Ward-Takahashi identities and restoration of supersymmetries in lattice $\mathcal{N} = 4$ super Yang-Mills

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SUSY on the lattice : Motivation

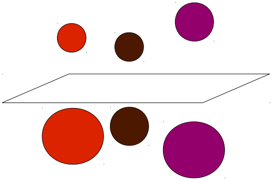
Supersymmetry (SUSY) - interesting subject in its own right.

New particles, hierarchy problem, unification ...

Expected to be significant in future experiments

Dark matter ...

- SUSY plays a significant role in many theories of BSM physics
  - MSSM - minimal extension of SM with $\mathcal{N} = 1$ SUSY
  - SUSY technicolor
  - Extra dimensional models
  - AdS/CFT duality, strings, black holes



- Many interesting features are non-perturbative. dynamical SUSY breaking, gaugino condensation, ...
- Need a non-perturbative definition of the theory $\rightarrow$ lattice construction.

SUSY on the lattice : Focusing on $\mathcal{N} = 4$ SYM

Supersymmetric Yang–Mills (SYM) theories: many interesting features/results.

- confinement
- spontaneous chiral symmetry breaking
- strong coupling/weak coupling duality
- gauge theory/string theory duality

Needs lattice to study the strong coupling dynamics.

We focus on the 4D $\mathcal{N} = 4$ SYM

- It is a fascinating theory in itself
- Plays crucial role in AdS/CFT correspondence

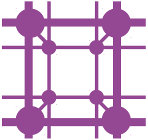
Lattice version of this theory would allow:

- Strong coupling calculations, Monte Carlo simulations
- New ideas/approaches eg. (quantum) string corrections from finite N, $\lambda \cdots$

SUSY and Lattice: Are they compatible?

- The SUSY algebra, which is an extension of the Poincaré algebra, is explicitly broken on the lattice.

$$\{\mathcal{Q}, \overline{\mathcal{Q}}\} = \gamma \cdot P$$



One cannot realize infinitesimal translation on the lattice. So this relation is broken by discretization [Dondi and Nicolai 1977].

- Folklore: Impossible to put SUSY on the lattice exactly.
- Leads to (very) difficult fine tuning - lots of relevant SUSY breaking counter-terms in effective action.
- $\mathcal{N} = 4$ SYM particularly difficult - contains scalar fields.

$\mathcal{N} = 4$ SYM on the Lattice

What if we could retain a subalgebra of the SUSY algebra on the lattice that does not generate translations?

Exact lattice SUSY allows this.

We could relabel the fields and supercharges of a class of SYM theories in a convenient way - a process called topological twisting.

E. Witten [Commun. Math. Phys. 117, 353 (1988)]

Twisting leads to exact lattice SUSY.

- Positive energy states
- E=0 ground state
- fermion-boson spectrum degenerate under $\mathcal{Q}$

In D=4 topological twisting results in a unique theory: the $\mathcal{N} = 4$ SYM.

Exact lattice SUSY : Twisting $\mathcal{N} = 4$ SYM

- $\mathcal{N} = 4$ SYM in 4D has additional flavor (R) symmetries: $SO_R(6)$.
- Twist: decompose fields under the diagonal subgroup of

$$SO(4)' = SO_{\rm Lorentz}(4) \times SO_R(4)$$

N. Marcus [Nucl. Phys. B452 (1995) 331-345]

- Fermions: spinors under both factors - become integer spin after twisting
- Scalars transform as vectors under R-symmetry - vectors after twisting
- Gauge fields remain vectors - combine with scalars to make complex gauge fields. Gauge symmetry is still just $U(N)$.

Twisting is just a change of variables in flat space.

Exact lattice SUSY : Twisting $\mathcal{N} = 4$ SYM [contd.]

- Field content: 4D gauge field, 6 scalars, 16 fermions
- Twisting the theory leads to a theory compactly expressed as dimensional reduction of a 5D theory:
  - 16 fermions: $\Psi = (\eta, \psi_a, \chi_{ab})$, $a, b = 1, \cdots, 5$
  - 10 bosons as 5 complex gauge fields: $\mathcal{A}_a = A_a + iB_a$, $a = 1, \cdots, 5$
  - 16 supercharges: $(\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab})$, $a = 1, \cdots, 5$
- Action:

$$
\begin{aligned}
S_{\rm SYM}^{\mathcal{N}=4, D=4} &= \mathcal{Q} \int {\rm Tr} \left(\chi_{ab} \mathcal{F}_{ab} + 2\eta [\overline{\mathcal{D}}_a, \mathcal{D}_a] - \eta d \right) + S_{\rm closed}, \\
S_{\rm closed} &= \frac{1}{2} \int {\rm Tr}\ \epsilon_{abcde} \chi_{de} \overline{\mathcal{D}}_c \chi_{ab}.
\end{aligned}
$$

- We have $\mathcal{Q} S_{\rm SYM}^{\mathcal{N}=4, D=4} = 0 \rightarrow$ action is $\mathcal{Q}$-invariant.

Exact lattice SUSY : Twisting $\mathcal{N} = 4$ SYM [contd.]

The supercharge $\mathcal{Q}$ is nilpotent: $\mathcal{Q}^2 = 0$. A property that can be easily transported on to the lattice.

$$\begin{aligned} \mathcal{Q}\mathcal{A}_a &= \frac{1}{2}\psi_a, & \mathcal{Q}\chi_{ab} &= -\overline{\mathcal{F}}_{ab}, \\ \mathcal{Q}\overline{\mathcal{A}}_a &= 0, & \mathcal{Q}\eta &= \frac{1}{2}d, \\ \mathcal{Q}\psi_a &= 0, & \mathcal{Q}d &= 0 \end{aligned}$$

Perform $\mathcal{Q}$-variation and integrate d to get more explicit form for the action:

$$S_{\rm SYM}^{\mathcal{N}=4,D=4} = \int {\rm Tr}\ \left(-\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} + \tfrac{1}{2}[\overline{\mathcal{D}}_a, \mathcal{D}_a]^2 - \tfrac{1}{2}\chi_{ab}\mathcal{D}_{[a}\psi_{b]} - \eta\overline{\mathcal{D}}_a\psi_a\right) + S_{\rm closed}.$$

Action on the lattice takes the form:

$$S_{\rm SYM}^{\mathcal{N}=4,D=4} = \int {\rm Tr}\ \Big[-(\mathcal{D}_a^{(+)}\mathcal{U}_b)^\dagger(\mathbf{x})(\mathcal{D}_a^{(+)}\mathcal{U}_b)(\mathbf{x}) + \tfrac{1}{2}(\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(\mathbf{x}))^2 - \tfrac{1}{2}\chi_{ab}(\mathbf{x})\mathcal{D}_{[a}^{(+)}\psi_{b]}(\mathbf{x}) - \eta(\mathbf{x})\overline{\mathcal{D}}_a^{(-)}\psi_a(\mathbf{x}) + \tfrac{1}{2}\epsilon_{abcde}\chi_{de}(\mathbf{x}+\widehat{\mathbf{e}}_a+\widehat{\mathbf{e}}_b+\widehat{\mathbf{e}}_c)\overline{\mathcal{D}}_c^{(-)}\chi_{ab}(\mathbf{x}+\widehat{\mathbf{e}}_c)\Big].$$

Outstanding question : How to restore full SUSY?

- We have a lattice formulation for $\mathcal{N} = 4$ SYM. It is:
  - Local
  - Gauge invariant
  - Doubler free
  - Invariant under ONE $\mathcal{Q}$
- There are 15 other SUSYs - $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ - that are broken on the lattice.
- What about restoration of the full set of SUSY as we take the continuum limit?
- Do we have to deal with operator renormalization/mixing? tuning of lattice parameters?
- Apparent solution: Construct SUSY Ward-Takahashi (WT) identities and examine the restoration of SUSYs as we approach the continuum limit.

Outstanding question : How to restore full SUSY? [contd.]

To construct WT identities we need to know the other SUSY transformations.

We can make use of the discrete subgroups of the R-symmetries and the $\mathcal{Q}$ symmetry.

We can find $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ SUSY transformations of the fields.
Let us look at the following field interchanges (with index a fixed):

$$2\eta \to \psi_a, \quad \psi_a \to 2\eta, \quad \psi_b \to -\chi_{ab}$$

$$\chi_{ab} \to -\psi_b, \quad \chi_{bc} \to \frac{1}{2}\epsilon_{bcagh}\chi_{gh}$$

$$\mathcal{D}_a \to \mathcal{D}_a, \quad \overline{\mathcal{D}}_a \to \overline{\mathcal{D}}_a, \quad \mathcal{D}_b \to \overline{\mathcal{D}}_b, \quad \overline{\mathcal{D}}_b \to \mathcal{D}_b$$

The action is invariant under these field interchanges.

The $\mathcal{Q}_a$ SUSY transformations

This leads to the SUSY transformations associated with $\mathcal{Q}_a$:

$$
\begin{aligned}
\mathcal{Q}_a \mathcal{A}_b &= \delta_{ab}\eta, \\
\mathcal{Q}_a \overline{\mathcal{A}}_b &= -\frac{1}{2}\chi_{ab}, \\
\mathcal{Q}_a \psi_b &= \delta_{ab} d + (1-\delta_{ab})[\overline{\mathcal{D}}_a, \mathcal{D}_b], \\
\mathcal{Q}_a \chi_{bc} &= -\frac{1}{2}\epsilon_{abcgh}\mathcal{F}_{gh}, \\
\mathcal{Q}_a \eta &= 0, \\
\mathcal{Q}_a d &= 0.
\end{aligned}
$$

Note: $\mathcal{Q}_a^2 = 0$ in the continuum but not true on the lattice.

The $\mathcal{Q}_{ab}$ SUSY transformations

The action is invariant under another set of field interchanges (with indices a, b fixed).

$$2\eta \to \chi_{ab}, \quad \psi_a \to \psi_b, \quad \psi_b \to -\psi_a, \quad \psi_c = \frac{1}{2}\epsilon_{cabgh}\chi_{gh}$$

$$\chi_{ab} \to -2\eta, \quad \chi_{ac} \to \chi_{bc}, \quad \chi_{bc} \to -\chi_{ac}, \quad \chi_{cd} \to -\epsilon_{cdabe}\psi_e$$

$$\mathcal{D}_{a,b} \to \overline{\mathcal{D}}_{a,b}, \quad \overline{\mathcal{D}}_{a,b} \to \mathcal{D}_{a,b}, \quad \mathcal{D}_c \to \mathcal{D}_c, \quad \overline{\mathcal{D}}_c \to \overline{\mathcal{D}}_c$$

This leads to the SUSY transformations associated with $\mathcal{Q}_{ab}$:

$$\begin{aligned}
\mathcal{Q}_{ab}\mathcal{A}_c &= \frac{1}{4}\epsilon_{abcgh}\chi_{gh}, \\
\mathcal{Q}_{ab}\overline{\mathcal{A}}_c &= \frac{1}{2}(\delta_{ac}\psi_b - \delta_{bc}\psi_a), \\
\mathcal{Q}_{ab}\psi_c &= \epsilon_{abcgh}\overline{\mathcal{F}}_{gh}, \\
\mathcal{Q}_{ab}\chi_{cd} &= \delta_{ac}\delta_{bd}d - \delta_{bc}[\mathcal{D}_a, \overline{\mathcal{D}}_d] + \delta_{ac}[\mathcal{D}_b, \overline{\mathcal{D}}_d], \\
\mathcal{Q}_{ab}\eta &= \frac{1}{2}\mathcal{F}_{ab}, \\
\mathcal{Q}_{ab}d &= 0.
\end{aligned}$$

$Q_{ab}^2 = 0$ in the continuum but broken on the lattice.

WT identities and supercurrents

WT identities involving supercurrent and a local (or multi-local) operator.
Consider a composite operator $O(y)$. Its expectation value is

$$\langle O(y) \rangle = \frac{1}{Z} \int [d\Phi] \exp(-S[\Phi]) O(y).$$

Consider infinitesimal transformations of the fields

$$\Phi(x) \rightarrow \Phi'(x) = \Phi(x) + \delta_\kappa \Phi(x), \quad \delta_\kappa \Phi(x) = \delta\kappa \Delta\Phi(x),$$

$\delta\kappa$: infinitesimal Grassmann odd parameter, Δ: deformation of the field.
The functional integral is independent of relabeling of integration variables:

$$\langle O'(y) \rangle - \langle O(y) \rangle = 0.$$

This gives the relation:

$$\langle \delta_\kappa S O(y) \rangle = \langle \delta_\kappa O(y) \rangle.$$

Making the transformation position dependent:

$$\Phi(x) \to \Phi'(x) = \Phi(x) + \delta_{\kappa(x)}\Phi(x), \qquad \delta_{\kappa(x)}\Phi(x) = \delta\kappa(x)\Delta\Phi(x),$$

In the twisted theory we have

$$\begin{aligned} \delta_{\kappa(x)} &= \delta\kappa_A(x)\mathcal{Q}_A, \\ \delta\kappa_A(x) &= (\delta\kappa_0(x), \delta\kappa_a(x), \delta\kappa_{ab}(x)), \\ \mathcal{Q}_A &= (\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab}). \end{aligned}$$

The infinitesimal variation of the action gives:

$$\delta_{\kappa(x)} S = \int d^4x - (\partial_m \delta_{\kappa_A(x)})\mathcal{S}^m_A(x), \qquad \mathcal{S}^m_A(x) = (\mathcal{S}^m_0(x), \mathcal{S}^m_a(x), \mathcal{S}^m_{ab}(x)).$$

Here $\mathcal{S}^m_A(x)$ are the supercurrents (Noether currents).

WT identities and supercurrents [contd.]

The associated WT relations are:

$$\langle \partial_m \mathcal{S}_A^m(x) O(y) \rangle = \delta^{(4)}(x-y) \langle \mathcal{Q}_A O(y) \rangle.$$

We can check how strongly these relations hold on the lattice for a given operator O.

A way of measuring the amount of SUSY breaking by the lattice.

The scalar supercurrent: $\delta_\kappa^{\rm scalar} S \to \mathcal{S}_0^m$.

In the continuum it is:

$$\mathcal{S}_0^m = {\rm Tr} \sum_n \overline{\mathcal{F}}_{mn} \psi_n - \frac{1}{2} d \psi_m - \frac{1}{2} \sum_{n,c,g,h} \epsilon_{cnmgh} \overline{\mathcal{F}}_{cn} \chi_{gh}.$$

WT identities on the lattice

Additional terms appear in the lattice WT equations: $I_0^m(\mathbf{x})$, $I_a^m(\mathbf{x})$, $I_{ab}^m(\mathbf{x})$.

WT identities on the lattice:

$$\partial_m \langle \mathcal{S}_A^m(\mathbf{x}) O(0) \rangle + \langle I_A^m(\mathbf{x}) O(0) \rangle = \delta^{(4)}(\mathbf{x}) \langle \mathcal{Q}_A O(0) \rangle .$$

We can write down expressions for the supercurrents on the lattice after performing a variation of the lattice action.

Symmetry breaking terms $I_A^m(\mathbf{x})$ also follow from such variation.

The scalar supercurrent on the lattice:

$$\begin{aligned} \mathcal{S}_0^m(\mathbf{x}) &= \mathrm{Tr}\, \mathcal{F}_{mn}^\dagger(\mathbf{x}) \mathcal{U}_m(\mathbf{x}) \psi_n(\mathbf{x} + \widehat{\mathbf{e}}_m) - \frac{1}{2} d(\mathbf{x} + \widehat{\mathbf{e}}_m) \mathcal{U}_m^\dagger(\mathbf{x}) \psi_m(\mathbf{x}) \\ &\quad - \frac{1}{2} \epsilon_{cnmgh} \chi_{gh}(\mathbf{x} + \widehat{\mathbf{e}}_c + \widehat{\mathbf{e}}_n + \widehat{\mathbf{e}}_m) \mathcal{U}_m^\dagger(\mathbf{x} + \widehat{\mathbf{e}}_m) \mathcal{F}_{cn}^\dagger(\mathbf{x} + \widehat{\mathbf{e}}_m). \\ I_0^m(\mathbf{x}) &= 0 \ , \ \text{for scalar SUSY}. \end{aligned}$$

WT identities on the lattice : Restoration of SUSY

Breaking terms are $\mathcal{O}(a)$ artifacts.

$\langle I_a^m(\mathbf{x})\rangle \to 0$, $\langle I_{ab}^m(\mathbf{x})\rangle \to 0$ as we take the continuum limit.

Things to remember while choosing the operator O:

- It is a composite operator.
- May or may not be gauge invariant.
- Operator mixing can happen.
- Ultraviolet effects in correlators.

We could take an appropriate lattice supercurrent as the operator and study the WT relation.

$$O(x) = \mathcal{S}_A^m(x)$$

$$\partial_m \langle \mathcal{S}_A^m(\mathbf{x})\mathcal{S}_B^m(0)\rangle + \langle I_A^m(\mathbf{x})\mathcal{S}_B^m(0)\rangle = \delta^{(4)}(\mathbf{x})\langle \mathcal{Q}_A \mathcal{S}_B^m(0)\rangle.$$

Conclusions/Further developments

- We have a lattice $\mathcal{N} = 4$ SYM that preserves exactly one supersymmetry.
- Studying WT identities on the lattice is needed to examine the restoration of other SUSYs in the continuum limit.
- Need expressions for 1-form and 2-form lattice supercurrents and appropriately chosen operators.
- What can we say about the renormalization of the supercurrent operators?
- There are subtleties like operator mixing, ultraviolet effects etc. that one has to address.