MATH228   Classical Mechanics

Kinematics of Rigid Bodies

A rigid body is one in which the distance between each pair of points does not change regardless of the forces acting on the body. The configuration of a rigid body has six degrees of freedom, three of which are needed to determine the position

$$\bar{r} = \frac{1}{M} \int_B \rho r \, dV$$

of the centroid, and three of which are needed to determine its orientation relative to the centroid. For example, the centroid of the Hubble Space Telescope follows an elliptical orbit about the earth determined by the gravitational field of the earth. Successful operation of the telescope is however crucially dependent on an ability to control its orientation. This is achieved by the use of small thruster jets to apply a torque about its centroid. Before discussing the dynamics of a rigid body, however, we first need to develop tools for describing its configuration in space and time.

Rotation of a rigid body

Suppose now that OXYZ is a set of Cartesian axes fixed in space. The displacement of the body \( B \) during a time \( t \) will then be characterised by the displacement of a given reference point \( O_B \) fixed in the body plus a rotation \( R \) mapping the original position vector of any point \( P \) of the body relative to \( O_B \) onto its current position vector relative to \( O_B \). Letting \( c(t) \) be the position vector of \( O_B \) relative to the fixed origin \( O \) at time \( t \), the position vector of the point \( P \) at time \( t \) can thus be written

$$\mathbf{r}(t) = \mathbf{c}(t) + \mathbf{R}(t)(\mathbf{r}_0 - \mathbf{c}_0)$$

where \( \mathbf{r}_0 \) and \( \mathbf{c}_0 \) are the initial positions of \( P \) and \( O_B \).
**Axis of rotation**

It is readily shown that any non-trivial rotation matrix $R$ has exactly one real and positive eigenvalue $\lambda = 1$. The corresponding eigenvector has the property

$$Rn = n.$$  

This eigenvector defines an axis through $O_B$ called the axis of rotation. Position vectors relative to $O_B$ of points on this line are invariant under the rotation $R$. This axis will in general vary with time.

When there is a fixed axis, choosing this to coincide with the axis $OZ$, lines in the $xy$-plane all undergo rotation through a fixed angle $\theta$. Since the images of the base vectors $i, j, k$ are

$$\cos \theta i + \sin \theta j, \quad -\sin \theta i + \cos \theta j, \quad k,$$

the rotation matrix is

$$R = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

**Angular velocity**

Differentiating the equation

$$r(t) = c(t) + R(t)(r_o - c_o)$$

the velocity of the point $P$ in the body relative to the fixed observer $O$ can be written

$$\dot{r} = \dot{c} + \dot{R}(r_o - c_o).$$

Noting that

$$\dot{R}(r_o - c_o) = \dot{RR}^T (r - c) = \Omega (r - c) = \omega \times (r - c),$$

where $\Omega = \dot{RR}^T$ is the antisymmetric spin matrix and $\omega$ the angular velocity of the rotation, this can be written

$$\dot{r} = \dot{c} + \omega \times (r - c).$$
The vector $\mathbf{\omega}$ is called the \textit{angular velocity} of $B$. At each instant this vector defines an axis $L$ through the body reference point $O_B$ called the \textit{instantaneous axis of rotation}. (It does not in general coincide with the axis of rotation associated with the rotation matrix $\mathbf{R}$). Points on $L$ are instantaneously at rest relative to $O_B$. The motion relative to $O_B$ of any point $P$ not on this line is (instantaneously) at right angles to $\mathbf{\omega}$ and so lies in a plane perpendicular to $L$. The motion must also be perpendicular to $\mathbf{r}$, in accordance with the requirement that it must lie at a fixed distance from $O_B$.

![Diagram](image)

The speed of $P$ in the plane perpendicular to $L$ is

$$v = \mathbf{\omega} r \sin \theta$$

where $\theta$ is the angle between $\mathbf{r}$ and $\mathbf{\omega}$. Since this can also be written

$$v = \mathbf{\omega} d$$

where $d = r \sin \theta$ is the perpendicular distance of $P$ from $L$, the magnitude of $\mathbf{\omega}$ is the angular speed of the instantaneous rotation of $B$ about $L$.

\textbf{Angular momentum}

The angular momentum about $O$ of a particle $P$ of mass $m$ moving with velocity $\mathbf{\dot{r}}$ is defined to be

$$\mathbf{h} = m \mathbf{r} \times \mathbf{\dot{r}}$$

where $\mathbf{r}$ is the position vector of $P$ relative to the given origin $O$. The angular momentum of a system of $N$ particles $P_i (i = 1, ..., N)$ is

$$\mathbf{h} = \sum_{i=1}^{N} m_i \mathbf{r}_i \times \mathbf{\dot{r}}_i .$$

Applying this notion to the rigid body $B$, the angular momentum of a small element of $B$ of volume $\delta V$ is $\delta \mathbf{h} = (\rho \delta V) \mathbf{r} \times \mathbf{\dot{r}}$, where $\rho$ is the mass density. Summing over the whole body and taking the limit as the dimensions of each element tend to zero, the angular momentum of $B$ about the given origin $O$ is

$$\mathbf{h} = \int_B \rho \mathbf{r} \times \mathbf{\dot{r}} dV .$$
The angular momentum \( h \) can be resolved into two parts, the angular momentum about \( O \) of a particle of mass \( M \) placed at the centroid of the body plus the angular momentum of the body about its own mass centre.

To show this, choosing the centroid \( C \) as body reference point, the position vector of the point \( P \) relative to \( O \) can be written

\[
\mathbf{r} = \mathbf{r}_C + \mathbf{s}
\]

where \( \mathbf{r}_C \) is the position vector of \( C \) relative to \( O \), and \( \mathbf{s} \) is the position vector of \( P \) relative to \( C \). Since then

\[
\dot{\mathbf{r}} = \dot{\mathbf{r}}_C + \dot{\mathbf{s}}
\]

we have

\[
\mathbf{r} \times \dot{\mathbf{r}} = (\mathbf{r}_C + \mathbf{s}) \times (\dot{\mathbf{r}}_C + \dot{\mathbf{s}}) = \mathbf{r}_C \times \dot{\mathbf{r}}_C + \mathbf{s} \times \dot{\mathbf{s}} + \mathbf{r}_C \times \dot{\mathbf{s}} + \mathbf{s} \times \dot{\mathbf{r}}_C
\]

whence

\[
\mathbf{h} = (\mathbf{r} \times \dot{\mathbf{r}}) \left( \int_B \rho \, dV \right) + \int_B \rho \mathbf{s} \times \dot{\mathbf{s}} \, dV + \left( \int_B \rho \, dV \right) \times \dot{\mathbf{r}}_C + \mathbf{r}_C \times \frac{d}{dt} \left( \int_B \rho \, dV \right)
\]

Given that \( C \) is the centroid of the body, so that

\[
\int_B \rho \mathbf{s} \, dV = 0,
\]

and noting that the mass of the body is

\[
M = \int_B \rho \, dV,
\]

this expression reduces to

\[
\mathbf{h} = \mathbf{r}_C \times \mathbf{p} + \mathbf{h}_C
\]

where

\[
\mathbf{p} = M \dot{\mathbf{r}}_C
\]

is the linear momentum of \( B \) and

\[
\mathbf{h}_C = \int_B \rho \mathbf{s} \times \dot{\mathbf{s}} \, dV
\]

is the angular momentum of \( B \) about \( C \).
The inertia matrix

With the centroid as body reference point the velocity of a point of the body with position vector \( \mathbf{r} \) can be written
\[
\dot{\mathbf{r}} = \dot{\mathbf{r}} + \mathbf{\omega} \times (\mathbf{r} - \bar{\mathbf{r}}) = \dot{\mathbf{r}} + \mathbf{\omega} \times \mathbf{r},
\]
whence \( \dot{\mathbf{s}} = \mathbf{\omega} \times \mathbf{s} \). Hence the angular momentum about C can be written
\[
\mathbf{h}_C = \int_B \rho \mathbf{s} \times (\mathbf{\omega} \times \mathbf{s}) \, dV.
\]
Using one of the standard vector identities,
\[
\mathbf{s} \times (\mathbf{\omega} \times \mathbf{s}) = s^2 \mathbf{\omega} - (\mathbf{s} \cdot \mathbf{\omega}) \mathbf{s}.
\]
Introducing the notation
\[
\mathbf{s} \otimes \mathbf{s} = \begin{pmatrix} x & y & z \\ y & x & z \\ z & y & x \end{pmatrix} = \begin{pmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{pmatrix}
\]
where \( x, y, z \) are Cartesian coordinates relative to axes with an origin at C, noting that
\[
s^2 \mathbf{\omega} = (s^2 \mathbf{1}) \mathbf{\omega}, \quad (\mathbf{s} \cdot \mathbf{\omega}) \mathbf{s} = (\mathbf{s} \otimes \mathbf{s}) \mathbf{\omega}
\]
the RHS of this equation can be written in the matrix form
\[
(s^2 \mathbf{1} - \mathbf{s} \otimes \mathbf{s}) \cdot \mathbf{\omega} = \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}.
\]
It follows that the angular momentum of a rigid body about its centroid C can be written
\[
\mathbf{h}_C = \mathbf{I}_C \mathbf{\omega}
\]
where
\[
\mathbf{I}_C = \int_B \rho (s^2 \mathbf{1} - \mathbf{s} \otimes \mathbf{s}) \, dV = \int_B \rho \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} \, dV.
\]
The symmetric matrix \( \mathbf{I}_C \) is called the \textit{inertia matrix} of \( B \) relative to the centroid C. The components
\[
I_{C,xx} = \int_B \rho (y^2 + z^2) \, dV \\
I_{C,yy} = \int_B \rho (x^2 + z^2) \, dV \\
I_{C,zz} = \int_B \rho (x^2 + y^2) \, dV
\]
on the leading diagonal are called the \textit{moments of inertia}. The off-diagonal components
\begin{align*}
I_{xy} &= -\int_B \rho xy dV \\
I_{xz} &= -\int_B \rho xz dV \\
I_{yz} &= -\int_B \rho yz dV
\end{align*}

are called the \textit{products of inertia}.

The inertia matrix \( I_C \) defines a linear relation between the angular velocity \( \omega \) and the rotational angular momentum \( h_C \), and in this respect plays a role for angular momentum corresponding to that of the mass \( M \) for the linear momentum. With this result, the total angular momentum of the body about \( O \) can now be written

\[ h = \mathbf{r} \times \mathbf{p} + I_C \omega. \]

\textit{Angular momentum about fixed pivot}

If the body has a fixed pivot, choosing that point as the fixed origin \( O \) the velocity of a point of the body with position vector \( \mathbf{r} \) is

\[ \dot{\mathbf{r}} = \omega \times \mathbf{r}, \]

and the angular momentum about \( O \) can be written

\[ h = \int_B \rho (\mathbf{r} \times (\omega \times \mathbf{r})) dV. \]

Following exactly the same arguments as for the angular momentum about the centroid but replacing \( \mathbf{s} \) by \( \mathbf{r} \), we obtain

\[ h = \mathbf{I} \omega \]

where, now letting \( x, y, z \) be Cartesian coordinates relative to axes with an origin at \( O \),

\[ I = \int_B \rho (r^2 \mathbf{1} - \mathbf{r} \otimes \mathbf{r}) dV = \int_B \rho \begin{pmatrix} y^2 + z^2 & -xy & -xz \\
-xy & x^2 + z^2 & -yz \\
-xz & -yz & x^2 + y^2 \end{pmatrix} dV. \]

The symmetric matrix \( \mathbf{I} \) is called the \textit{inertia matrix} of \( B \) relative to the origin \( O \). The components

\[ I_{xx} = \int_B \rho (y^2 + z^2) dV, \quad I_{yy} = \int_B \rho (x^2 + z^2) dV, \quad I_{zz} = \int_B \rho (x^2 + y^2) dV \]

on the leading diagonal are the moments of inertia relative to \( O \), and the off-diagonal components

\[ I_{xy} = -\int_B \rho xydV, \quad I_{xz} = -\int_B \rho xzdV, \quad I_{yz} = -\int_B \rho yzdV, \]

the products of inertia.
The inertia matrices relative to O and the centroid C, although distinct, are connected by a simple relationship. Writing \( \vec{r} \) for the position vector of C relative to O and \( \vec{s} \) for the position vector of a point P of the body relative to C, so that

\[ \vec{r} = \vec{r} + \vec{s}, \]

substituting for \( \vec{r} \) in the expression

\[ I = \int_B \rho (r^2 \mathbf{1} - \vec{r} \otimes \vec{r}) dV, \]

for the inertia matrix relative to O, since

\[ r^2 = (\vec{r} + \vec{s}) \cdot (\vec{r} + \vec{s}) = \vec{r}^2 + 2\vec{r} \cdot \vec{s} + \vec{s}^2, \]

and

\[ \vec{r} \otimes \vec{r} = (\vec{r} + \vec{s}) \otimes (\vec{r} + \vec{s}) = \vec{r} \otimes \vec{r} + \vec{r} \otimes \vec{s} + \vec{s} \otimes \vec{s}, \]

we obtain

\[ I = \int_B \rho (s^2 \mathbf{1} - \vec{s} \otimes \vec{s}) dV + (r^2 \mathbf{1} - \vec{r} \otimes \vec{r}) \int_B \rho dV + 2\vec{r} \cdot \int_B \rho \vec{s} dV - \vec{r} \otimes \int_B \rho \vec{s} dV - \int_B \rho \vec{s} dV \otimes \vec{r}. \]

Making use of the results

\[ M = \int_B \rho dV, \quad \int_B \rho \vec{s} dV = 0, \]

this reduces to

\[ I = I_C + M(\vec{r}^2 \mathbf{1} - \vec{r} \otimes \vec{r}) \]

where \( I_C \) is the inertia matrix relative to C. When calculating the inertia matrix relative to an origin O it is often simpler to first calculate the inertia matrix \( I_C \) (making use of any symmetries of the body) and then use this formula to calculate I.

**Principal axes of inertia**

Since \( I \) (and \( I_C \)) are symmetric they each have three real eigenvalues and three mutually orthogonal eigenvectors. The eigenvalues, called the principal moments of inertia, are the solutions of the cubic equation

\[ \det(I - \lambda I) = 0 \]

(where \( I \) is the unit matrix). The eigenvectors, called the principal axes of inertia, are the solutions of the equation

\[ I \cdot \vec{n} = \lambda \vec{n}. \]

The principal axes of the inertia matrix about the centroid can often be identified from the symmetries of the body.
Kinetic energy of a rigid body

The kinetic energy of a rigid body is

\[ T = \int_B \frac{1}{2} \rho \mathbf{r}^2 \, dV. \]

This too can be resolved into two parts, the kinetic energy of a particle of mass \( M \) placed at the centroid of the body plus the rotational kinetic energy of the body about its own mass centre. To show this, writing the position vector of the point \( P \) relative to fixed point in space \( O \) as

\[ \mathbf{r} = \mathbf{\bar{r}} + \mathbf{s} \]

where \( \mathbf{s} \) is its position vector relative to the centroid \( C \), and \( \mathbf{\bar{r}} \) the position vector of \( C \) relative to \( O \), since

\[ \dot{\mathbf{r}}^2 = (\mathbf{\bar{r}} + \dot{\mathbf{s}}) \cdot (\mathbf{\bar{r}} + \dot{\mathbf{s}}) = \mathbf{\bar{r}}^2 + 2 \mathbf{\bar{r}} \cdot \dot{\mathbf{s}} + \dot{\mathbf{s}}^2, \]

the kinetic energy can be written

\[ T = \int_B \frac{1}{2} \rho (\dot{\mathbf{\bar{r}}} + \dot{\mathbf{s}}) \cdot (\dot{\mathbf{\bar{r}}} + \dot{\mathbf{s}}) \, dV = \frac{1}{2} \mathbf{\bar{r}}^2 \left( \int_B \rho \, dV \right) + \mathbf{\bar{r}} \cdot \left( \int_B \rho \mathbf{s} \, dV \right) + \int_B \frac{1}{2} \rho \dot{\mathbf{s}}^2 \, dV. \]

Again making use of the results

\[ M = \int_B \rho \, dV, \quad \int_B \rho \mathbf{s} \, dV = 0, \]

it follows that

\[ T = \frac{1}{2} M \mathbf{\bar{r}}^2 + T_c \]

where

\[ T_c = \int_B \frac{1}{2} \rho \dot{\mathbf{s}}^2 \, dV. \]

Since \( \dot{\mathbf{s}} = \mathbf{\omega} \times \mathbf{s} \), the rotational kinetic energy of the motion about the mass centre can be written

\[ T_c = \int_B \frac{1}{2} \rho (\mathbf{\omega} \times \mathbf{s}) \cdot (\mathbf{\omega} \times \mathbf{s}) \, dV. \]

Using the vector identity for the scalar product of two cross products,

\[ (\mathbf{\omega} \times \mathbf{s}) \cdot (\mathbf{\omega} \times \mathbf{s}) = s^2 \mathbf{\omega}^2 - (\mathbf{s} \cdot \mathbf{\omega})^2 = \mathbf{\omega}^T (s^2 \mathbf{1} - \mathbf{s} \otimes \mathbf{s}) \mathbf{\omega}, \]

we further have

\[ T_c = \frac{1}{2} \mathbf{\omega}^T \left( \int_B \rho (s^2 \mathbf{1} - \mathbf{s} \otimes \mathbf{s}) \, dV \right) \mathbf{\omega} = \frac{1}{2} \mathbf{\omega}^T \mathbf{I}_c \mathbf{\omega}. \]

The total kinetic energy can thus be written

\[ T = \frac{1}{2} M \mathbf{\bar{r}}^2 + \frac{1}{2} \mathbf{\omega}^T \mathbf{I}_c \mathbf{\omega}. \]
In the particular case when there is a fixed pivot $O$, since $\mathbf{r} = \omega \times \mathbf{r}$ where $\omega$ is the angular velocity, the total kinetic energy can be written

$$T = \int_B \frac{1}{2} \rho (\omega \times \mathbf{r}) \cdot (\omega \times \mathbf{r}) \, dV .$$

Using the same arguments as above but with $s$ replaced by $r$ we obtain

$$T = \frac{1}{2} \mathbf{\Omega}^T \mathbf{I} \mathbf{\Omega} .$$

**Two-dimensional rigid body motion**

Two-dimensional rigid body motions are exemplified by the cases of a plane lamina or uniform cylinder moving in such a way that the axis of rotation has a fixed direction, usually taken to be parallel to the $z$-axis. The motion can then be regarded as taking place in the $xy$ plane. When the body has extension in the $z$-direction, for example a cylinder or a sphere, we suppose that the body is symmetric on reflection in the $xy$-plane.

By a “point” $P(x,y)$ we mean the set of all points of the body on a line normal to the plane whose projection onto the $xy$-plane is $P(x,y)$. The locus of all point $P$ defines an area $A$ of the $xy$-plane. A small element $\delta A$ of this area will define a small cylindrical volume element $\delta V$ the generators of which are normal to the plane. If this volume has mass $\delta M$ we call

$$\sigma = L t_{\delta t \to 0} \frac{\delta M}{\delta A}$$

the mass per unit area of the body. The mass of the body will be

$$M = \int_A \sigma \, dA .$$

Its centroid will lie in the $xy$-plane at the point

$$\overline{r} = \frac{1}{M} \int_A \sigma r \, dA .$$

The position vector $\mathbf{r}$ is understood to be the position vector of the point $(x,y)$ in the $xy$-plane so that

$$\mathbf{r} = xi + yj .$$

The displacement of the body during a time $t$ will again be characterised by the displacement of a given reference point $O_B$ fixed in the body plus a rotation $\mathbf{R}$ mapping the original position vector of any point $P$ of the body relative to $O_B$ onto its current position vector relative to $O_B$. Letting $\mathbf{e}(t)$ be the position vector of $O_B$ relative to the origin fixed $O$ at time $t$, the position vector of the point $P$ at time $t$ can thus be written

$$\mathbf{r}(t) = \mathbf{e}(t) + \mathbf{R}(t)(\mathbf{r}_0 - \mathbf{e}_0) .$$
Recalling that the rotation matrix for rotation through an angle $\theta$ about the $z$-axis is

$$
R = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix},
$$

it follows that the coordinates at time $t$ of the point initially at $(x_0, y_0)$ will be given by

$$
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_0 - a_0 \\ y_0 - b_0 \end{bmatrix} = \begin{bmatrix} a + (x_0 - a_0) \cos \theta - (y_0 - b_0) \sin \theta \\ b + (x_0 - a_0) \sin \theta + (y_0 - b_0) \cos \theta \end{bmatrix},
$$

where $(a,b)$ are the coordinates of the body reference point $O_b$ initially at $(a_0,b_0)$, that is,

- $x = a + (x_0 - a_0) \cos \theta - (y_0 - b_0) \sin \theta,$
- $y = b + (x_0 - a_0) \sin \theta + (y_0 - b_0) \cos \theta.$

**Angular velocity**

The angular velocity of the motion is

$$
\mathbf{\omega} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} = \dot{\theta} \mathbf{k}.
$$

The velocity of $P$ can be written

$$
\mathbf{r} = \mathbf{\dot{r}} + \mathbf{\omega} \times (\mathbf{r} - \mathbf{c}),
$$

or equivalently,
\[
\begin{align*}
\dot{x} &= \dot{a} - \omega(y - b), \\
\dot{y} &= \dot{b} + \omega(x - a).
\end{align*}
\]

The velocity of \( P \) relative to the reference point \( O_B \) is perpendicular to the line joining \( O_B \) to \( P \) axis of rotation and of speed \( v = d\omega \) where \( d \) is the distance of \( P \) from \( O_B \).

**Angular momentum**

Since we assume symmetry of the body on reflection in the \( xy \)-plane the components

\[
I_{czz} = -\int_V \rho x z dV, \quad I_{cyy} = -\int_V \rho y z dV
\]

of the inertia matrix about the centroid \( C \) are clearly both zero. The inertia matrix thus reduces to

\[
I_c = \begin{pmatrix}
I_{cxx} & I_{cxy} & 0 \\
I_{cxy} & I_{cyy} & 0 \\
0 & 0 & I_{czz}
\end{pmatrix}
\]

The only non-zero component of the equation

\[
h_c = I_c \omega
\]

for the angular momentum about \( C \) is thus the \( z \)-component

\[
h_c = I_{czz} \omega
\]

where \( I_c = I_{czz} \) is simply called the moment of inertia about \( C \). It is easily seen that the volume integral for \( I_c \) reduces to a double integral

\[
I_c = \int_A \sigma(x^2 + y^2) dA
\]

over the projection onto the \( xy \)-plane, where \( \sigma \) is the mass per unit area.

Since the moment \( \vec{r} \times \vec{p} = \vec{r} \times M\vec{\omega} \) of the linear momentum is perpendicular to plane of the motion the total angular momentum

\[
h = \vec{r} \times \vec{p} + h_c
\]

about \( C \) is also parallel to the \( z \)-axis. Its magnitude is

\[
h = M(\vec{x} \vec{y} - \vec{x} \vec{y}) + I_c \omega.
\]

**Motion about a fixed pivot**

When the body has a fixed point \( O \), choosing that point as the origin the displacement equations for the point \( P \) reduce to

\[
x = x_0 \cos \theta - y_0 \sin \theta, \\
y = x_0 \sin \theta + y_0 \cos \theta.
\]
The velocity of P can be written
\[ \dot{x} = -\omega y, \quad \dot{y} = \omega x, \]
or equivalently
\[ \mathbf{v} = \omega \mathbf{r} \]
where \( \mathbf{i} = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \) is a unit vector perpendicular to the position vector \( \mathbf{r} \).

The angular momentum about a fixed pivot can be written
\[ \mathbf{h} = I \mathbf{\omega} \]
where the scalar quantity
\[ I = \int_D \sigma(x^2 + y^2) \, dA \]
is the moment of inertia about O. It follows immediately from the equation
\[ I = I_c + M \mathbf{r}^2 \mathbf{1} - \mathbf{r} \otimes \mathbf{r} \]
that the moments of inertia about O and C are related by
\[ I = I_c + Md^2 \]
where \( d \) is the distance of O from the centroid C. This result is known as the parallel axis theorem.

**Kinetic energy**

For a two-dimensional motion the expression
\[ T = \frac{1}{2} M \mathbf{\dot{r}}^2 + \frac{1}{2} \mathbf{\omega}^T I \mathbf{\omega}. \]
for the kinetic energy of a rigid body reduces to
\[ T = \frac{1}{2} M \mathbf{\dot{v}}^2 + \frac{1}{2} I_c \mathbf{\dot{\omega}}^2 \]
where \( \mathbf{\dot{v}} \) is the velocity of the centroid. When there is a fixed pivot we can write
\[ T = \frac{1}{2} I \mathbf{\dot{\omega}}^2. \]