Breakdown of large-N reduction in the quenched Eguchi-Kawai model

Barak Bringoltz

Univ. of Washington, Seattle

In collaboration with S. Sharpe

Large-N volume reduction

Eguchi Kawai `82

- Starting point : pure SU(N) on a single point.

$$S_{EK} = Nb \sum_{\mu<\nu} 2\mathrm{Re}\,\mathrm{Tr}\,(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)$$ with $b = (g^2 N)^{-1}$ symmetric under $U_\mu \to U_\mu z_\mu \quad ; \quad z_\mu \in Z_N$

- Observables : $W_C = \frac{1}{N}\,\mathrm{tr}\,U_{x,\hat\mu}U_{x+\hat\mu,\hat\nu}\cdots U_{x-\hat\nu-\hat\rho,\hat\rho}U_{x-\hat\nu,\hat\nu}\,,$ $\Rightarrow$ $W_C^{\text{reduced}} = \frac{1}{N}\,\mathrm{tr}\,U_\mu U_\nu \cdots U_\rho U_\nu\,.$

$$\langle W_C\rangle_{\text{gauge theory}} = \langle W_C^{\text{reduced}}\rangle_{\text{reduced}} + O(1/N^2).$$

- Dyson-Schwinger Eqs.: $\left\langle \mathrm{tr}\,(U_\mu U_\nu^\dagger)\ \mathrm{tr}\,(U_\mu^\dagger U_\nu)\right\rangle_{\text{reduced}} = 0$

1. $\langle W_{C_1} W_{C_2}\rangle_{\text{reduced}} = \langle W_{C_1}\rangle_{\text{reduced}}\langle W_{C_2}\rangle_{\text{reduced}} + O(1/N^2),$
2. $\langle W_{\text{open}}\rangle_{\text{reduced}} = 0.$

if Z_N intact

- However, weak-coupling analysis : Bhanot, Heller & Neuberger `82 (also later Kazakov & Migdal `82)

$$\mathrm{Eig}\,(U_\mu) = \left(e^{ip_\mu^1}, e^{ip_\mu^2}, \ldots, e^{ip_\mu^N}\right)$$ attract and break of Z_N

Alternatives to EK

Name of the game : cause p to repel each other

* Quench the p's to be uniform - the QEK
  BHN `82, Migdal `82, Gross and Kitazawa `82, Parisi `82, Bars `83, Okawa `82, Parsons `84, Carlson `83, Lewis '84, Greensite and co. `83-`86

* Twisted BC's - the TEK
  Gonzalez-Arroyo & Okawa `82
  but
  Teper and Vairinhos `06, Ishikawa et al.`07

* Partial reduction - $\mathbf{L}^4$ instead of $\mathbf{1}^4$
  Neuberger, Narayanan, Kiskis `04-`07

* Adjoint fermions - the AEK
  Kovtun, Unsal and Yaffe `07

* Deform the action - the DEK
  Unsal and Yaffe `08

The QEK model

Bhanot Heller & Neuberger `82,
(see also Migdal `82)

- Definition of the model :

(I) $$\langle \mathcal{O}(\mathcal{U}) \rangle_p = \frac{\int \prod_\mu DV_\mu \; e^{S_{\rm QEK}(p)} \; \mathcal{O}(\mathcal{U})}{\int \prod_\mu DV_\mu \exp(S_{\rm QEK}(p))}$$

(II) $$S_{\rm QEK}(p) = Nb \sum_{\mu<\nu} 2 {\rm Re\,Tr}\,(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)\,.$$

(III) $$\langle \mathcal{O}(\mathcal{U}) \rangle_{\rm QEK} = \int dp \;\; \langle \mathcal{O}(U) \rangle_p$$

invariant to $p_\mu \longrightarrow p_\mu + 2\pi k_\mu/N \quad ; \quad k_\mu = 1, 2, \ldots, N$

with

$$U_\mu = V_\mu^\dagger \Lambda_\mu V_\mu \,,$$

and

$$\Lambda_\mu(p) = \begin{pmatrix} e^{ip_\mu^1} & 0 & \ldots & 0 \\ 0 & e^{ip_\mu^1} & \ldots & 0 \\ \vdots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & e^{ip_\mu^N} \end{pmatrix}$$

$$p_\mu^a \in [0, 2\pi)\,.$$

Formal proofs

Gross & Kitazawa `82,
Parisi `82

✓ Planar perturbation theory.

- Perturbation theory : ***integrands*** of all planar diagrams in gauge theory.
- ∫ dp ⇒ all planar diagrams in gauge theory

✓ W-loop's Dyson-Schwinger equations.

- ∫ dp is Z_N invariant $p_\mu \longrightarrow p_\mu + 2\pi k_\mu/N \quad ; \quad k_\mu = 1, 2, \ldots, N$
- But W_{open} is not and so

$$\langle W_{\text{open}} \rangle_{QEK} = \int dp \, \langle W_{\text{open}} \rangle_p = 0$$

Is that enough ? No ! due to non-perturbative effects.

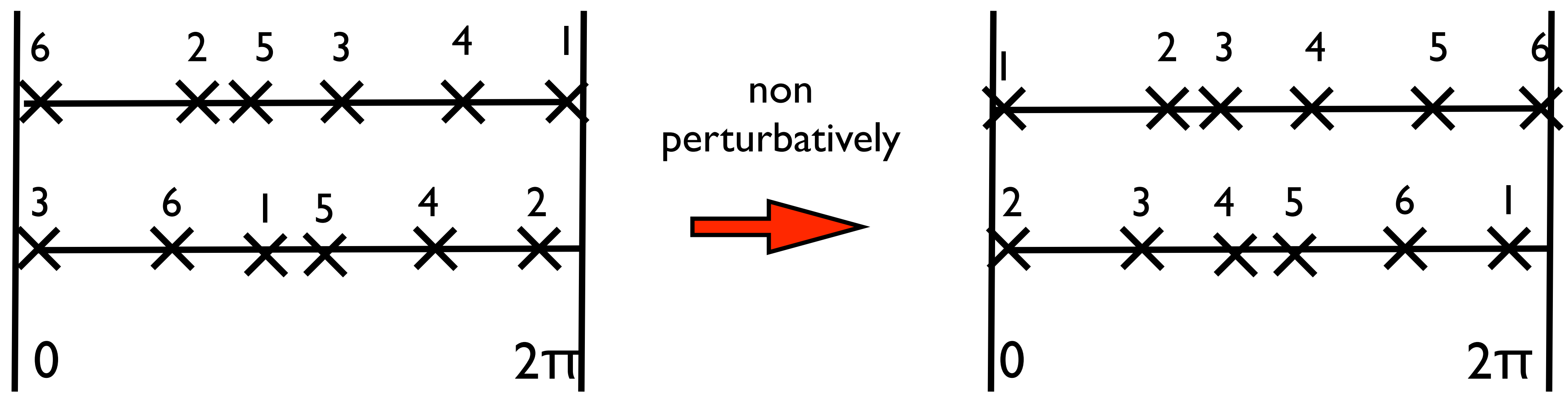
$\exists\, V_\mu \in SU(N)$
such that :

$$V_\mu \begin{pmatrix} e^{ip_\mu^1} & 0 & \cdots & 0 \\ 0 & e^{ip_\mu^2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{ip_\mu^N} \end{pmatrix} V_\mu^\dagger = \begin{pmatrix} e^{ip_\mu^{\sigma(1)}} & 0 & \cdots & 0 \\ 0 & e^{ip_\mu^{\sigma(2)}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{ip_\mu^{\sigma(N)}} \end{pmatrix}$$

- Focus on one point in $\int \prod_\mu \prod_i dp_\mu^i$ (imagine N=6)

$p_x^{i=1,2,3,4,5,6}$:

$p_y^{i=1,2,3,4,5,6}$:



- "Locking" occurs in weak-coupling : minima defined by $p^i_\mu - p^i_\nu \simeq \alpha_{\mu\nu}$

$M_{\mu,\nu} \equiv \mathrm{tr}\,(U_\mu U_\nu)/N$ and $M_{\mu,-\nu} \equiv \mathrm{tr}\,(U_\mu U_\nu^\dagger)/N \quad (\mu > \nu)$ $\Rightarrow$

$$\boxed{\begin{aligned} (M_{\mu,-\nu})_{\text{locked}} &\simeq e^{+i\alpha_{\mu\nu}} \\ |M_{\mu,-\nu}|_{\text{locked}} &\simeq 1 \end{aligned}}$$

Implications on Gross-Kitazawa-Parisi analysis

Planar perturbation theory

- Perturbatively fix p and integrate uniformly over 4D BZ of p.
- Non-perturbatively can get locking and p integral is not uniform.

DS equations: consider $W_{\text{open}} = M_{\mu,\nu} = \text{tr}\left(U_\mu U_\nu\right)/N$

usually : $\left\langle M_{\mu\nu} M^\star_{\mu\nu} \right\rangle_{QEK} = 0$, because $\left\langle M_{\mu\nu} \right\rangle_{QEK} = \left\langle M^\star_{\mu\nu} \right\rangle_{QEK} = 0$

but if locked : $$\underbrace{\int dp \left\langle M_{\mu\nu} M^\star_{\mu\nu} \right\rangle_p}_{\sim O(1)} \neq \underbrace{\int dp \left\langle M_{\mu\nu} \right\rangle_p}_{= 0} \underbrace{\int dp \left\langle M^\star_{\mu\nu} \right\rangle_p}_{= 0} + O(1/N)$$

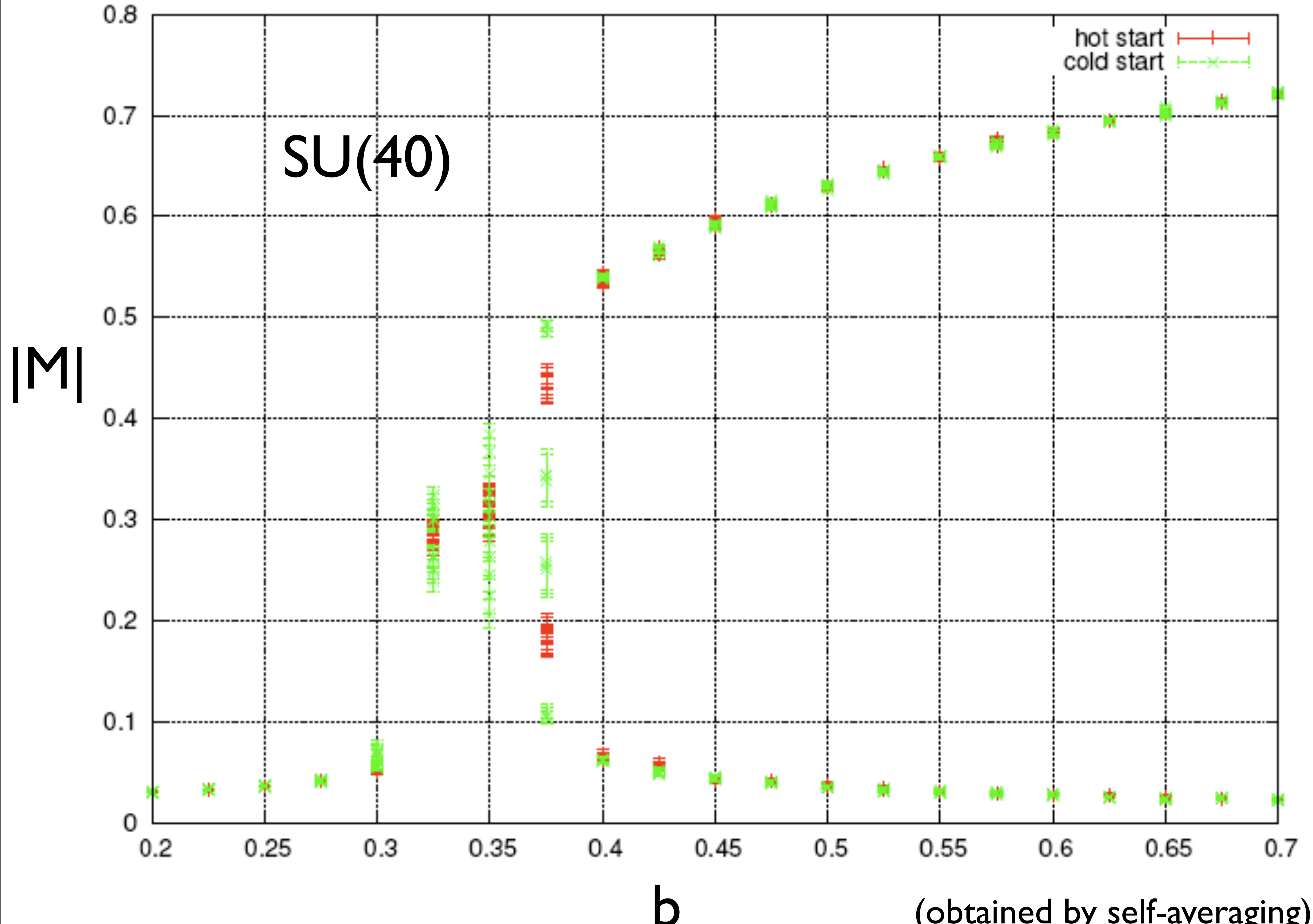
Does locking occurs ? Non-perturbative studies

1. Fix p, do MC to evaluate
$$\langle \mathcal{O}(\mathcal{U}) \rangle_p = \frac{\displaystyle\int \prod_\mu DV_\mu \; e^{S_{\mathrm{QEK}}(p)} \; \mathcal{O}(\mathcal{U})}{\displaystyle\int \prod_\mu DV_\mu \exp(S_{\mathrm{QEK}}(p))}$$

2. Integrate over p
$$\langle \mathcal{O}(\mathcal{U}) \rangle_{\mathrm{QEK}} = \int dp \; \langle \mathcal{O}(U) \rangle_p$$

1. N=20,30,40,50,80,100,125,150,200, @ 100K MEASUREMENTS.
2. USED : METROPOLIS/HYBRID HEAT BATH/HEAT BATH/OVER-RELAXATIONS.
3. VARIOUS CHOICES FOR P DISTRIBUTIONS (UNIFORM, "CLOCK" MOMENTA, "BARS")

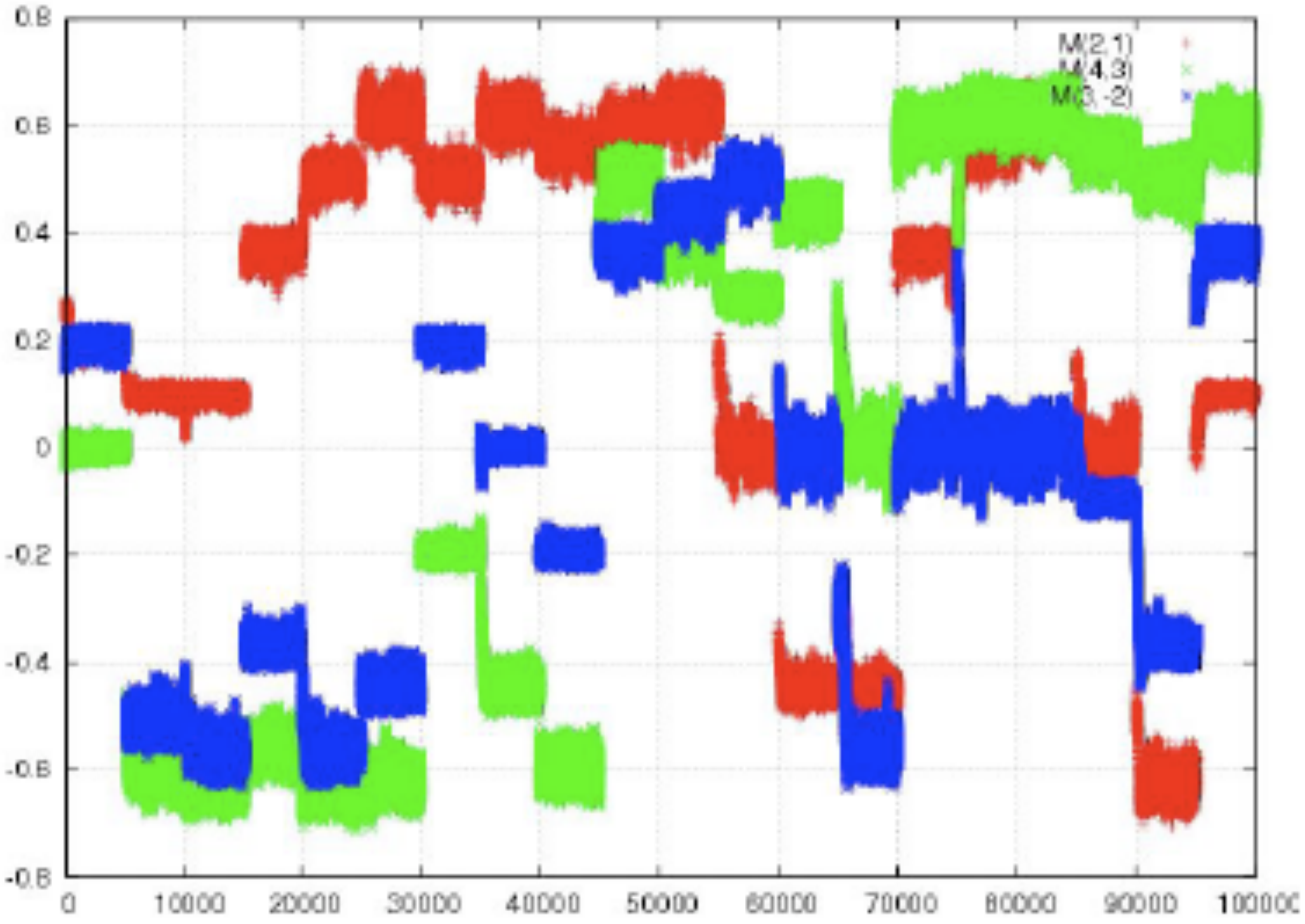
Results : MC lattice studies of QEK

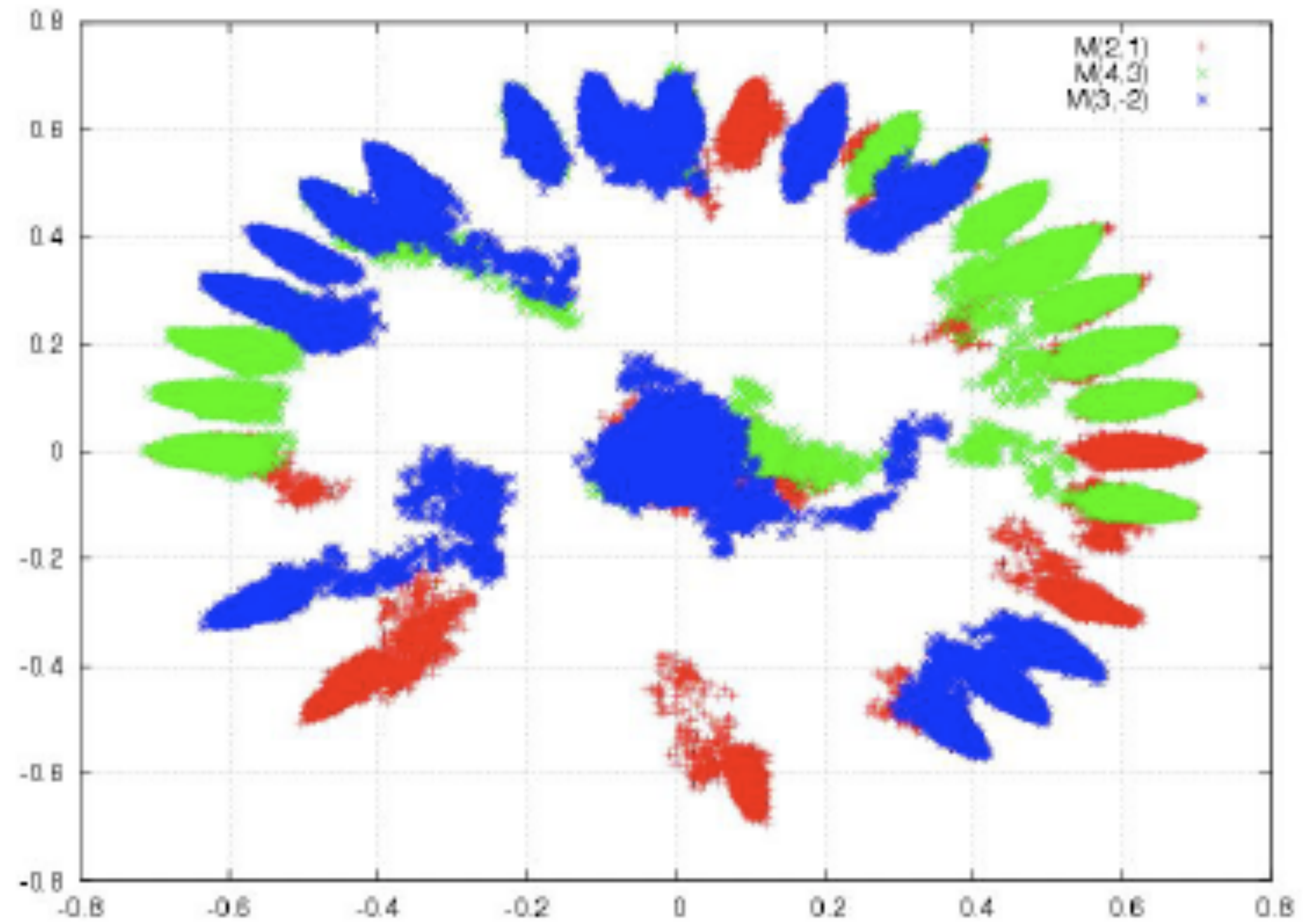


(obtained by self-averaging)

Results : MC lattice studies of QEK

SU(40), b=0.5





Real($M_{\mu\nu}$)

$M_{\mu\nu}$ in complex plane

Similar results $\forall$N, b, dp

Evidence for breakdown of quenched large-N reduction

- Weak-coupling : breakdown of Gross-Kitazawa-Parisi :
  - p’s chosen by non-perturbatively are locked, and not what your put in.
- Monte-Carlo studies of $N \leq 200$:
  1. Locking.
  2. Large discrepancies in plaquette of QEK vs. gauge.
  3. Large discrepancies in strong-to-weak transition coupling:
     - $b_{transition} = 0.3148(2)$ in QEK
     - $b_{bulk} \cong 0.36$

Other Large-N reductions on the lattice

- DEK : deform Yang-Mills action

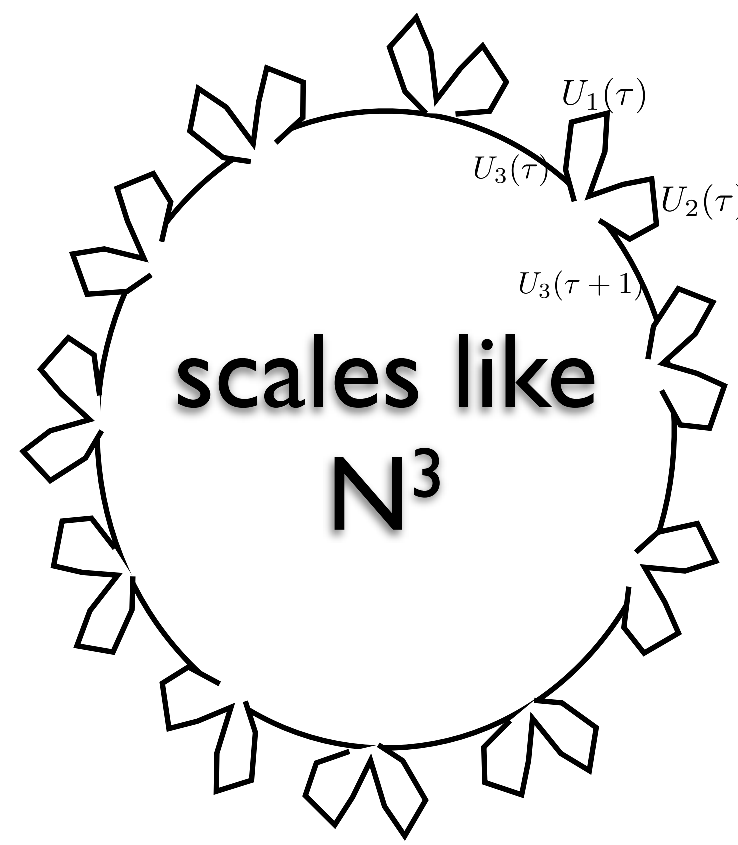
Unsal and Yaffe `08

$$S_{DEK} = S_{\rm YM} + \sum_{\substack{n_1,n_2\\ n_3,n_4}} a_{n_1,n_2,n_3,n_4} \left|\mathrm{tr}\left(U_\mu^{n_\mu} \cdot U_\nu^{n_\nu} \cdots U_\rho^{n_\rho}\right)\right|^2$$

  - Numerically hard : naively scales like N^7 !!!!!

- partial DEK : for example 2+1 dimensions

$$S_{DEK} = S_{\rm YM} + \int_0^{1/T} d\tau \sum_{n_1,n_2} a_{n_1,n_2} \left|\mathrm{tr}\left(U_1^{n_1}(\tau) \cdot U_2^{n_2}(\tau)\right)\right|^2$$



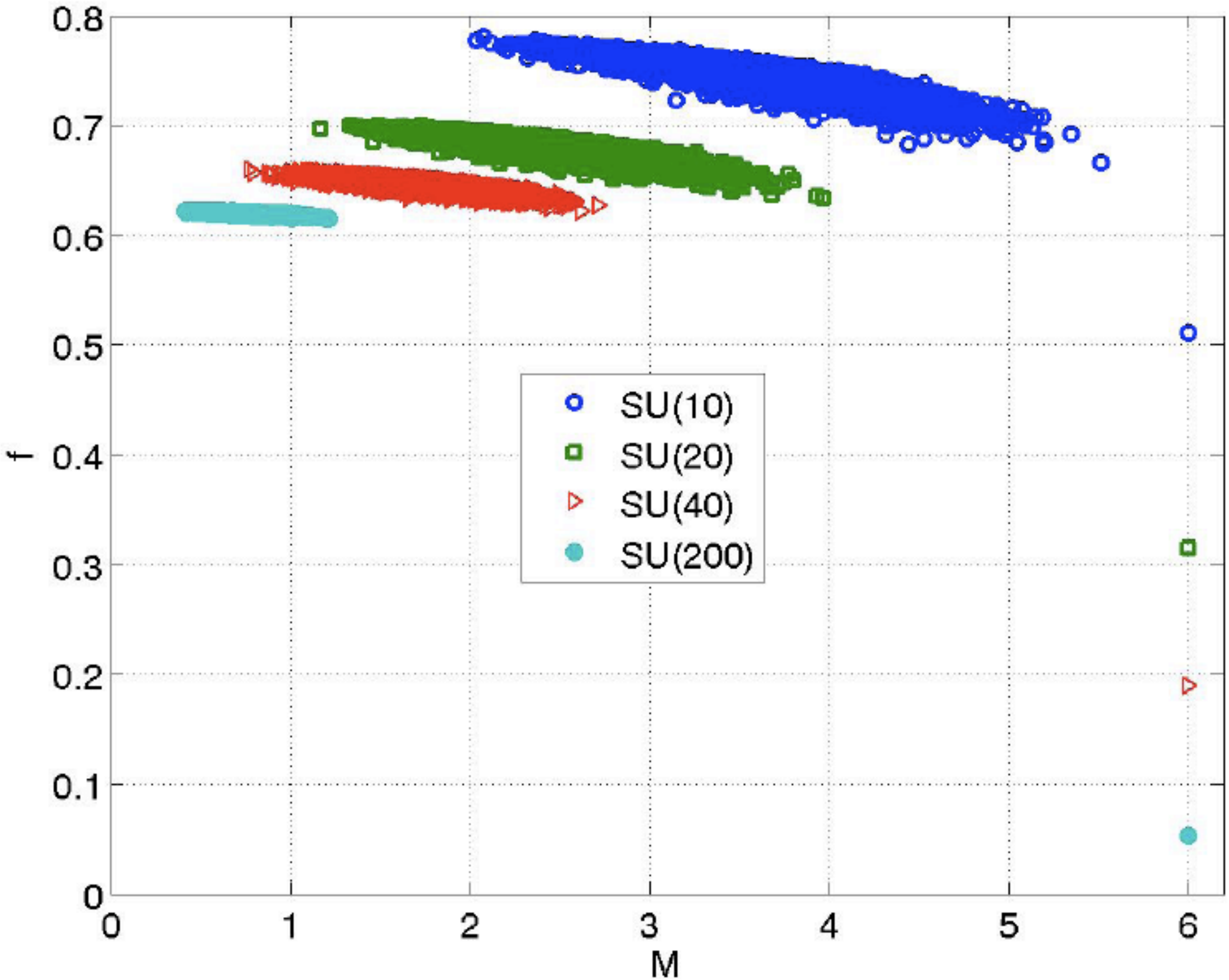
- AEK : dynamical adjoint fermions.

Kovtun, Unsal and Yaffe `07

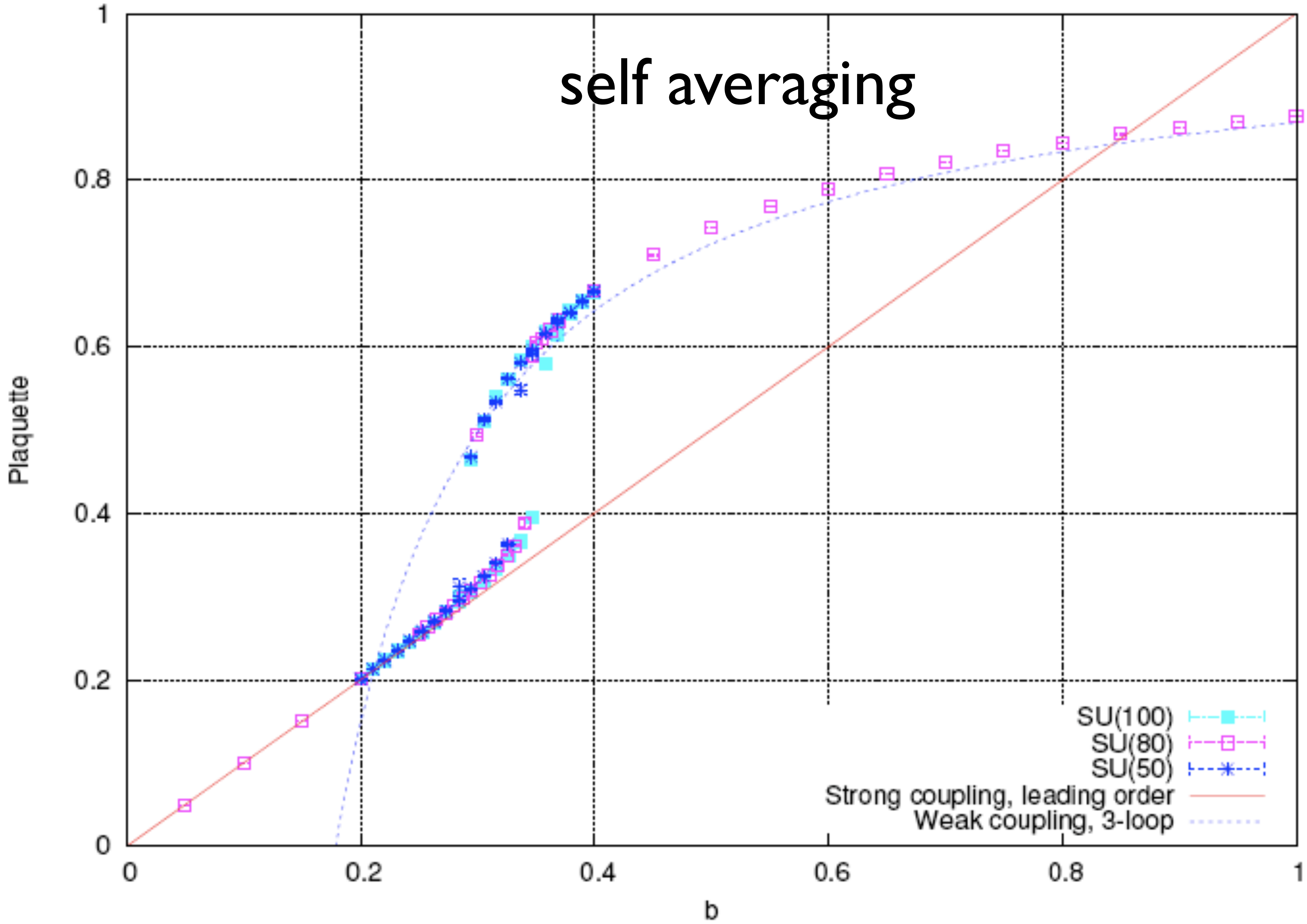
$$F_{\rm EK}(p) \overset{b\to\infty}{\longrightarrow} (d-2) \sum_{a<b} \log\left[\sum_\mu \sin^2\left(\frac{p_\mu^a - p_\mu^b}{2}\right)\right].$$

flips sign and p's reject

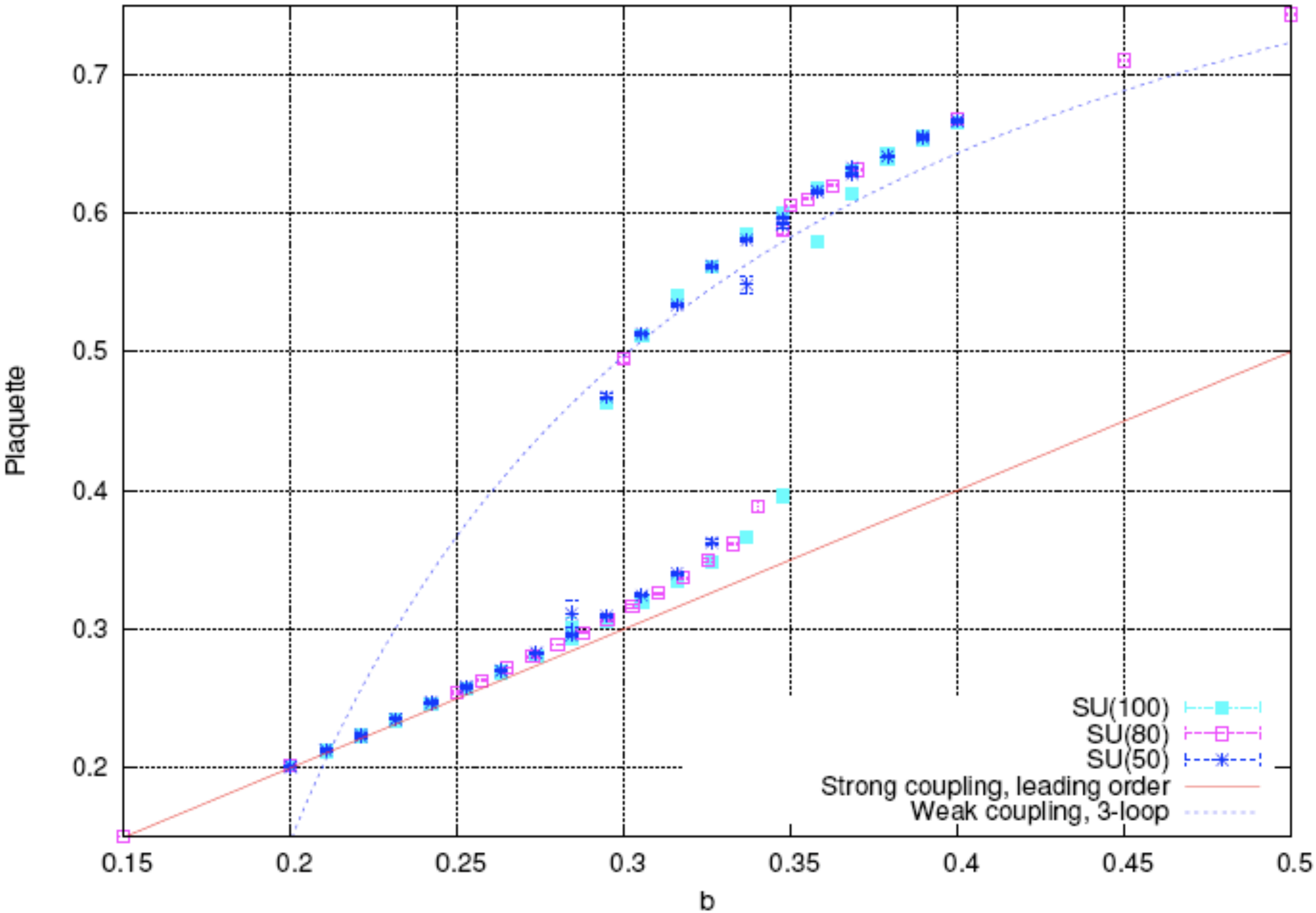
Minimizing $F_{QEK}(p_\mu^{\sigma(a)}) \sim \sum_{a<b} \log \left[\sum_\mu \sin^2 \left(\frac{p_\mu^{\sigma(a)} - p_\mu^{\sigma(b)}}{2} \right) \right]$



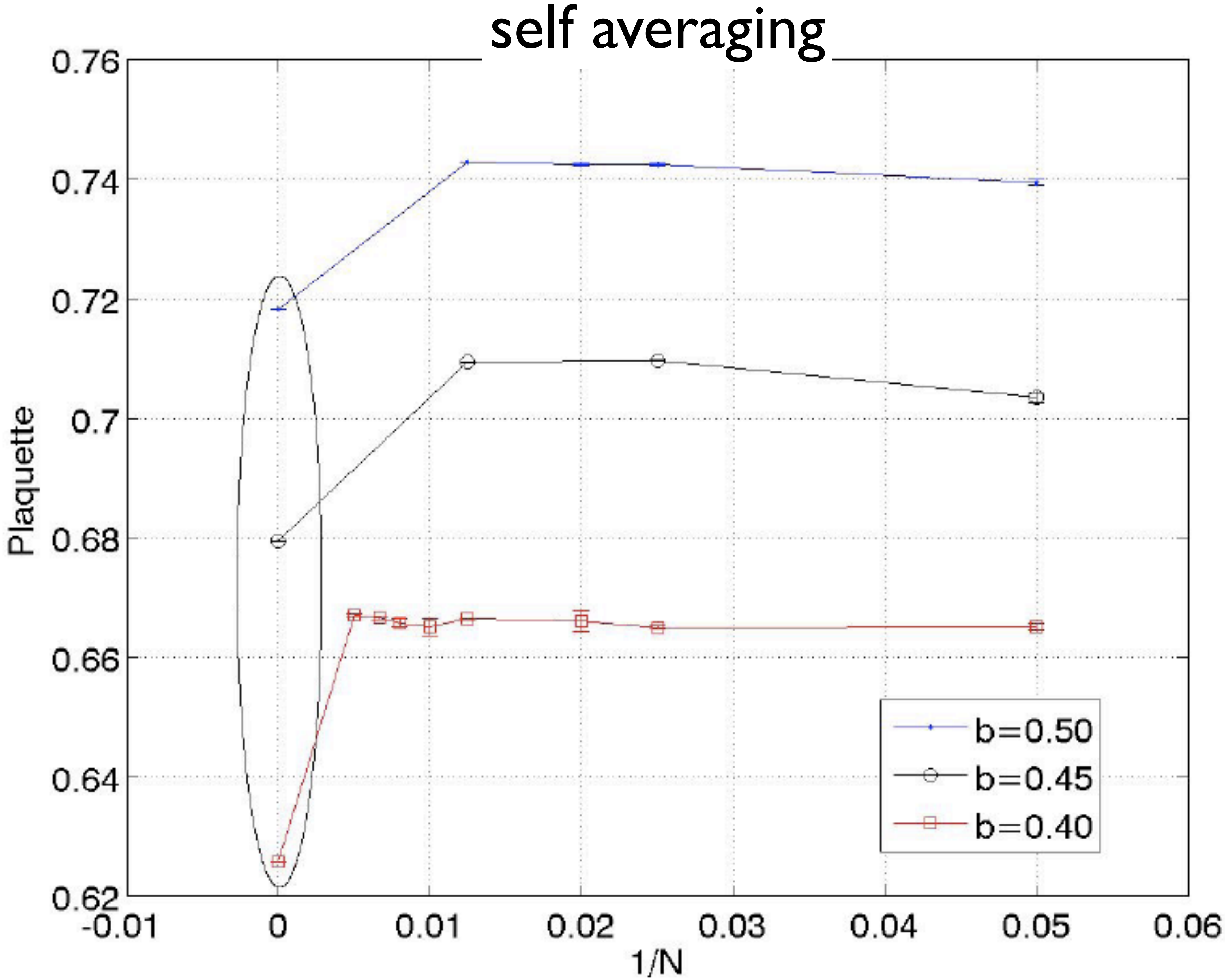
Results : MC lattice studies of QEK



Results : MC lattice studies of QEK



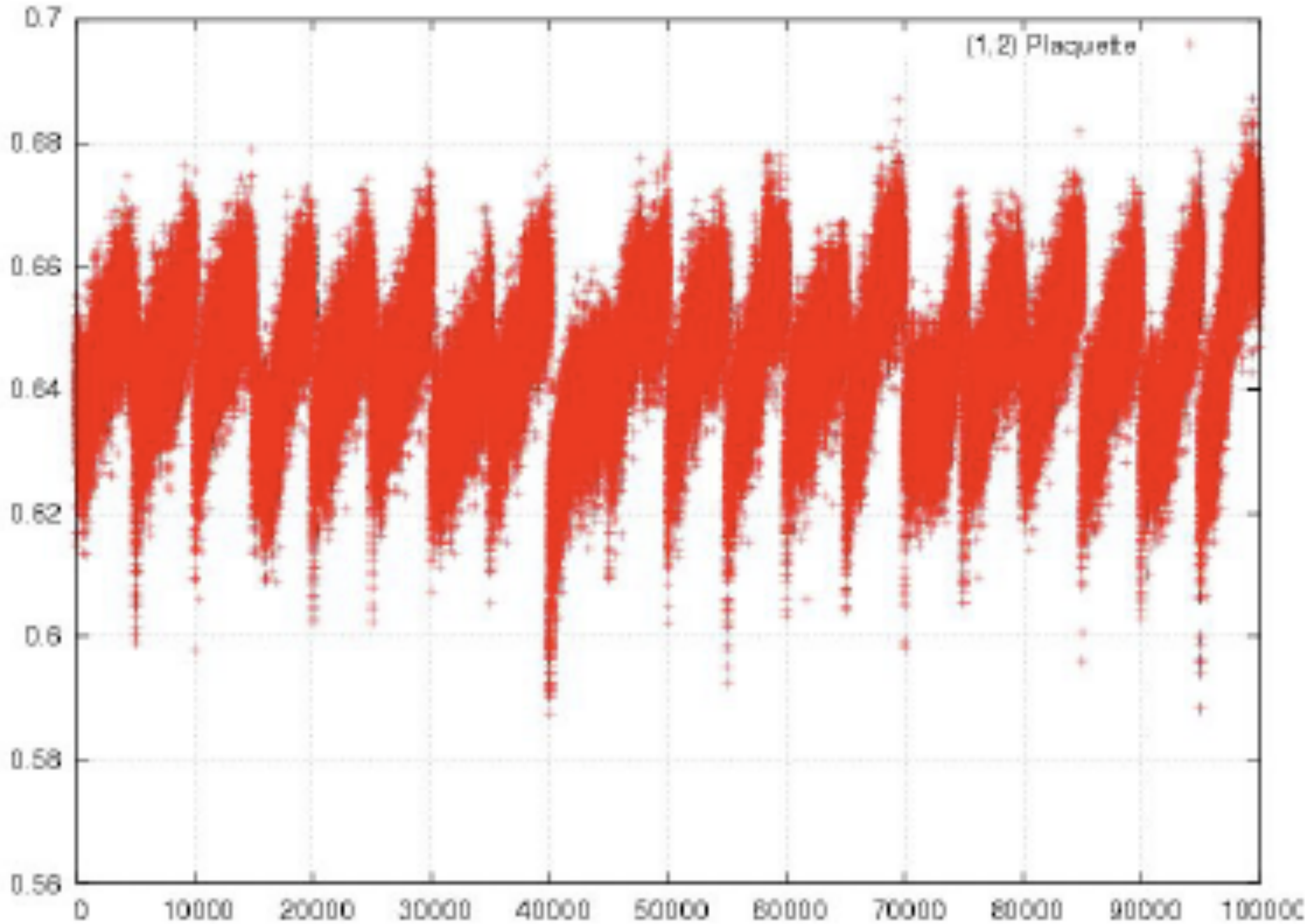
Results : MC lattice studies of QEK



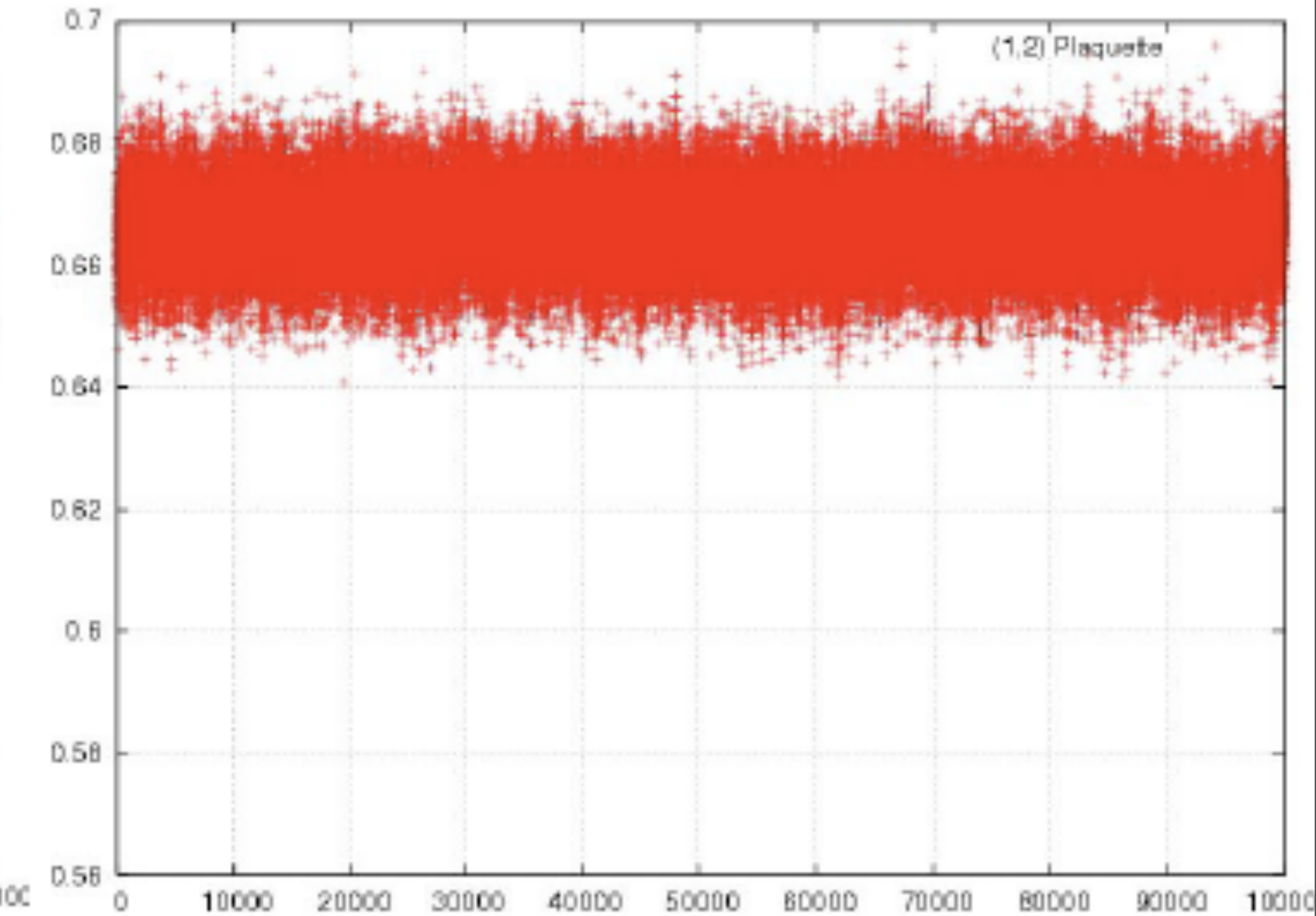
Results : MC lattice studies of QEK

Results : MC lattice studies of QEK

SU(80), b=0.4



Each 5000 have randomize p



self-averaging, p = locked start

QEK slowly tunnels to the locked state (20K updates)

Why ? (Intuitive)

E.g. Migdal `82

The QEK model		The original EK model	
(I)	$$\langle \mathcal{O}(\mathcal{U})\rangle_p = \frac{\int \prod_\mu DV_\mu \; e^{S_{\mathrm{QEK}}(p)} \; \mathcal{O}(\mathcal{U})}{\int \prod_\mu DV_\mu \exp(S_{\mathrm{QEK}}(p))}$$	(I)	$$\langle \mathcal{O}(\mathcal{U})\rangle_p = \frac{\int \prod_\mu DV_\mu \; e^{S_{EK}(p)} \; \mathcal{O}(\mathcal{U})}{\int \prod_\mu DV_\mu \exp(S_{EK}(p))}$$
(II)	$$S_{\mathrm{QEK}}(p) = Nb \sum_{\mu<\nu} 2\mathrm{Re}\,\mathrm{Tr}\,(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)\,.$$	(II)	$$S_{EK}(p) = Nb \sum_{\mu<\nu} 2\mathrm{Re}\,\mathrm{Tr}\,(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)\,.$$
(III)	$$\langle \mathcal{O}(\mathcal{U})\rangle_{\mathrm{QEK}} = \int dp \;\; \langle \mathcal{O}(U)\rangle_p$$	(III)	$$\langle \mathcal{O}(\mathcal{U})\rangle_{\mathrm{EK}} = \frac{\int dp \;\; \langle \mathcal{O}(U)\rangle_p \, Z(p)}{\int dp \; Z(p)}$$
Uniform :	$$\int dp$$	Non-uniform:	$$\int dp \;\; Z(p) \sim \int \prod_{\mu,a} \frac{dp_\mu^a}{2\pi} \;\; e^{-F_{EK}(p)}$$

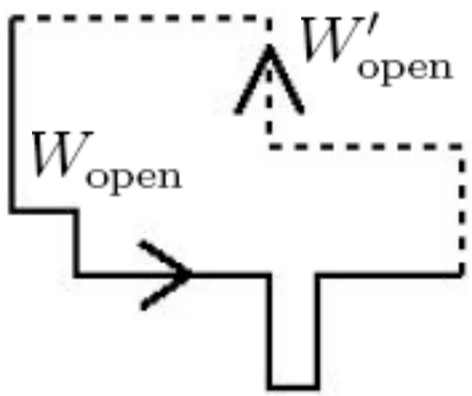
W-loop's Dyson-Schwinger Equations

- Do a change of variables $U_\mu \rightarrow U_\mu + i\mathcal{O}(\epsilon U_\mu)$

- Again get source terms, which with the p-integral are

$$\left\langle W_{\text{open}} W'_{\text{open}} \right\rangle_{\text{QEK}} = \int dp \, \left\langle W_{\text{open}} \, W'_{\text{open}} \right\rangle_p$$

with



- These are zero if quenched large-N factorization holds

$$\int dp \, \left\langle W_{\text{open}} \, W'_{\text{open}} \right\rangle_p = \int dp \, \langle W_{\text{open}} \rangle_p \int dp' \left\langle W'_{\text{open}} \right\rangle_{p'} + O(1/N)$$

Because $\int dp \, \langle W_{\text{open}} \rangle_p$ vanishes.

Quenched factorization - why ?

$$\int dp\, \left\langle W_{\rm open}\, W'_{\rm open} \right\rangle_p \stackrel{?}{=} \int dp\, \left\langle W_{\rm open} \right\rangle_p \int dp' \left\langle W'_{\rm open} \right\rangle_{p'} + O(1/N)$$

- Perturbation theory to (L+M)-loop order

$$\int dp \sum_{\substack{a_1,a_2,\ldots,a_L \\ b_1,b_2,\ldots,b_M}} f(p_{a_1},p_{a_2},\ldots,p_{a_L})\, g(p_{b_1},p_{b_2},\ldots,p_{b_M}).$$

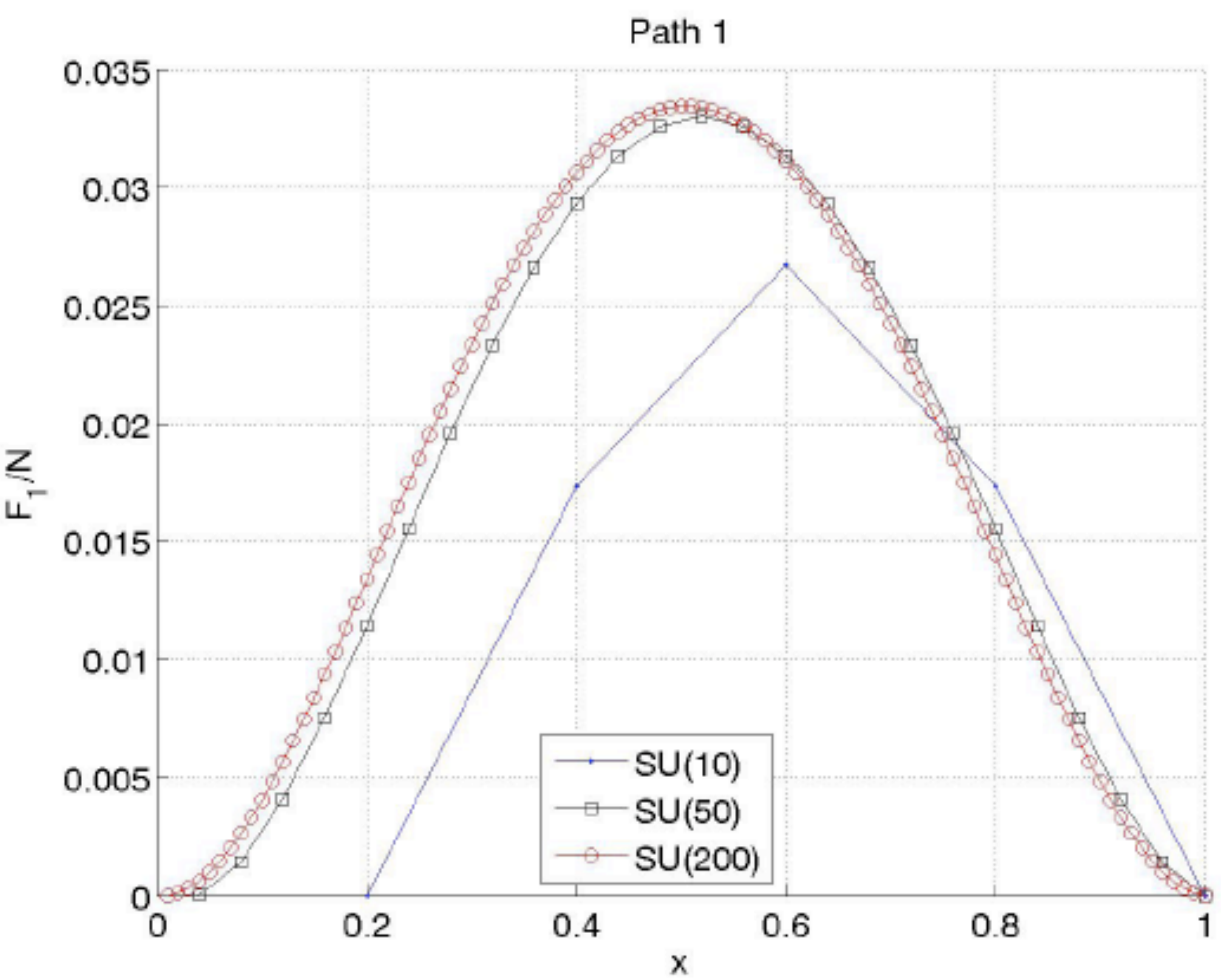
- For most terms in the sum, with the exception of O(1/N) have

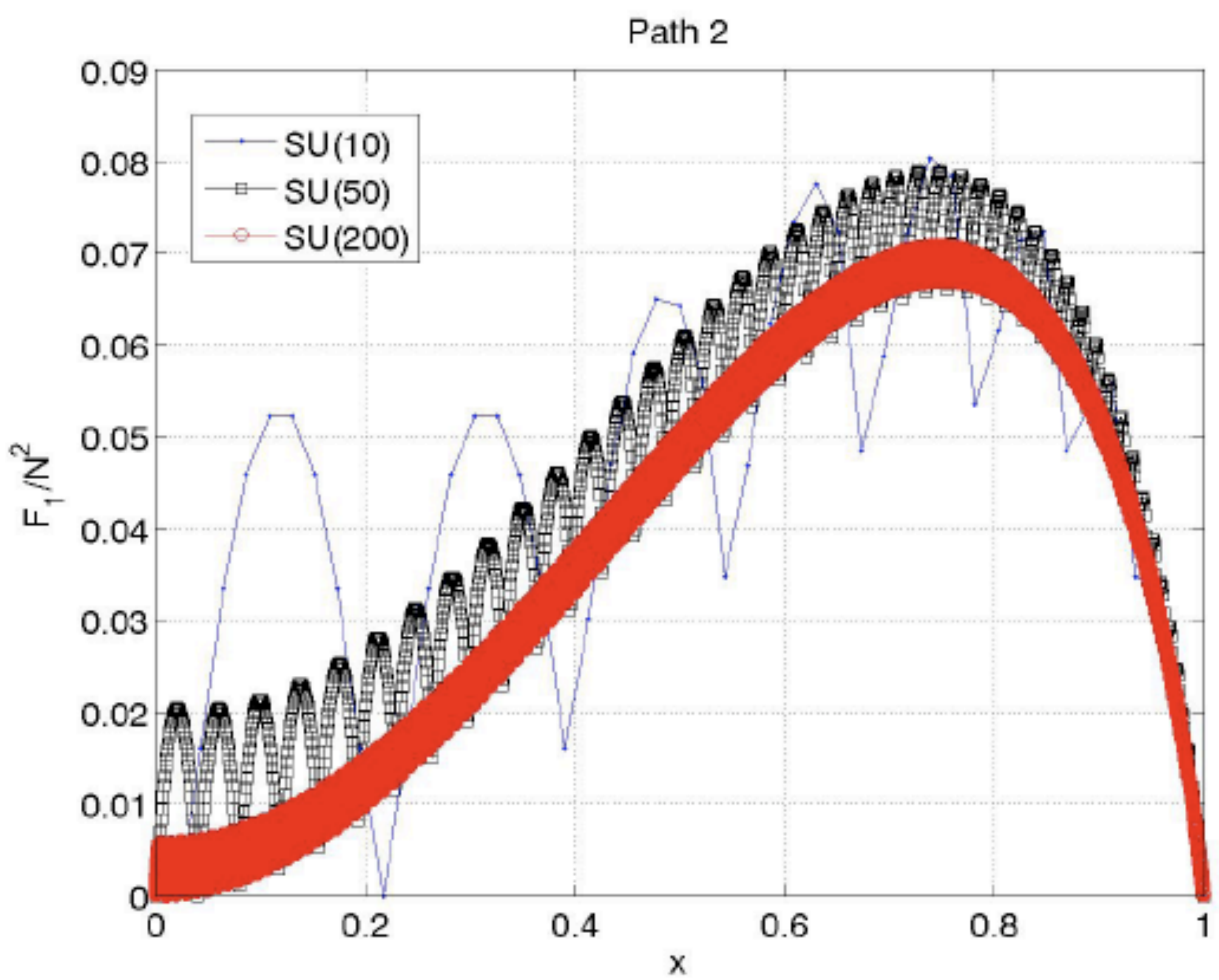
$$(a_1,a_2,a_3,\ldots,a_L) \neq (b_1,b_2,b_3,\ldots,b_M)$$

$$\Downarrow$$

$$\int dp\, f(p_{a_1},p_{a_2},\ldots,p_{a_L})\, g(p_{b_1},p_{b_2},\ldots,p_{b_M}) = \int dp\, f(p_{a_1},p_{a_2},\ldots,p_{a_L}) \int dq\, g(q_{b_1},q_{b_2},\ldots,q_{b_M}).$$

Free energy along path

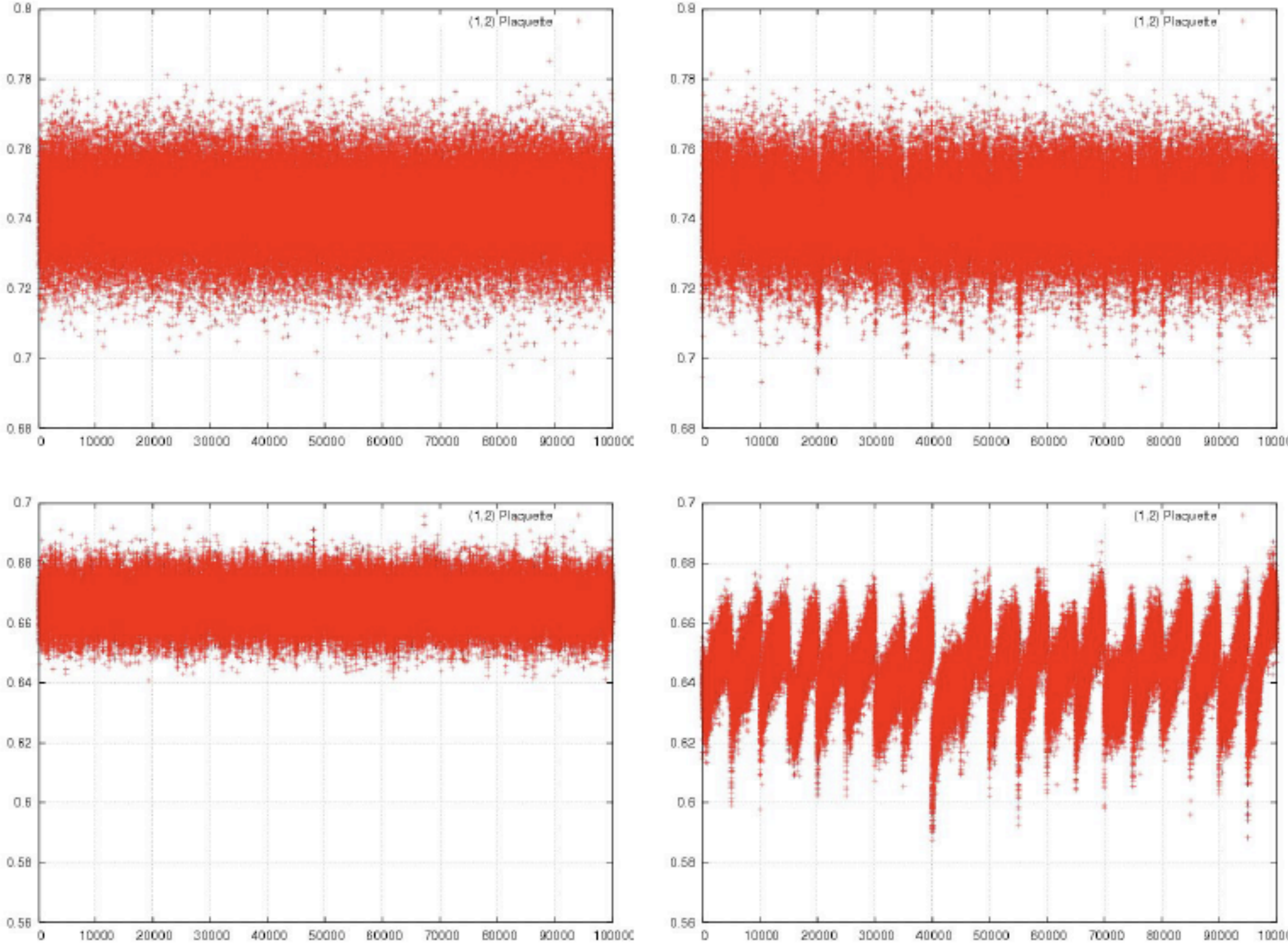




Plaquette vs MC time

SU(40)
at
b=0.5

SU(80)
at
b=0.4



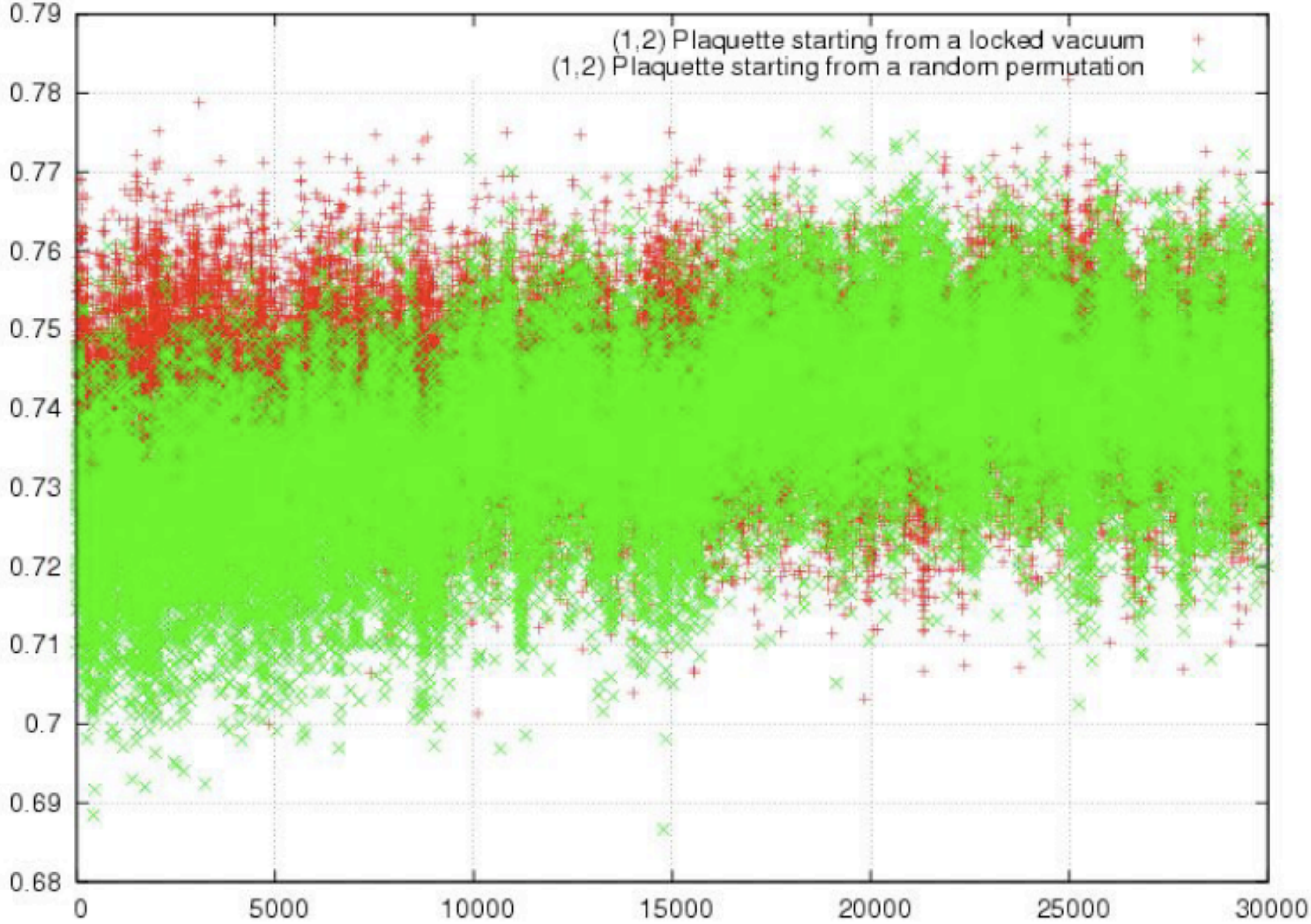
fixed p

randomize p every 5000

Tunneling event unlocked to locked

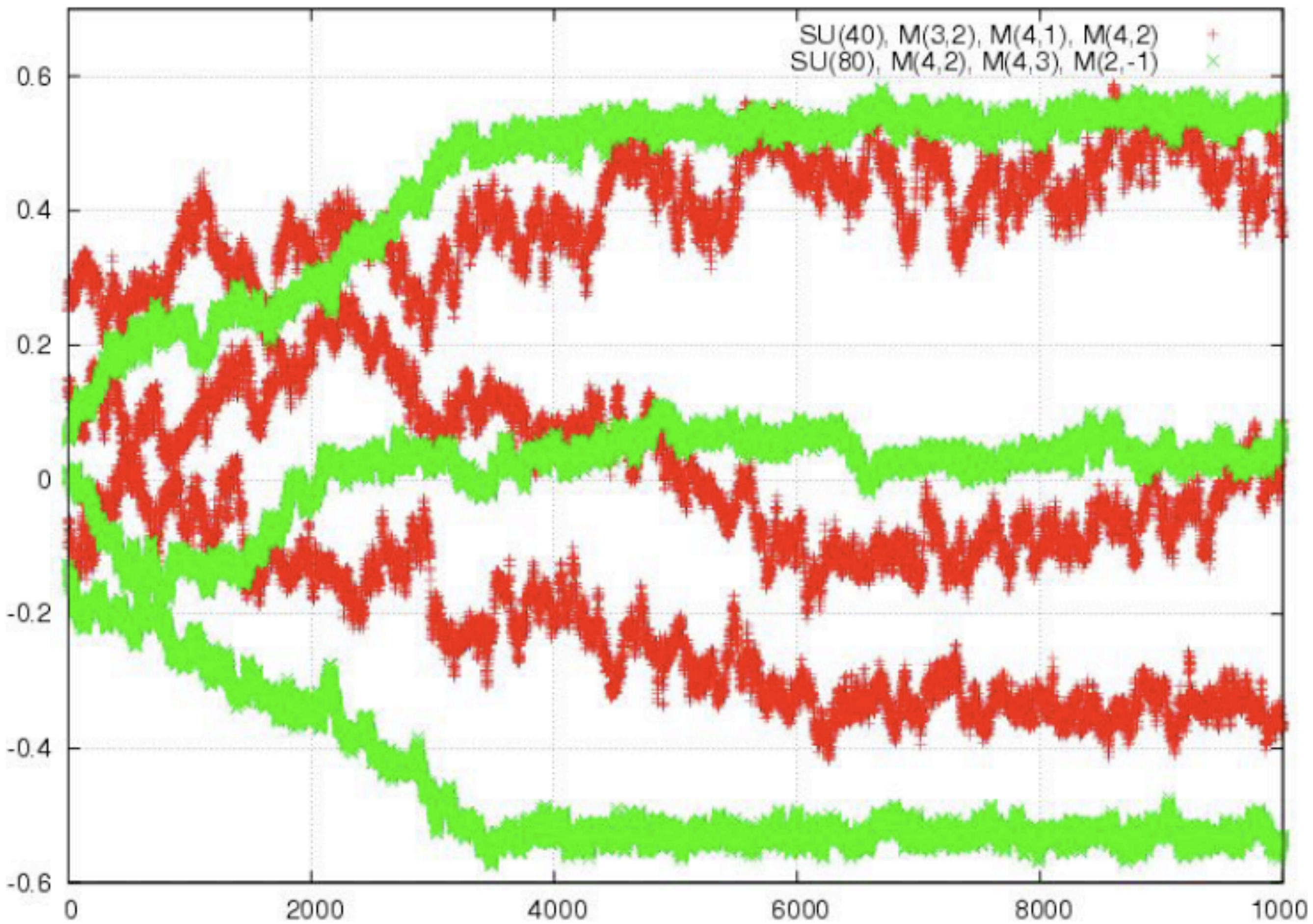
plaquette :

SU(40)
at
b=0.5

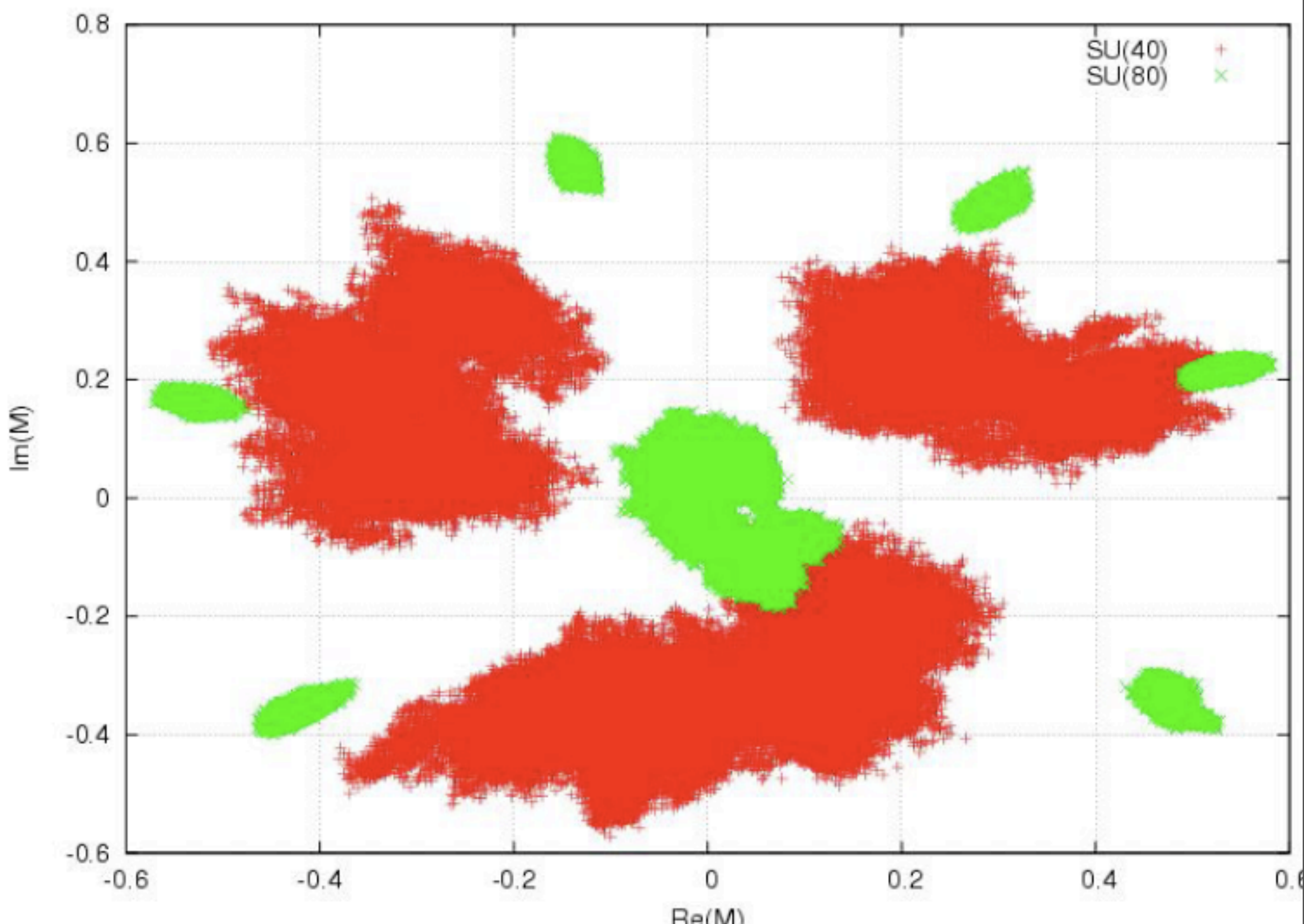


Non-perturbative locking

SU(40,80) at b=0.5, uniform dist.



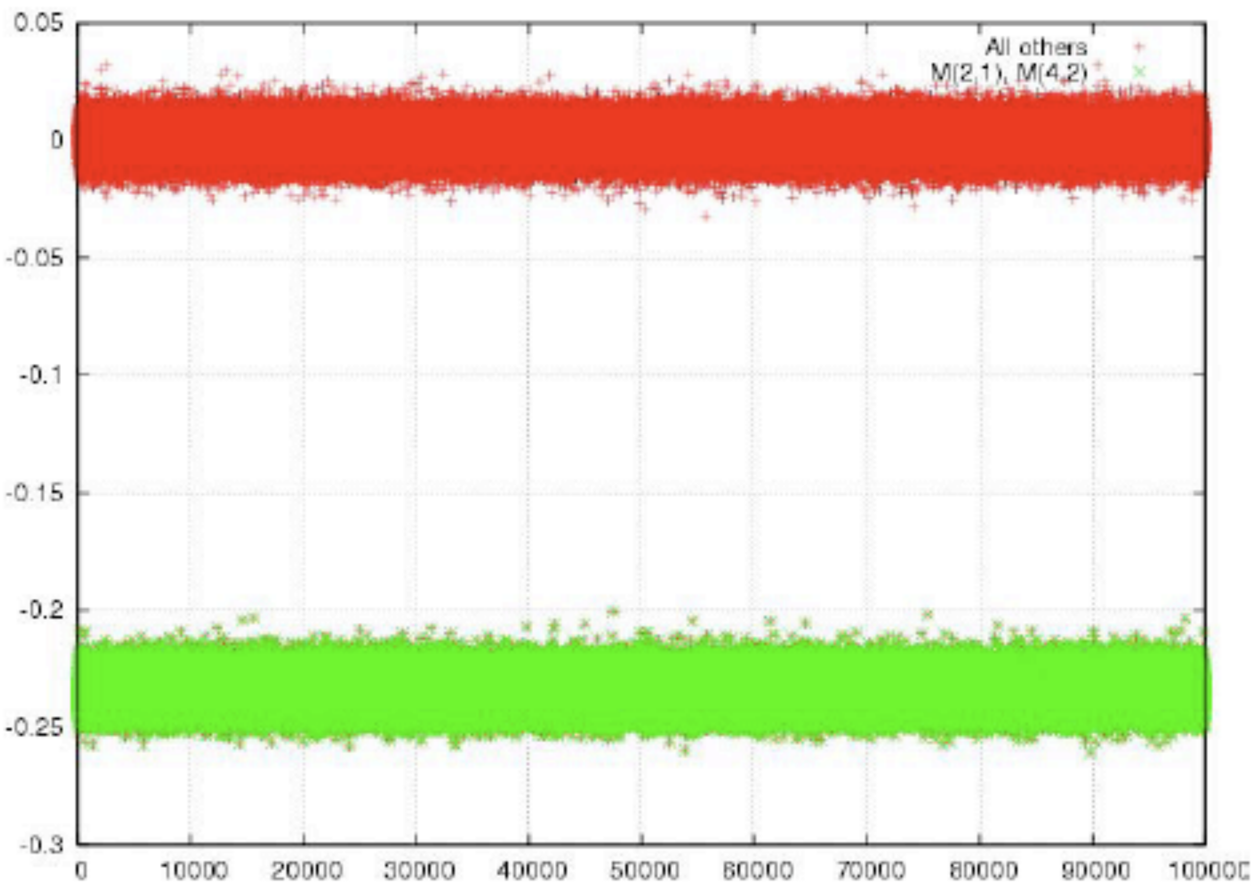
Real($M_{\mu\nu}$)



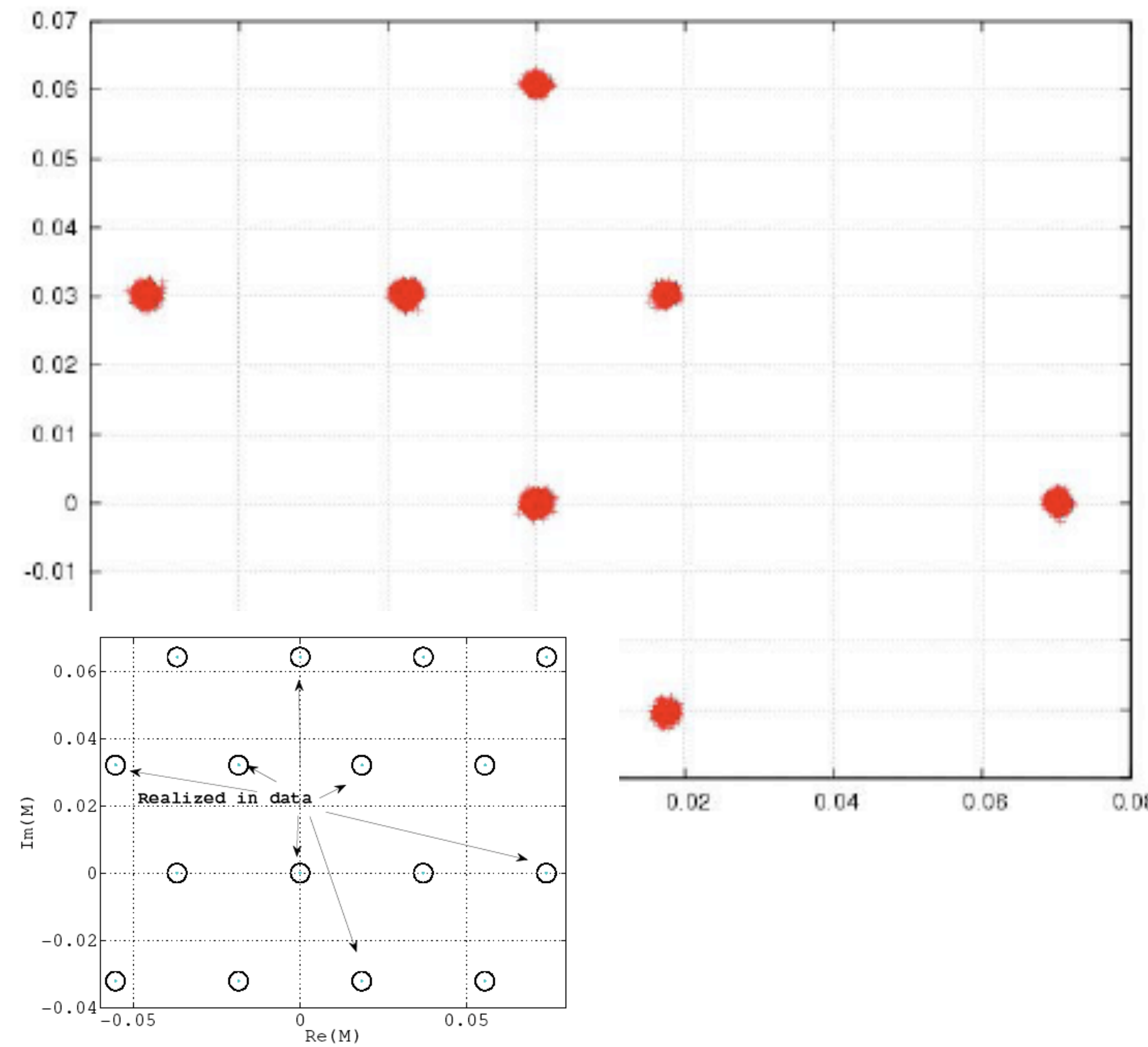
$M_{\mu\nu}$ in complex plane

Non-perturbative locking

SU(16,81) at b=0.7, dist. a la Bars



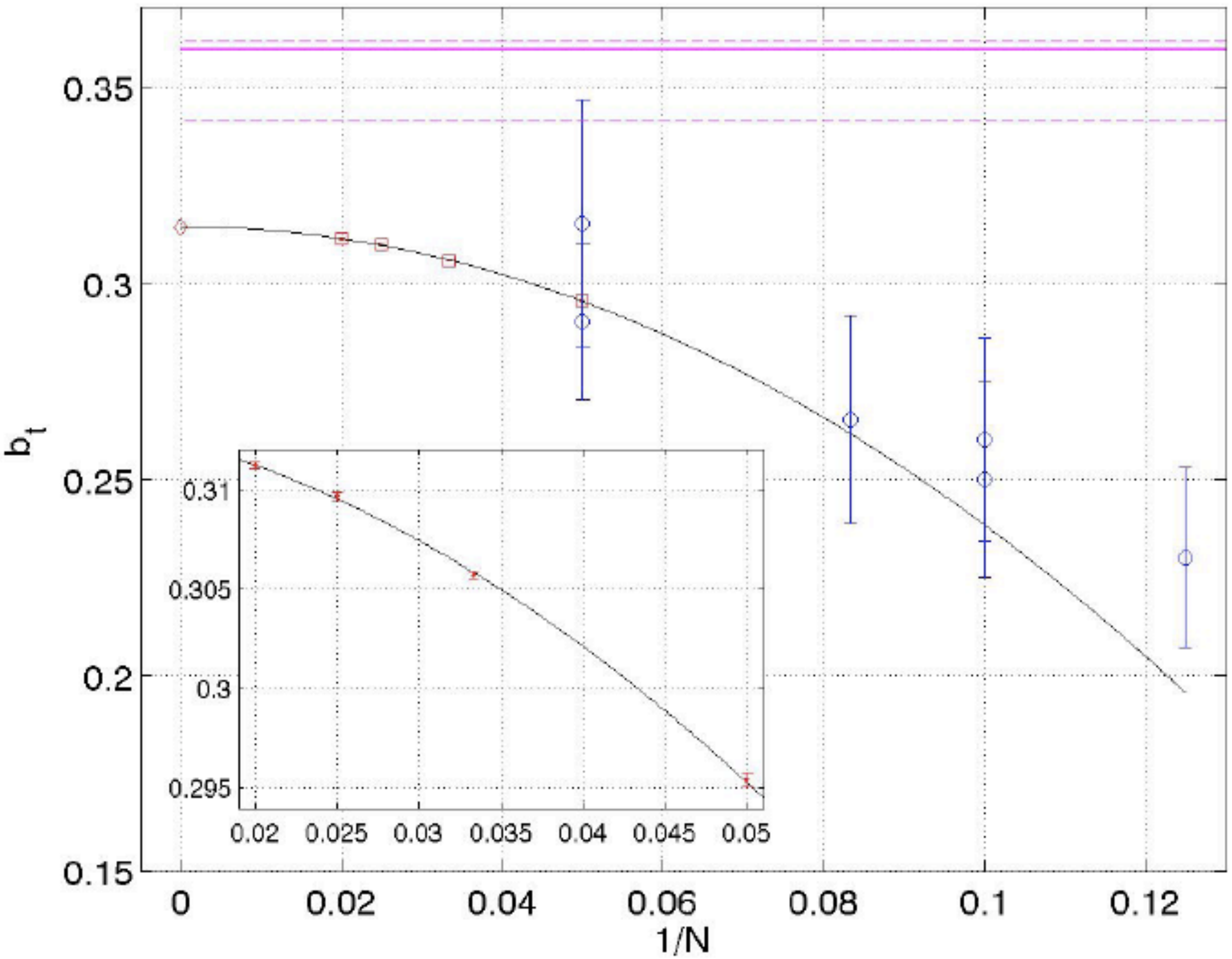
Real($M_{\mu\nu}$), SU(16)



$M_{\mu\nu}$ in complex plane, SU(81)

Transition is very strongly 1st order

- FIrst implementation of Wang-Landau algorithm for gauge theory



Large-N transition at

b = 0.3148(2) QEK

b ≅ 0.36 gauge theory