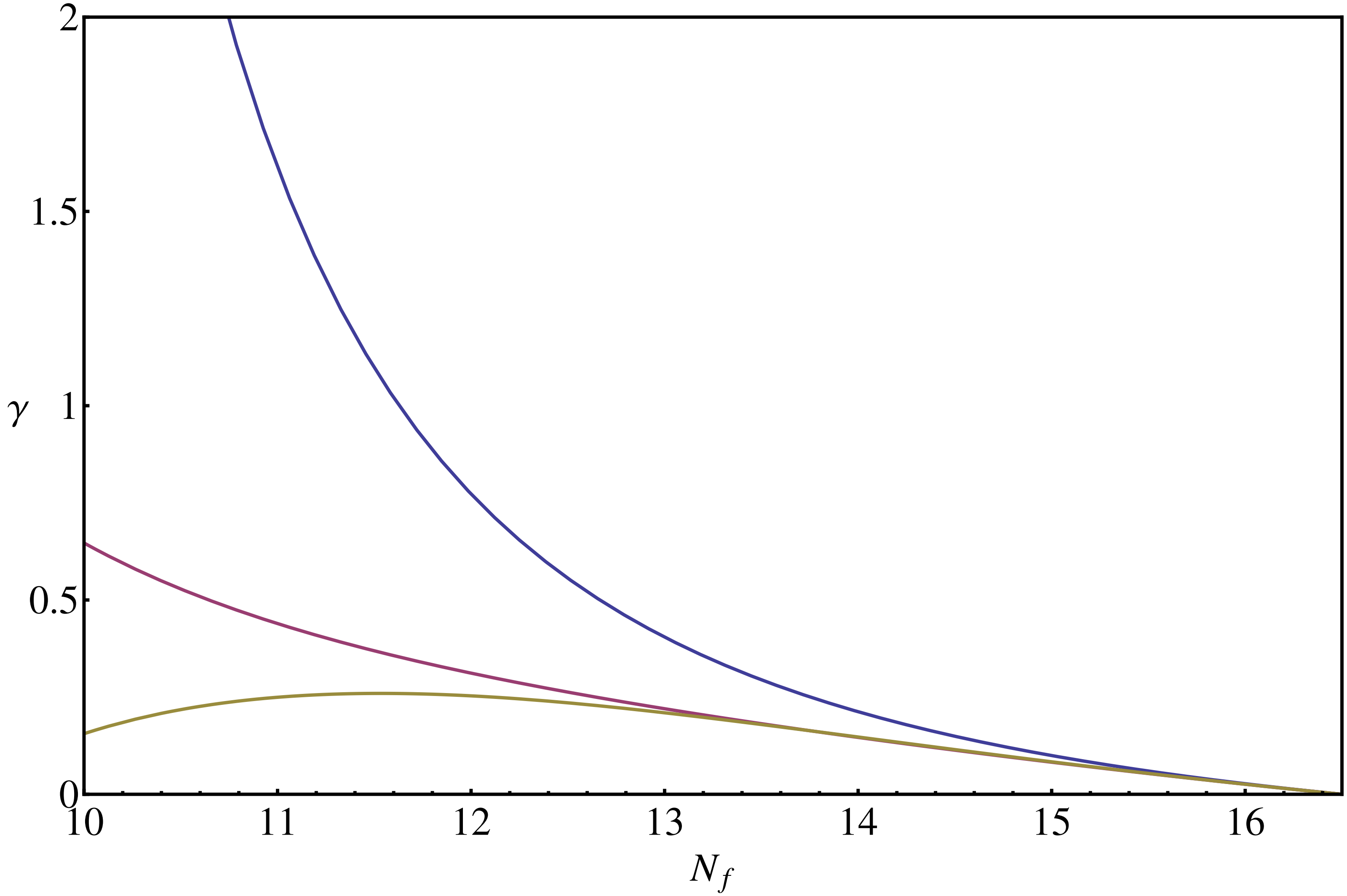

Figure 2: Anomalous dimension $\gamma_m \equiv \gamma$ for SU(3) for N_f fermions in the fundamental representation; (i) blue: $\gamma_{IR,2\ell}$; (ii) red: $\gamma_{IR,3\ell}$; (iii) brown: $\gamma_{IR,4\ell}$.

We have also performed these higher-loop calculations for higher fermion reps. R. In general, we find that, for a given N, R, and N_f, the values of $\gamma_{IR,n\ell}$ calculated to 3-loop and 4-loop order are smaller than the 2-loop value.

The value of these higher-loop calcs. to 3-loop and 4-loop order is evident from the figures. A necessary condition for a perturbative calculation to be reliable is that higher-order contribs. do not modify the result too much. One sees from the tables and figures that, especially for smaller N_f, there is a substantial decrease in $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$ when one goes from 2-loop to 3-loop order, but for a reasonable range of N_f, the 3-loop and 4-loop results are close to each other.

Thus, our higher-loop calculations of α_{IR} and γ allow us to probe the theory reliably down to smaller values of N_f and thus stronger couplings. Of course, for sufficiently small N_f in interval I, α_{IR} becomes too large for perturbative calc. to be reliable.

Some Comparisons with Lattice Measurements

For SU(3) with $N_f = 12$, from table above,

$$\gamma_{IR,2\ell} = 0.77, \qquad \gamma_{IR,3\ell} = 0.31, \qquad \gamma_{IR,4\ell} = 0.25$$

Some lattice results (N.B.: some error estimates do not include all syst. uncertainties)

$\gamma = 0.414 \pm 0.016$ (Appelquist, Fleming, Lin, Neil, Schaich, PRD 84, 054501 (2011), arXiv:1106.2148, analyzing data of Kuti et al., PLB 703, 348 (2011), arXiv:1104.3124, inferring conformality [Kuti et al. find SχSB])

$\gamma \sim 0.35$ (DeGrand, PRD 84, 116901 (2011), arXiv:1109.1237, also analyzing data of Kuti et al., finding conformality)

$0.2 \lesssim \gamma \lesssim 0.4$ (Fodor, Holland, Kuti, Nogradi, Schroeder, Wong, arXiv:1205.1878, finding SχSB)

$\gamma = 0.4 - 0.5$ (Y. Aoki et al., (LatKMI) PRD 86, 054506 (2012), arXiv:1207.3060, finding IR-conformality)

$\gamma = 0.27 \pm 0.03$ (Hasenfratz, Cheng, Petropoulos, Schaich, arXiv:1207.7162, finding IR-conformality)

So here the 2-loop value is larger than, and the 3-loop and 4-loop values closer to, these lattice measurements. Thus, our higher-loop calcs. of γ yield better agreement with these lattice measurements than the two-loop calculation.

This SU(3) theory with $N_f = 12$ fermions in fund. rep. was found to be in the IR-conformal phase by Appelquist et al. (PRL, 100, 171607 (2008)); other studies by Deuzeman, Lombardo, Pallante; Hasenfratz et al.; Degrand et al.; Aoki et al. also find IR-conformality, while Kuti et al. and Jin and Mawhinney argue for SχSB.

For SU(3) with $N_f = 10$ fermions in fund. rep., Appelquist et al., LSD Collab., arXiv:1204.6000 get $\gamma_{IR} \sim 1$, consistent with idea that $\gamma_{IR} \simeq 1$ at lower end of IR-conformal phase.

Similar comparisons can be carried out for SU(2) with N_f fermions in fund. rep. Lattice studies indicate that for SU(2), $N_f = 10$ is in IR-conformal phase and $N_f = 4$ is in SχSB phase; $N_f = 6,\ 8$ are also being considered, e.g., Bursa et al., PRD 84, 034506 (2011), arXiv:1104.4301; Karavirta, Rantaharju, Rummukainen, Tuominen, JHEP 1205, 003 (2012), arXiv:1111.4104; Hayakawa, Ishikawa, Osaki, Takeda, Yamada, arXiv:1210.4985; G. Voronov and LSD Collab., in progress.

Our results for some higher fermion reps.: For $R =$ adj. rep., interval I contains only the integer $N_f = 2$. For this we get

N	$\gamma_{IR,2\ell,adj}$	$\gamma_{IR,3\ell,adj}$	$\gamma_{IR,4\ell,adj}$
2	0.820	0.543	0.500
3	0.820	0.543	0.523
4	0.820	0.543	0.532

For SU(2) with $N_f = 2$ fermions in the adjoint rep., lattice results include (N.B.: various groups quote uncertainties differently):

$\gamma = 0.31 \pm 0.06$ DeGrand, Shamir, Svetitsky, PRD 83, 074507 (2011), arXiv:1102.2843

$\gamma = 0.17 \pm 0.05$ (Appelquist et al., PRD 84, 054501 (2011) (analyzing data of Bursa, Del Debbio et al.), arXiv:1106.2148)

$-0.6 < \gamma < 0.6$ (Catterall, Del Debbio, Giedt, Keegan, PRD 85, 094501 (2012), arXiv:1108.3794)

Case of SU(N) with fermions in symmetric rank-2 tensor rep. (for SU(2), this is equiv. to adjoint rep.) Here,

$$N_{f,b1z} = \frac{11N}{2(N+2)} \ , \quad N_{f,b2z} = \frac{17N^2}{(N+2)(8N+3-6N^{-1})}$$

and interval I is $N_{f,b2z} < N_f < N_{f,b1z}$;

$$N=3: \quad 1.22 < N_f < 3.30 \ , \quad \Longrightarrow \ N_f = 2, \ 3$$

$$N=4: \quad 1.35 < N_f < 3.67 \ , \quad \Longrightarrow \ N_f = 2, \ 3$$

(as $N \to \infty$, $2.125 < N_f < 4.5, \quad \Longrightarrow \ N_f = 3, \ 4$).

Analytic expressions are given in our paper; here, only list numerical values.

N	N_f	$\alpha_{IR,2\ell,S2}$	$\alpha_{IR,3\ell,S2}$	$\alpha_{IR,4\ell,S2}$
3	2	0.842	0.500	0.470
3	3	0.085	0.079	0.079
4	2	0.967	0.485	0.440
4	3	0.152	0.129	0.131

N	N_f	$\gamma_{IR,2\ell,S2}$	$\gamma_{IR,3\ell,S2}$	$\gamma_{IR,4\ell,S2}$
3	2	(2.44)	1.28	1.12
3	3	0.144	0.133	0.133
4	2	(4.82)	(2.08)	1.79
4	3	0.381	0.313	0.315

Some lattice results for $N_f = 2$ fermions in this symmetric rank-2 tensor rep.:

e.g., SU(3), $N_f = 2$: here, need to resolve a difference between two groups on the presence of absence of SχSB and value of γ before comparison with our continuum higher-loop calculations:

$\gamma \lesssim 0.45$ (Degrand, Shamir, Svetitsky, arXiv:1201.0935, find IR-conformality)

$\gamma \sim 1.5$ (Fodor, Holland, Kuti, et al., arXiv:1205.1878, find SχSB)

It is of interest to carry out a similar analysis in an asymptotically free $\mathcal{N} = 1$ supersymmetric gauge theory with vectorial chiral superfield content $\Phi_i, \ \tilde{\Phi}_i$, $i = 1, ..., N_f$ in the $R, \ \bar{R}$ reps. for various R, since here $N_{f,cr}$ is known exactly.

We have done this in Ryttov and Shrock, Phys. Rev. D 85, 076009 (2012), arXiv:1202.1297.

This will be discussed as part of Thomas Ryttov's talk.

Study of Scheme-Dependence in Calculation of IR Fixed Point

Since the coeffs. in β at 3-loops and higher are scheme-dependent, so is the resultant value of $\alpha_{IR,n\ell}$ calculated to a (finite-loop order) of $n \geq 3$ loops. It is important to assess quantitatively the uncertainty due to this scheme dependence.

A way to do this is to perform scheme transformations and determine how much of a change there is in $\alpha_{IR,n\ell}$. We have carried out this study in Ryttov and Shrock, PRD 86, 065032 (2012), arXiv:1206.2366; PRD 86, 085005 (2012), arXiv:1206.6895.

A scheme transformation (ST) is a map between α and α' or equivalently, a and a', where $a = \alpha/(4\pi)$, which can be written as

$$a = a' f(a')$$

with $f(0) = 1$ to keep the UV properties unchanged. Considering STs analytic about $a = 0$, we write

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s = 1 + \sum_{s=1}^{s_{max}} \bar{k}_s (\alpha')^s \ ,$$

where the k_s are constants, $\bar{k}_s = k_s/(4\pi)^s$, and s_{max} may be finite or infinite.

Hence, the Jacobian $J = da/da' = d\alpha/d\alpha'$ satisfies $J = 1$ at $a = a' = 0$. We have

$$\beta_{\alpha'} \equiv \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha}\frac{d\alpha}{dt} = J^{-1}\beta_\alpha \ .$$

This has the expansion

$$\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell = -2\alpha' \sum_{\ell=1}^{\infty} \bar{b}'_\ell (\alpha')^\ell \ ,$$

where $\bar{b}'_\ell = b'_\ell/(4\pi)^\ell$.

Using these two equiv. expressions for $\beta_{\alpha'}$, one can solve for the b'_ℓ in terms of the b_ℓ and k_s. This leads to the well-known result that

$$b'_1 = b_1 \ , \quad b'_2 = b_2$$

i.e, the one-loop and two-loop terms in β are scheme-independent.

To assess the scheme-dependence of an IRFP, we have calculated the relations between the b'_ℓ and b_ℓ for higher ℓ values. For example, for $\ell = 3,\ 4,\ 5$, we obtain

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1 \ ,$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1$$

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2) b_3 + (-k_1^3 + 3k_1 k_2 - k_3) b_2$$
$$+ (4k_1^4 - 11k_1^2 k_2 + 6k_1 k_3 + 4k_2^2 - 3k_4) b_1$$

Since the β function coefficients are scheme-dependent, there should exist a ST in which one can make all coeffs. at $\ell \geq 3$ loops vanish ('t Hooft). We constructed an explicit ST that can does this at a UVFP.

To be physically acceptable, a ST must satisfy several conditions, C_i. For finite s_{max}, the ST is is an algebraic eq. of degree $s_{max} + 1$ for α' in terms of α. We require that at least one of the $s_{max} + 1$ roots must satisfy these conditions. For $s_{max} = \infty$, the eq. for α' in terms of α is generically transcendental, and again we require that the relevant sol. must satisfy these conditions, which are:

- C_1: the ST must map a real positive α to a real positive α', since a map taking $\alpha > 0$ to $\alpha' = 0$ would be singular, and a map taking $\alpha > 0$ to a negative or complex α' would violate the unitarity of the theory.
- C_2: the ST should not map a moderate value of α, for which pert. theory may be reliable, to an excessively large value of α' where pert. theory is inapplicable

- C_3: J should not vanish in the region of α and α' of interest, or else there would be a pole in the relation between β_α and $\beta_{\alpha'}$.
- C_4: The existence of an IR zero of β is a scheme-independent property, depending (in an AF theory) only on the condition that $b_2 < 0$. Hence, a ST should satisfy the condition that β_α has an IR zero if and only if $\beta_{\alpha'}$ has an IR zero.

These four conditions can always be satisfied by STs in the vicinity of a UV fixed point, and hence in applications to pert. QCD calculations, since α is small, and one can choose the k_s to be small also, so $\alpha' \simeq \alpha$.

However, these conditions C1-C4 are not automatically satisfied, and are a significant constraint, on a ST applied in the vicinity of an IRFP, where α may be O(1).

For example, consider the ST

$$\alpha = \tanh(\alpha')$$

with inverse

$$\alpha' = \frac{1}{2} \ln\Big(\frac{1+\alpha}{1-\alpha}\Big)$$

If $\alpha << 1$, as at a UVFP, this is acceptable, but if α exceeds 1, even if by a small amount, then it is unacceptable, since it maps a real positive α to a complex α'.

We have studied scheme dependence of the IR zero of β using several STs. For example, we have used the ST (depending on a parameter r)

$$S_{sh,r}: \quad a = \frac{\sinh(ra')}{r}$$

Since $\sinh(ra')/r$ is an even function of r, we take $r > 0$ with no loss of generality.

This has the inverse

$$a' = \frac{1}{r} \ln\left[ra + \sqrt{1+(ra)^2}\,\right]$$

and the Jacobian

$$J = \cosh(ra')$$

For this ST,

$$f(a') = \frac{\sinh(ra')}{ra'} .$$

This has a series expansion with $k_s = 0$ for odd s and for even s,

$$k_2 = \frac{r^2}{6}, \quad k_4 = \frac{r^4}{120}$$

$$k_6 = \frac{r^6}{5040}, \quad k_8 = \frac{r^8}{362880},$$

etc. for higher s.

Substituting these results for k_s into the general eq. for b'_ℓ, we obtain

$$b'_3 = b_3 - \frac{r^2 b_1}{6}$$

$$b'_4 = b_4$$

$$b'_5 = b_5 + \frac{r^2 b_3}{6} + \frac{31 r^4 b_1}{360}$$

etc. for higher ℓ.

We apply this S_{sh_r} ST to the β function in the $\overline{MS}$ scheme, calculated up to $\ell = 4$ loop level. For N_f in the interval I where the 2-loop β function has an IR zero, we then calculate the resultant IR zeros in $\beta_{\alpha'}$ at the 3- and 4-loop order and compare the values with those in the $\overline{MS}$ scheme.

We list some numerical results for illustrative values of r and for $N = 2,\ 3$. We denote the IR zero of $\beta_{\alpha'}$ at the n-loop level as $\alpha'_{IR,n\ell} \equiv \alpha'_{IR,n\ell,r}$.

For example, for $N=3$, $N_f=10$, $\alpha_{IR,2\ell}=2.21$, and:

$$\alpha_{IR,3\ell,\overline{MS}}=0.764, \quad \alpha'_{IR,3\ell,r=3}=0.762, \quad \alpha'_{IR,3\ell,r=6}=0.754,$$
$$\alpha'_{IR,3\ell,r=9}=0.742, \quad \alpha'_{IR,3\ell,r=4\pi}=0.723$$
$$\alpha_{IR,4\ell,\overline{MS}}=0.815, \quad \alpha'_{IR,4\ell,r=3}=0.812, \quad \alpha'_{IR,4\ell,r=6}=0.802,$$
$$\alpha'_{IR,4\ell,r=9}=0.786, \quad \alpha'_{IR,4\ell,r=4\pi}=0.762$$

In general, the effect of scheme dependence tends to be reduced (i) for a given N and N_f, as one calculates to higher-loop order, and (ii) for a given N, as $N_f \to N_{f,b1z}$, so that the value of $\alpha_{IR} \to 0$.

The results provide a quantitative measure of scheme dependence of the location of an IR zero of β.

Application of Quasiconformal Gauge Theories to Models of Dynamical Electroweak Symmetry Breaking and Implications for LHC Data

Models with dynamical electroweak symmetry breaking (EWSB) have been of interest as one way to avoid the hierarchy (fine-tuning) problem with the Standard Model (SUSY being another way).

These models make use of an asymptotically free vectorial gauge interaction, technicolor (TC), with a set of massless technifermions $\{F\}$ and a gauge coupling $\alpha_{TC}(\mu)$ that gets large at TeV scale, producing condensates $\langle \bar{F}F \rangle = \langle \bar{F}_L F_R \rangle + h.c. \sim \Lambda_{TC}^3$ (Susskind, Weinberg, 1979).

These dynamically break EW symmetry, since the technifermions include a left-handed SU(2)$_L$ doublet with corresponding right-handed SU(2)$_L$ singlets. Their condensates transform as EW $I = 1/2$, $Y = 1$, same as SM Higgs, and give masses to W and Z satisfying $m_W^2/(m_Z^2 \cos^2\theta_W) = 1$ to leading order.

Indeed quark condensates $\langle \bar{q}q \rangle$ also dynamically break EW symmetry (at much smaller scale, Λ_{QCD}), also transform as $I = 1/2$, $Y = 1$.

The TC theory is embedded in extended technicolor (ETC) to give masses to SM fermions via exchanges of ETC gauge bosons, which transform SM fermions to technifermions and vice versa, communicate EWSB in TC sector to SM fermions.

Resultant SM fermion mass matrices

$$M_{ii}^{(f)} \sim \frac{\eta\,\Lambda_{TC}^3}{\Lambda_{ETC,i}^2}$$

where $i = 1, 2, 3$ is generation index, $\Lambda_{ETC,i}$ is a corresponding ETC mass scale, and

$$\eta_i = \exp\left[\int_{\Lambda_{TC}}^{\Lambda_i} \frac{d\mu}{\mu}\gamma(\alpha_{_{TC}}(\mu))\right] \quad \text{is RG factor}$$

Typical values: $\Lambda_1 \simeq 10^3$ TeV, $\Lambda_2 \simeq 50 - 100$ TeV, $\Lambda_3 \simeq$ few TeV. Hierarchy in ETC symmetry breaking scales $\Lambda_{ETC,1} > \Lambda_{ETC,2} > \Lambda_{ETC,3}$ produces inverse generational hierarchy in SM fermion masses.

The running mass $m_{f_i}(p)$ of a SM fermion of generation i is constant up to the ETC scale $\Lambda_{ETC,i}$ and has the power-law decay (Christensen and Shrock, PRL 94, 241801 (2005))

$$m_{f_i}(p) \propto p^{-2} \quad \text{for} \;\; p >> \Lambda_{ETC,i}$$

Original TC models were scaled-up versions of QCD and were excluded by their inability to produce sufficiently large SM fermion masses without having ETC scales so low as to cause excessively large flavor-changing neutral current (FCNC) effects.

TC models after mid 1980s have been built to have a coupling that gets large but runs very slowly (walking, quasiconformal TC, WTC) (Holdom, Yamawaki et al., Appelquist, Wijewardhana...). This quasiconformal behavior arises naturally from an approx. IR zero of the TC β function, with α_{IR} slightly greater than α_{cr}.

If γ_{IR} is approx. const. near this IRFP, then, e.g., third-gen. SM fermion masses are increased by factor

$$\eta_3 \simeq \left(\frac{\Lambda_3}{\Lambda_{TC}}\right)^{\gamma_{IR}}$$

which could give significant enhancement. Hence, one can raise ETC scales Λ_i, reducing FCNC effects.

Further, studies of reasonably UV-complete ETC models showed that approximate residual generational symmetries suppress FCNC effects (Appelquist, Piai, Shrock, PRD 69, 015002 (2004); PLB 593, 175 (2004); PLB 595, 442 (2004); Appelquist, Christensen, Piai, Shrock, PRD 70, 093010 (2004).)

ETC models still face challenges in trying to reproduce all features of SM fermion masses, such as $m_t >> m_b$, etc. Here focus on TC.

TC models that include color-nonsinglet technifermions, such as the one-family TC model, in which technifermions comprise one SM family, are disfavored at present, for several reasons, including (i) possibly excessive contributions to precision electroweak S parameter; (ii) prediction of pseudo-NGB's (PNGB's), some of which are color-nonsinglets, with O(100) GeV masses that they should have been observed at LHC; (ii) color-octet techni-vector mesons, with masses of order TeV, in tension with the current lower bound of ~ 2.5 TeV set by ATLAS and CMS.

But TC models need not have any color-nonsinglet technifermions; a TC model may have a minimal EW-nonsinglet technifermion content of one SU(2)$_L$ doublet with corresponding right-handed SU(2)$_L$ singlets, all of which are color-singlets.

TC models of this type can exhibit quasiconformal behavior. For models in which technifermions are in fund. rep. of TC group, one may add SM-singlet technifermions to get N_f slightly less than $N_{f,cr}$ (Christensen and Shrock, Phys. Lett. B632, 92 (2006); Ryttov and Shrock, Phys. Rev. D84, 056009 (2011), arXiv:1107.3572). Alternatively, one can use higher-dim. TC reps. (Dietrich, Tuominen, Ryttov; e.g., Dietrich, Sannino, and Tuominen, PRD 72, 055001 (2005); Sannino, arXiv:0911.0931).

In these minimal TC models, all NGBs with EW quantum numbers are eaten, so no left-over EW-nonsinglet NGBs, in contrast with one-family TC. Also, S parameter may be sufficiently reduced (also by walking) to satisfy precision EW constraints.

As noted, because quasiconformal TC has approx. scale invariance, dynamically broken by $\langle \bar{F} F \rangle$, this could plausibly lead to a light approx. NGB, the techidilaton (Yamawaki..Goldberger, Grinstein, and Skiba.. Fan; Sannino...; Appelquist and Bai; Elander, Nunez, and Piai; for different estimates of χ mass, see Bardeen, Leung, Love; Holdom and Terning). Approx. Bethe-Salpeter calc. finds $m_S/m_V \sim 0.3$ in WTC (Kurachi, Shrock, JHEP 12, 034 (2006)). Much recent work on estimates of the dilaton mass; e.g., Matsuzaki and Yamawaki, PRD85, 095020 (2012); arXiv:1201.4722. arXiv:1209.2017; Lawrance and Piai, arXiv:1207.0427; Elander and Piai, arXiv:1208.0546; Bellazzini, Csáki, Hubisz, Serra and Terning, arXiv:1209.3299. A technidilaton might be as light as 125 GeV.

Eventually, lattice gauge measurements may be able to determine the mass of a dilaton in a quasiconformal theory (a difficult calculation).

N.B. Technicolor gauge fields are color-singlets and all technifermions may be color-singlets as well, in which case a technidilaton χ may have no color-nonsinglet constituents.

The boson discovered at the LHC by ATLAS and CMS with mass of ~ 125 GeV is consistent with being the SM Higgs, although the diphoton rate is slightly high. However, it might also be explained as a technidilaton, χ, resulting from a quasiconformal TC theory; further experimental and theoretical work should settle this decisively.

A general TC collider signature is resonant scattering of longitudinally polarized W and Z bosons, via techni-vector mesons in s-channel. A decisive search at LHC may require $\int \mathcal{L} dt \sim 50 - 100$ fb^{-1} at $\sqrt{s} = 14$ TeV.

Conclusions

- Understanding the UV to IR evolution of an asymptotically free gauge theory and the nature of the IR behavior is of fundamental field-theoretic interest

- Our higher-loop calculations give new information on this UV to IR flow and on determination of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$; valuable to compare and combine results from higher-loop continuum calcs. with lattice measurements to gain insight into this flow

- Quantitative study of scheme-dependence in higher-loop calculations, noting that scheme transformations are subject to constraints that are easily satisfied at a UVFP but are quite restrictive at IRFP

- Application of quasiconformal gauge theories to models of dynamical EWSB

- Approx. dilatation-invariant (quasiconformal) behavior if $\alpha_{IR} \gtrsim \alpha_{cr}$; broken dilatation symmetry may yield sufficiently light dilaton with properties that may fit the LHC 125 GeV boson; key role of a future lattice calculation of the dilaton mass.

- Importance of these calculations in deciding the outstanding question of whether dynamical EWSB is realized in nature