Radial Quantization for Conformal Field Theories on the Lattice

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Lattice Meets Experiment 2012: Beyond the Standard Model
Boulder Colorado Oct 26-27, 2012

Motivation

- (near) Conformal Field Theories are important
  - BSM walking technicolor
  - AdS/CFT weak-strong duality
  - Model building

- Lattice difficulty: scales are (nearly) exponential.

- Hypercubic vs Radial Lattice

$$a < \Delta r < L \quad \text{vs} \quad a < \Delta log(r) < L$$

Early History

- S. Fubini, A. Hanson and R. Jackiw PRD 7, 1732 (1972)

  Abstract: A field theory is quantized covariantly on Lorentz-invariant surfaces. Dilatations replace time translations as dynamical equations of motion. This leads to an operator formulation for Euclidean quantum field theory. A covariant thermodynamics is developed, with which the Hagedorn spectrum can be obtained, given further hypotheses. The Virasoro algebra of the dual resonance model is derived in a wide class of 2-dimensional Euclidean field theories.

- J. Cardy J. Math. Gen 18 757 (1985).

  Abstract: The relationship between the correlation length and critical exponents in finite width strips in two dimensions is generalised to cylindrical geometries of arbitrary dimensionality d. For $d > 2$ these correspond however, to curved spaces. The result is verified for the spherical model

Outline

- Conformal Field Theory
- Lattice Radial Quantization
- 3-D Ising model at Tc
- Conclusion & Future Directions

Conformal Field Theories

O(d+1,1) adds Dilations and Inversion to Poincare transformations

$$x_\mu \to \lambda x_\mu \quad , \quad x_\mu \to \frac{x_\mu}{x^2}$$

Algebra:

$$K_\mu : (inv \to trans \to inv)$$

$$[K_\mu, \mathcal{O}(x)] = i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta)\mathcal{O}(x)$$

$$[D, \mathcal{O}(x)] = i(x^\mu \partial_\mu - \Delta)\mathcal{O}(x)$$

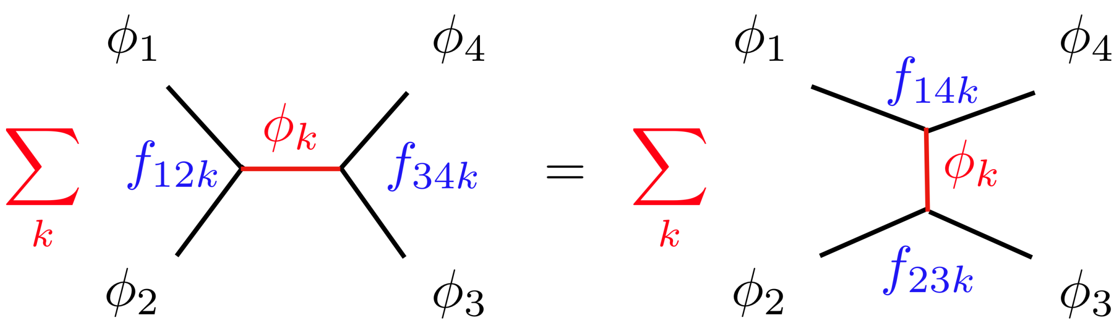
$$[D, P_\mu] = -iP_\mu \quad , \quad [D, K_\mu] = +iK_\mu \quad , \quad [K_\mu, P_\mu] = 2iD$$

CFT are highly constrained

1. More than hyper scaling (scale invariance).
2. 2 and 3 point correlators are determined.
3. OPE & factorization may fixed the theory completely*?

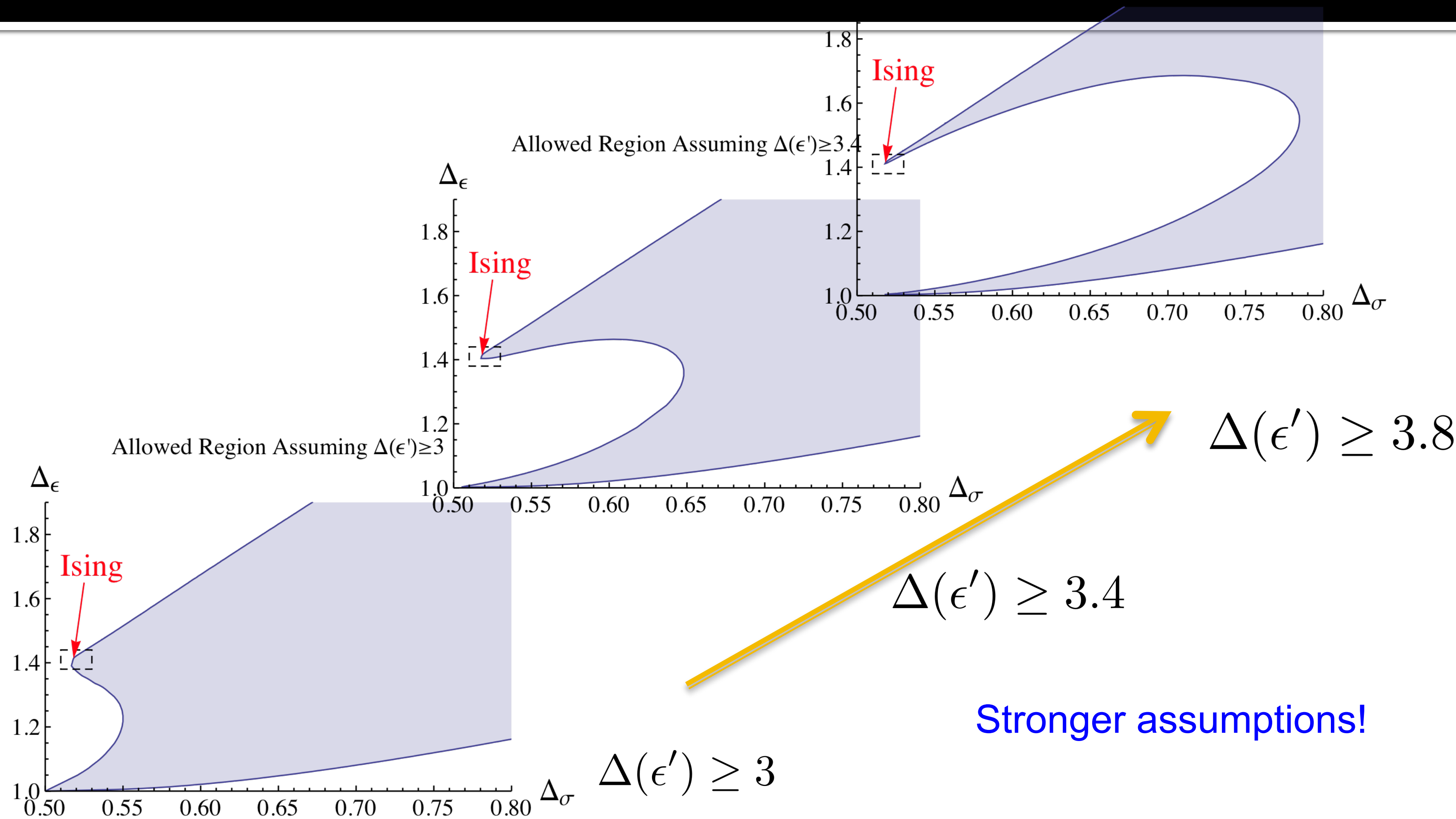
$$\langle \phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1 - x_2|^{2\Delta}}$$

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1 - x_2|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(0)$$



* "Solving the 3D Ising Model with the Conformal Bootstrap" (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

Inequalities from Bootstrap*



•"Solving the 3D Ising Model with the Conformal Bootstrap" (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

Radial Quantization

Evolution: $H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$

$$ds^2 = dx^\mu dx_\mu = e^{2\tau}[d\tau^2 + d\Omega^2]$$

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

$$\text{"time"} \quad \tau = log(r)\ , \quad \text{"mass"} \quad \Delta = d/2 - 1 + \eta$$

$$D \to x_\mu \partial_\mu = r\partial_r = \frac{\partial}{\partial \tau}$$

Power Law Correlator

Conformal correlator: $\langle \phi(x_1)\phi(x_2)\rangle = C\dfrac{1}{|x_1 - x_2|^{2\Delta}}$

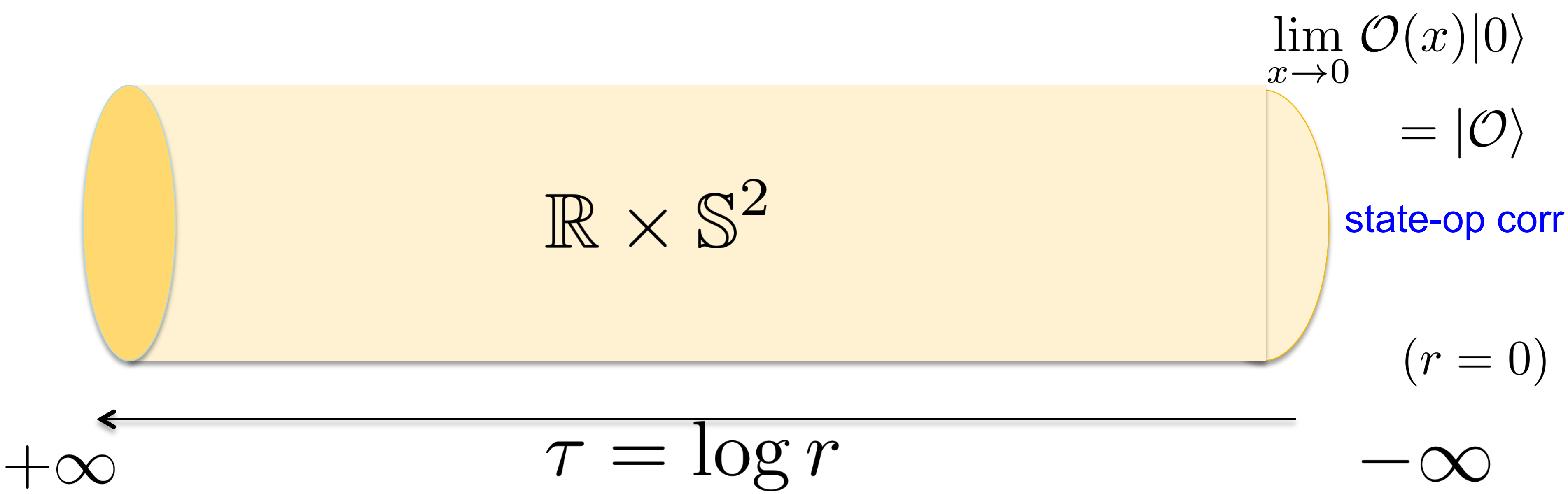
$$
\begin{aligned}
r_1^{\Delta} r_2^{\Delta} \langle \phi(\tau_1, \Omega_1)\phi(\tau_2, \Omega_2)\rangle &= C\frac{1}{[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]^{\Delta}} \\
&\simeq Ce^{-(\log(r_2) - \log(r_1)\Delta} \\
&= Ce^{-\tau\Delta}
\end{aligned}
$$

With $|x_1 - x_2|^2 = r_1 r_2 [r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]$

as $\tau = \log(r_2) - \log(r_1) \to \infty$

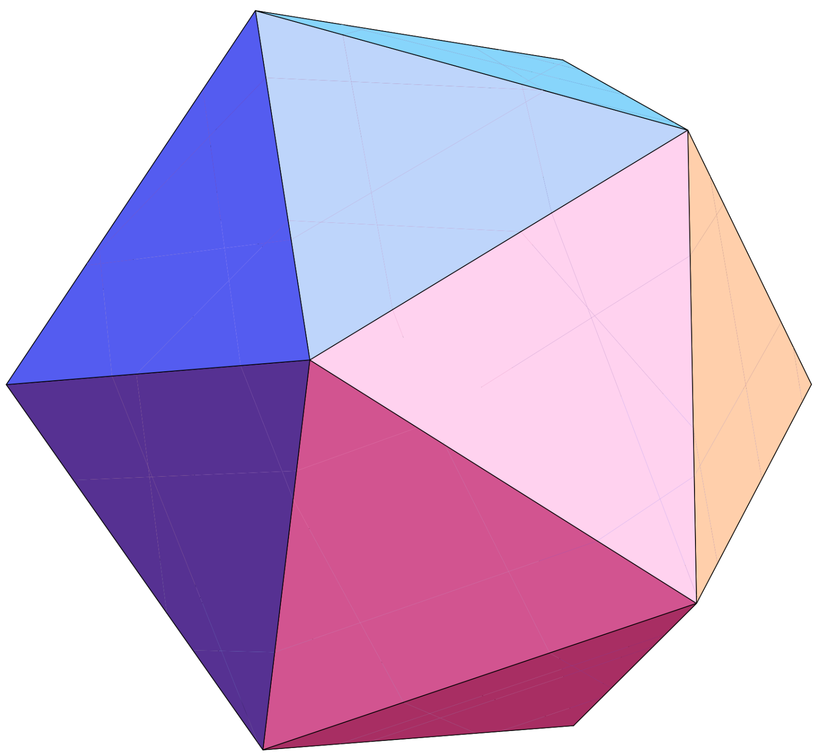
3-d Ising

$$Z_{Ising} = \sum_{\{s_{\tau,i}=\pm 1\}} e^{\beta \sum_{\tau,i} s_{\tau+1,i} s_{\tau,i} + \beta \sum_{\tau <i,j>} s_{\tau,i} s_{\tau,j}}$$

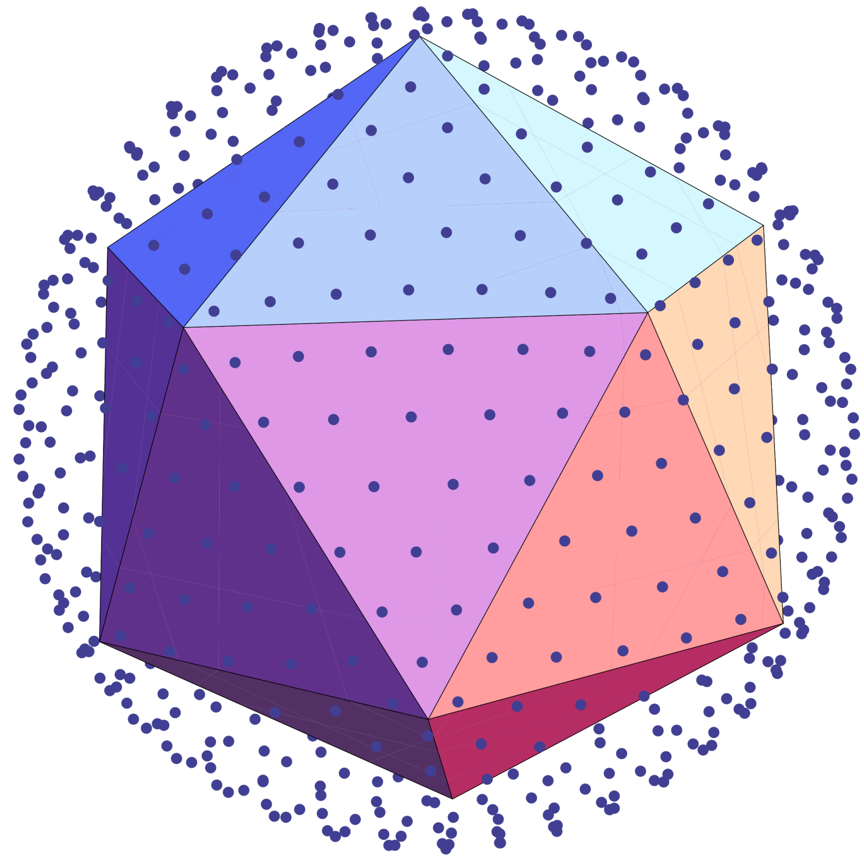


Order s Refined Triangulated Icosahedron

$s = 1$

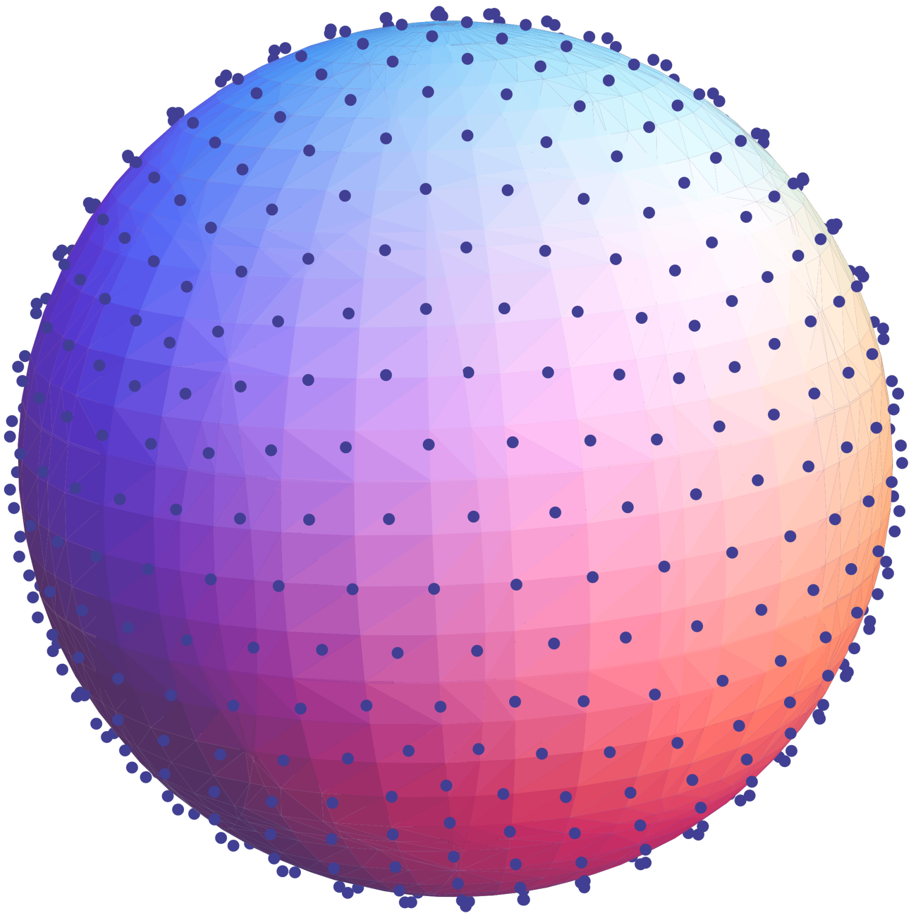


$s = 8$



l = 0,1, 2 irreducible under 120 Ih Iscosahedral subgroup of O(3)

Fixed t lattice are s refined Icosahedrons



s = 8

vertices:
N = 10 + 2*s*s = 138

edges:
E = 3*N – 6

faces:
F = E – N + 2 = 2*N - 4

$$\text{Continuum limit is } s \to \infty \text{ at } \beta = \beta_{critical}$$

Primary operators 3-d Ising Model

Operator	Spin l	$\mathbb{Z}$	Δ	Exponent
s	0	$-$	0.5182(3)	$\Delta = 1/2 + \eta/2$
s'	0	$-$	$\gtrsim 4.5$	$\Delta = 3 + \omega_A$
ε	0	$+$	1.413(1)	$\Delta = 3 - 1/\nu$
ε'	0	$+$	3.84(4)	$\Delta = 3 + \omega$
ε''	0	$+$	4.67(11)	$\Delta = 3 + \omega_2$
$T_{\mu\nu}$	2	$+$	3	$\Delta = 3$
$C_{\mu\nu\kappa\lambda}$	4	$+$	5.0208(12)	$\Delta = 3 + \omega_{\text{NR}}$

Low-lying primary operators of the 3D Ising model at criticality.

Primary l = 0

$$[K_\mu, \mathcal{O}(0)] = 0$$

Descendants l > 0

$$\mathcal{O}_{l+1}(x) = [P_\mu, \mathcal{O}(x)] = i\partial_\mu \mathcal{O}_l(x)$$

Preliminary Numerical Work

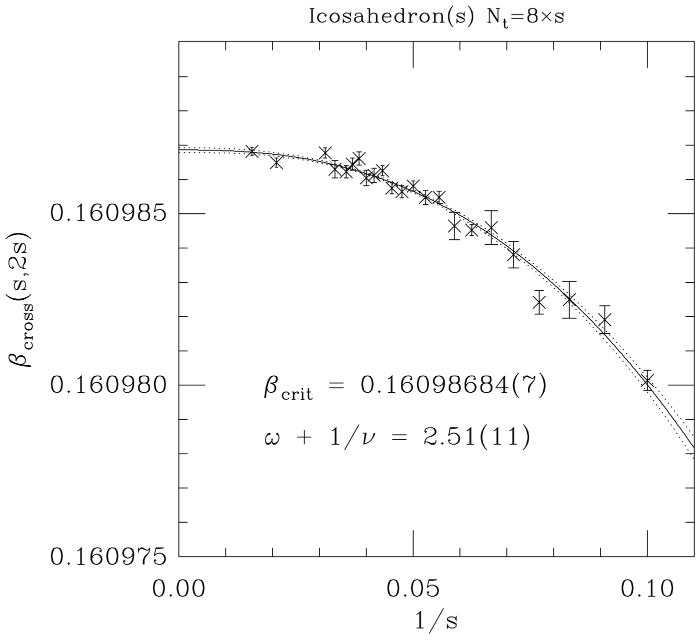
- Swendsen Wang & Wolff cluster algorithm
- Binder $$U(\beta, s = L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$
  - Fixes: $$\beta_{crit} = 0.16098703(3)$$
- Fix asymmetry of lattice by descendants.
  - $$\Delta_l = \Delta_0 + l \quad \text{for} \quad l = 0, 1, 2, \cdots$$
- Rough values of 3 primaries : η , ν , ω
- Much more is feasible with modest effort

Determining beta_critical

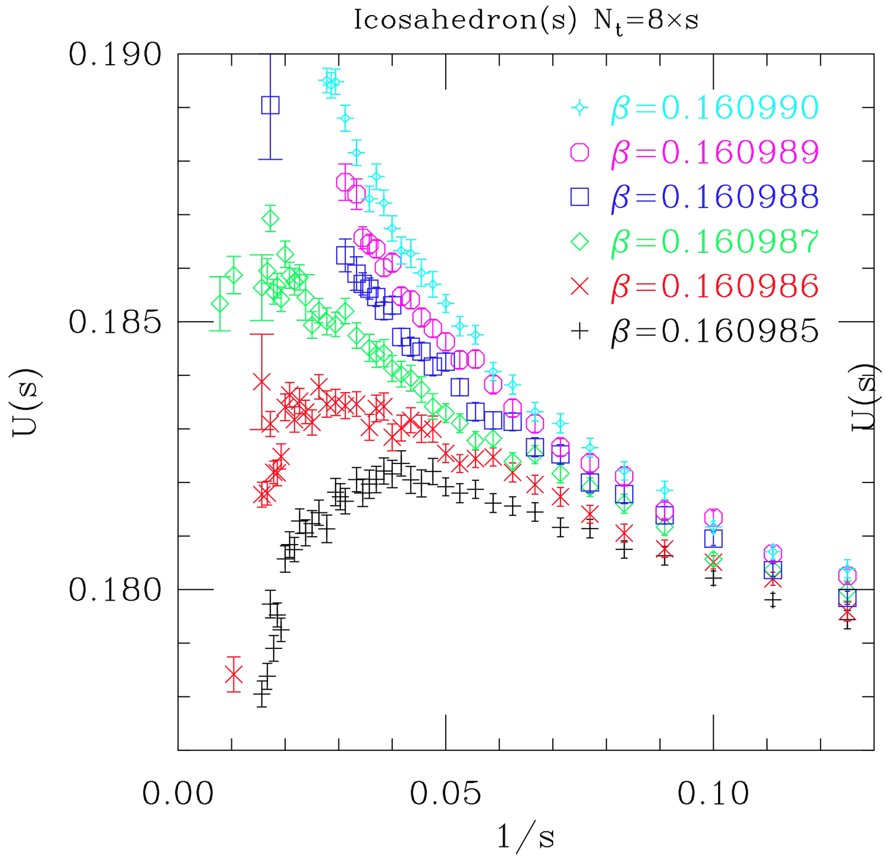
$$U_L(\beta) = \widetilde{U}(\beta-\beta_c)L^{1/\nu} + b_1 L^{-\omega} + b_2 L^{-\gamma/\nu} + \cdots$$
$$= U^* + a_1(\beta-\beta_c)L^{1/\nu} + b_1 L^{-\omega} + b_2 L^{-\gamma/\nu} + \cdots$$

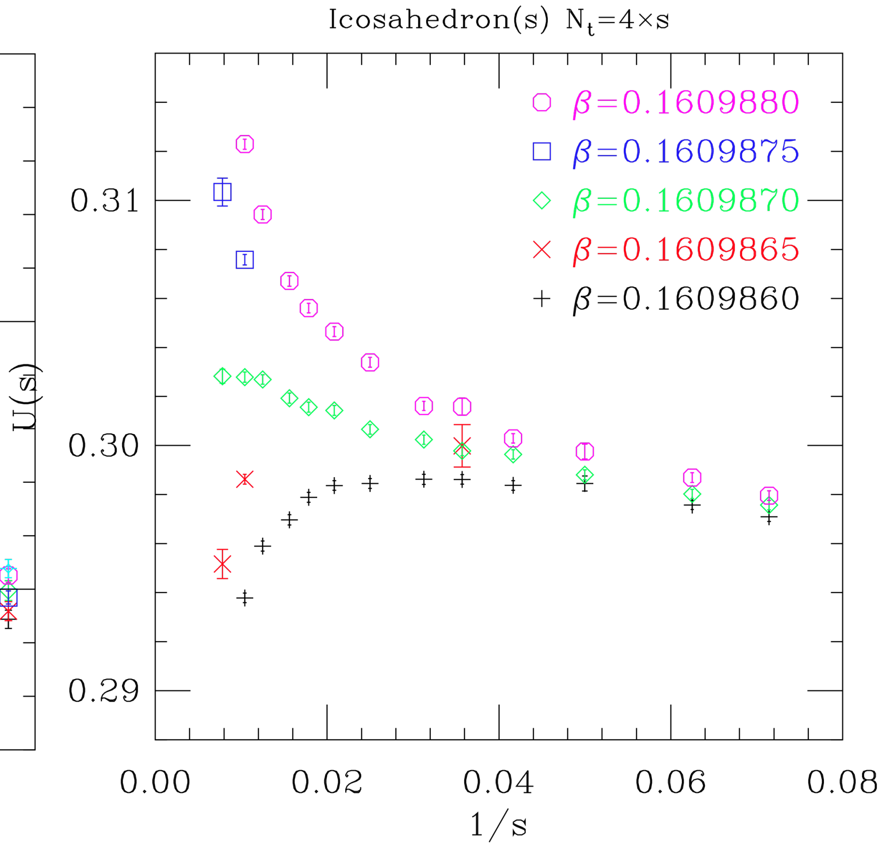
$$U(\beta, s=L) = 1 - \frac{\langle M^4\rangle}{3\langle M^2\rangle^2}$$

$$\beta_{crit} = 0.16098703(3)$$



Determining beta_critical





$$U(\beta, s = L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$

$$\beta_{crit} = 0.16098703(3)$$

Observables

$$C_{lm}(t) = \sum_{t_0,x,y} Y^*_{lm}(\Omega_x)\langle s_{t+t_0,x}\ s_{t_0,y}\rangle Y_{lm}(\Omega_y)$$

cosh fit:

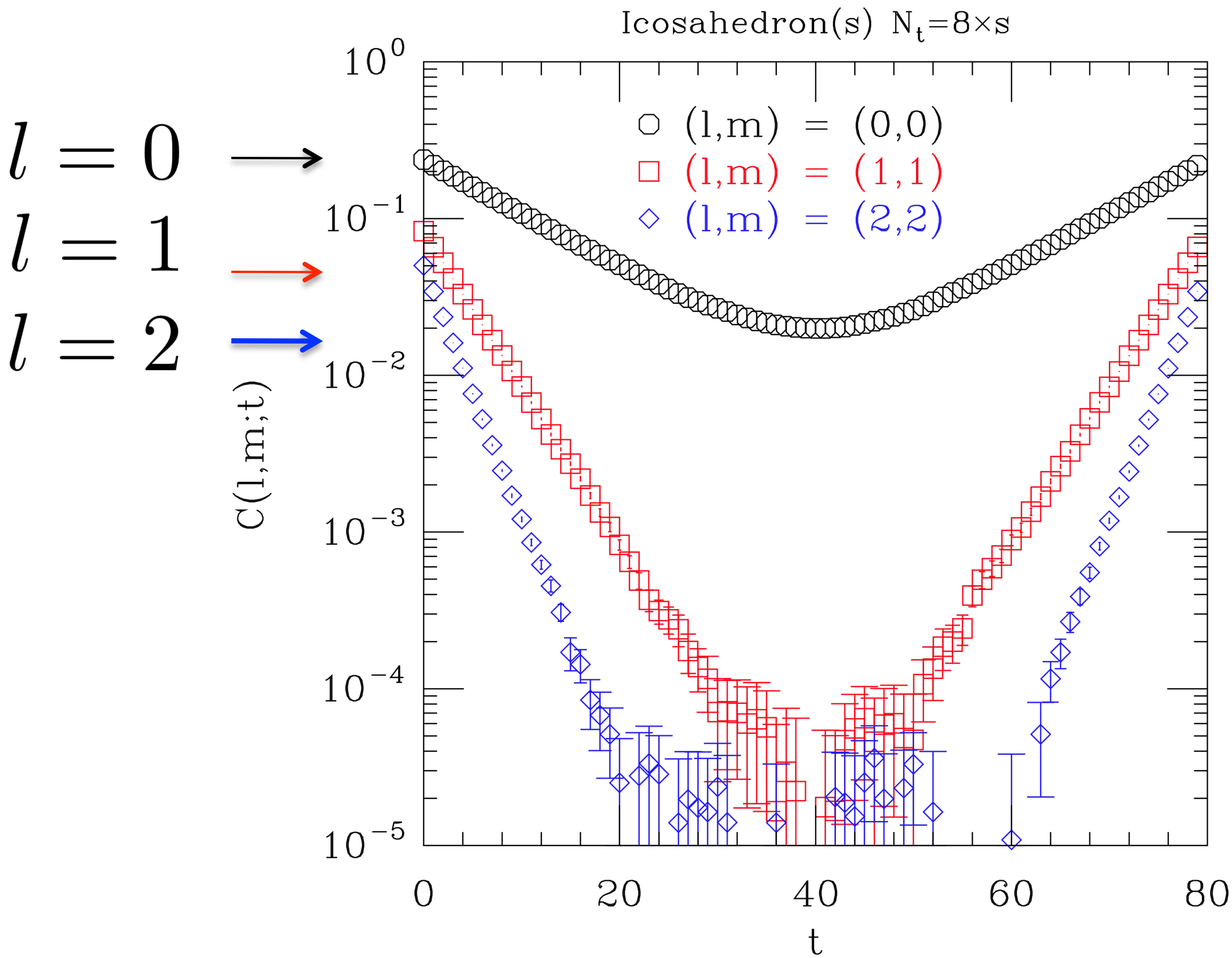
$$C_{lm}(t) = C[e^{-m_l t} + e^{-m_l(N_t-1-t)}]$$

$$\text{for} \quad t = 0, \cdots, N_t - 1\ ,\ l = 0, 1, 2\ ,\ m = -l, \cdots, l$$

$$m_l = \frac{c}{s}\Delta_l \qquad\qquad \Delta_l = \frac{1}{2} + \frac{\eta}{2} + l$$

After you adjust c = `speed of light` so $\Delta_{l+1} - \Delta_l = 1$

Early result for C(t)



Improved cluster Estimator

Swendsen-Wang: Real space

$$g(x-y) = \langle s_x s_y \rangle \simeq \frac{1}{N_{config}} \sum_{i=1}^{N_{config}} \sum_{C_i} \Delta_{C_i}(x) \Delta_{C_i}(y)$$

$$\Delta_C(x) = \quad 1 \text{ if } x \in C \text{ else } 0$$

Wolff single cluster

$$\widetilde{g}_{lm}(k) \simeq \frac{1}{N_{config}} \sum_{i=1}^{N_{config}} \frac{1}{|C|} \Big| \sum_{t,x \in C} e^{i2\pi kt/L_t} Y_{lm}(\Omega_x) \Big|^2$$

Note: All to All O(V) improved estimator in Momentum space *

*C. Ruge, P. Zhu and F. Wagner Physica A (1994) 431:

Numerical Test (so far)

- Equal spacing test of descendants:

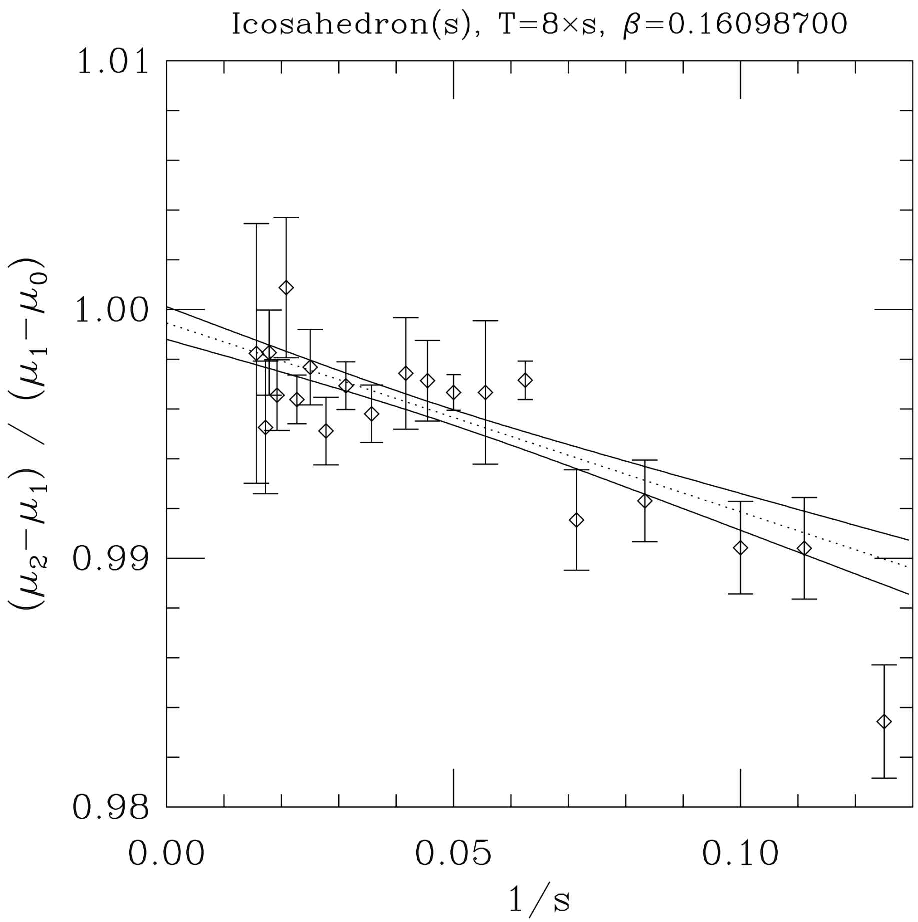
$$\frac{\mu_2 - \mu_1}{\mu_1 - \mu_0} = 0.999(1)$$

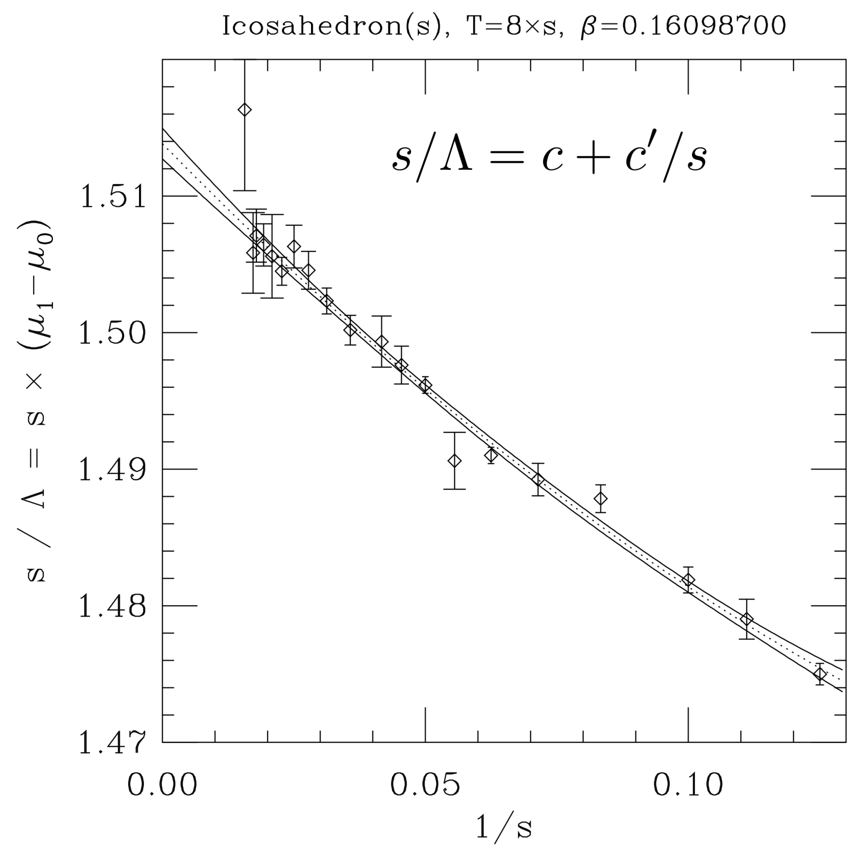
- "Speed of light" $c = 1.5105(7)$

- But critical point $\beta_{crit} = 0.16098703(3)$

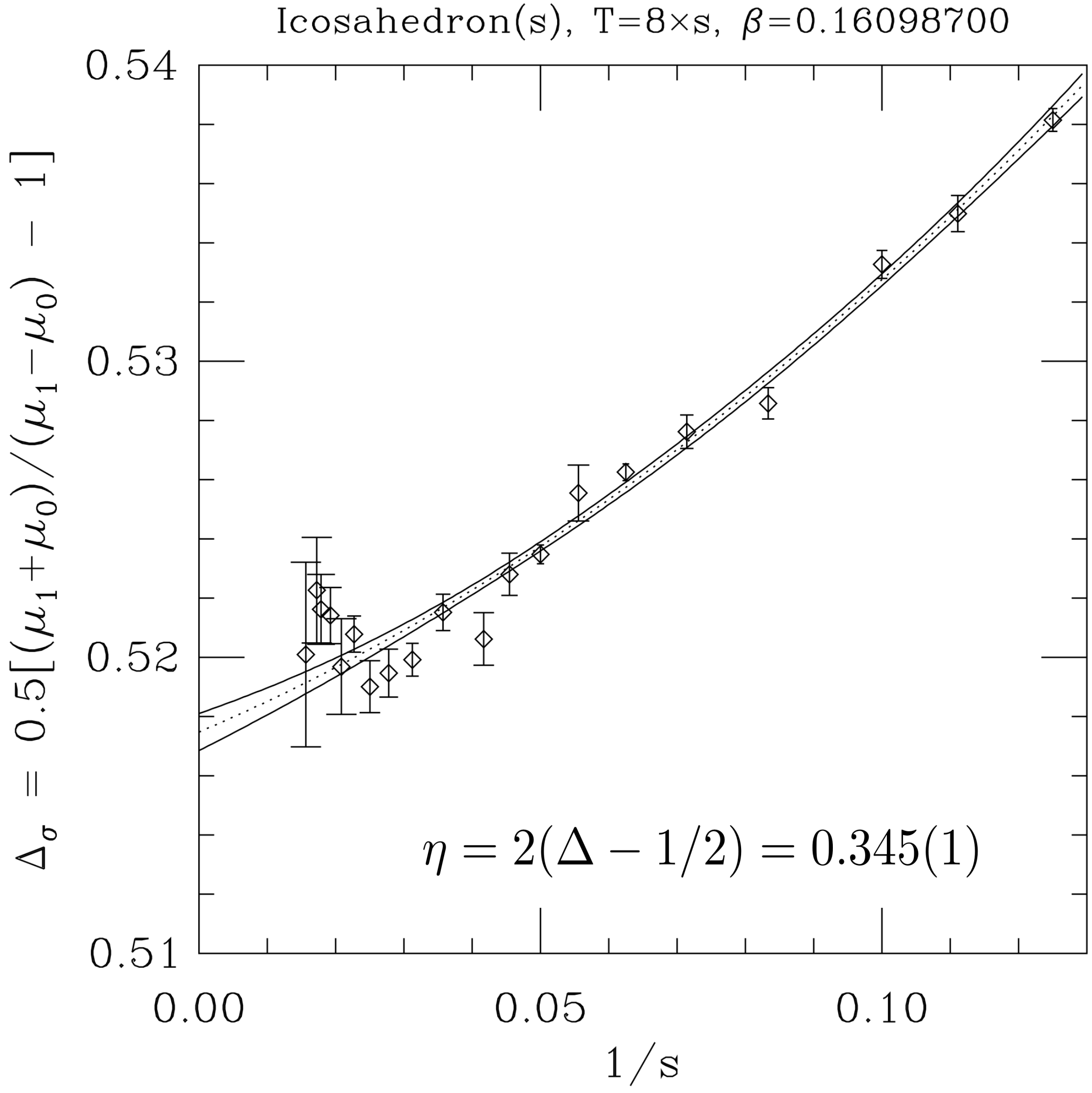
- Current anomalous dimensions (more soon)
  - from Binder: $\omega + 1/\nu = 2.51(11)$
  - from corr: $\Delta_\sigma = 1/2 + \eta/2 = 0.5175(6)$
  - Simulation are on going to reduce errors

Check Descendant Relation & rescale "log(r)"





Current Fit: $\Delta_\sigma = 0.5175(6)$



Fitting correlators

- Discrete states have exact cosh correlators

$$C_l(t) = A_l \cosh(-\mu_l(t - T/2))$$

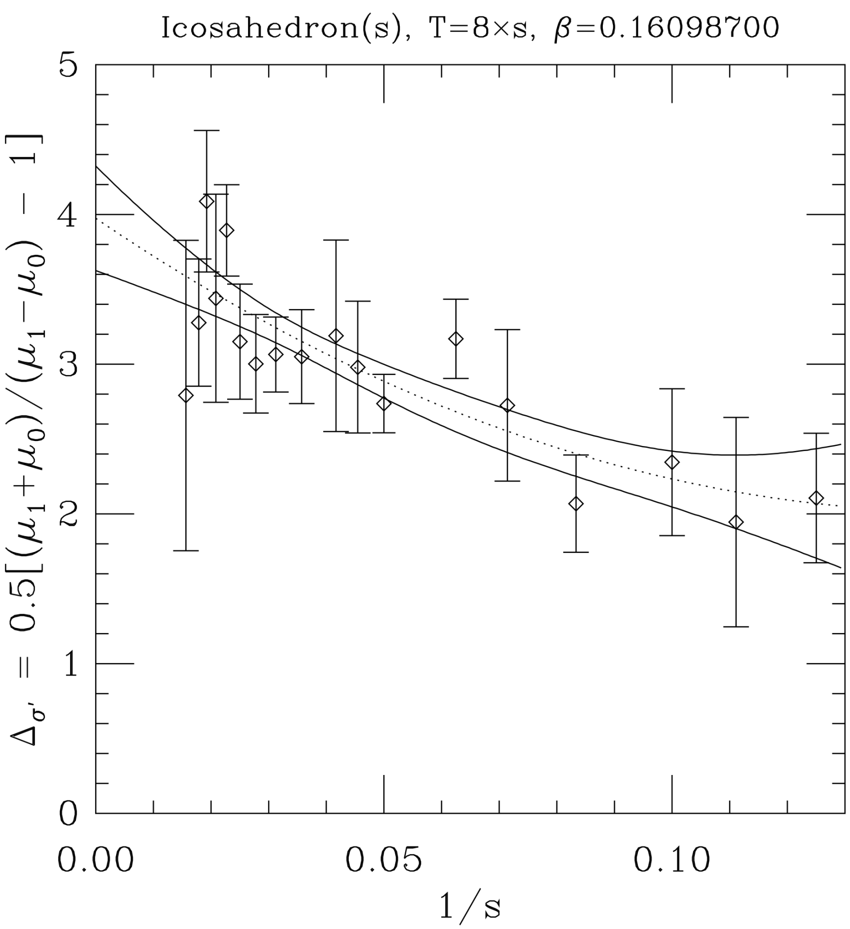
- Transform to k–space

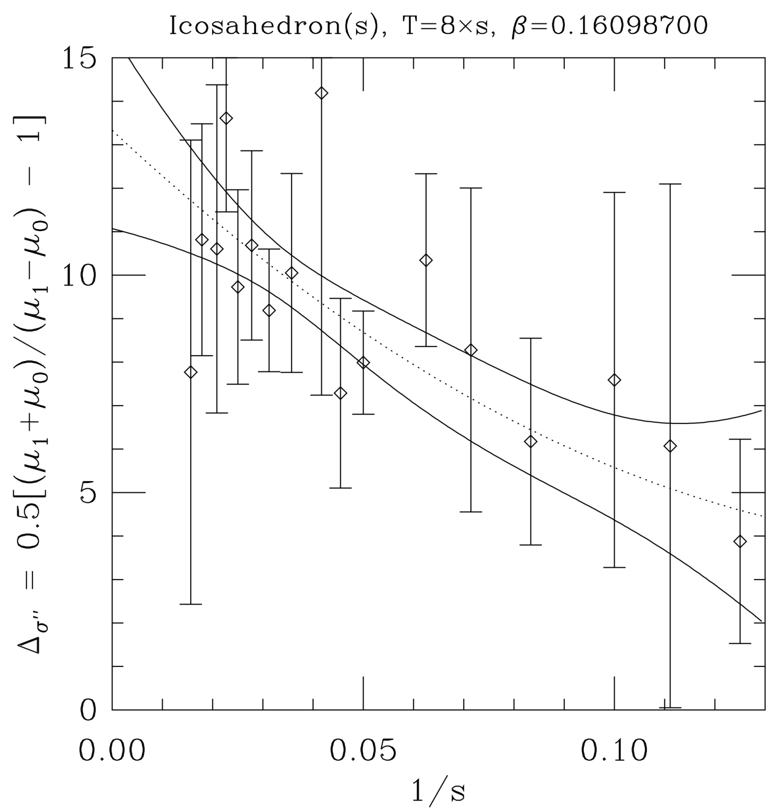
Disconnected piece

$$\begin{aligned} \widetilde{C}_l(k) &= \frac{1}{T} \sum_{t=0}^{T-1} e^{itk}\, C_l(t) \\ &= c_0 \delta_{l,0} \delta_{k,0} + a_l \frac{(1 - e^{-\mu_l T}) \sinh(\mu_l)}{\sinh^2(\mu_l/2) + \sin^2(k/2)}. \end{aligned}$$

- Good fits required 3 mass

Preliminary: Higher Primaries





Some Questions

- Restoration of full Conformal O(d+1,1) as 1/s → 0
  - Lattice only approximates the isometries of $\mathbb{R} \times \mathbb{S}^{d-1}$
  - Our action has a iscosahedron with flat faces

    (i.e. simplicial geometry from Regge calculus)
  - Do "conical" curvature defects relevant (Try tetrahedron)
  - Check 2-pt correlator for full conformal symmetry.
  - Check 3-point and 4-point functions as well?
  - "Hamiltonian Form": continuous time exact (dilation)

  + worm algorithms

  etc.

Future Challenges & Directions

- Other applications
  - O(N) model in 3-d compare with large N
  - Fermions on $\mathbb{R} \times \mathbb{S}^{d-1}$ (Dirac with Vierbein)
  - Maybe easiest with hypercubic shells.
  - Strengthen bootstrap inequalities.
  - Add mass deformations?
  - Study conformal IR fixed points for BSM.

    (Dilation operator is "time" dependent!)

Can you add Running coupling and mass deformations?

Callan-Symanzik Equation

$$\left[r\frac{\partial}{\partial r}+\beta(g)\frac{\partial}{\partial g}-(1+\gamma(g))m\frac{\partial}{\partial m}+2\Delta(g)\right]C(r,\Omega,g,m,\mu)=0$$

$$\beta(g)\simeq\omega(g^*-g)$$