Rotating Frames

Newton’s laws of motion were enunciated for motion relative to an inertial reference frame (or observer). In practice, many problems are formulated relative to a non-inertial reference frame, for example, a frame fixed relative to a laboratory located on the rotating surface of the earth.

Suppose now that \( O \) is a non-inertial observer, and \( O' \) an inertial observer. The configuration of \( O \) relative to \( O' \) will be characterised by a displacement vector \( \mathbf{c} \), and a rotation \( \mathbf{R} \) (preserving lengths and angles) that maps the basis vectors \( \mathbf{i}' \), \( \mathbf{j}' \), \( \mathbf{k}' \) for Cartesian axes \( O'X'Y'Z' \) relative to the inertial observer onto the basis vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) for Cartesian axes \( OXYZ \) relative to the rotating observer. Relative to the inertial observer, the rotation \( \mathbf{R} \) will be represented by a matrix

\[
\begin{pmatrix}
R_{xx} & R_{xy} & R_{xz} \\
R_{yx} & R_{yy} & R_{yz} \\
R_{zx} & R_{zy} & R_{zz}
\end{pmatrix}
\]

the columns of which are the components of the basis vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) of the rotating observer relative to the coordinate system \( O'X'Y'Z' \) of the inertial observer.

Consider now motion of \( O \) relative to \( O' \). A point \( P \) in space that is fixed relative to the rotating observer \( O \) will appear to \( O' \) to have a constant position vector

\[
\mathbf{r} = xi + yj + zk,
\]

where \( x, y, z \) are constants. Relative to the inertial observer the vector \( \mathbf{r} \) will appear to be moving and rotating according to the equation

\[
\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_o,
\]

where

\[
\mathbf{r}_o = xi' + yj' + zk'.
\]

If the two observers initially coincide, \( \mathbf{r}_o \) will be the initial position of \( \mathbf{r} \).
It is evident from this that time derivatives relative to the rotating observer need to be
distinguished from time derivatives relative to the inertial observer. Reserving the
superposed dot for time derivatives relative to the rotating observer, we write
\[ \frac{D}{Dt} \]
for time derivatives relative to the inertial observer. Using this notation, since
\[ \frac{D r_0}{D t} = 0 \]
the rate of change of \( r \) relative to \( O' \) is
\[ \frac{D r}{D t} = \left( \frac{D R}{D t} \right) r_0. \]
Since \( R \) is a rotation, as shown in the previous section,
\[ \frac{D r}{D t} = \left( \frac{D R}{D t} \right) R^T r = \Omega r = \omega \times r \]
where \( \Omega \) is the antisymmetric spin matrix, and \( \omega \) the angular velocity of the rotating observer
\( O \) relative to the inertial observer \( O' \).

If now \( P \) is moving relative to rotating observer \( O \), with position vector
\[ r(t) = x(t)i + y(t)j + z(t)k \]
relative to \( O \), the velocity of \( P \) relative to \( O \) will be the time derivative of \( r \) relative to \( O \), that is,
\[ \dot{r} = \dot{x}i + \dot{y}j + \dot{z}k. \]
Noting that
\[ \frac{D}{Dt} (x(t)i) = \dot{x}i + x \frac{D i}{D t} = \dot{x}i + x \omega \times i , \]
it follows that the time derivative of \( r \) relative to the inertial observer \( O \) will be
\[ \frac{D r}{D t} = (\dot{x}i + \dot{y}j + \dot{z}k) + \omega \times (x i + y j + z k) = \dot{r} + \omega \times r. \]
Letting
\[ r' = r + c \]
be the position vector of \( P \) relative to \( O' \), the velocity of \( P \) relative to \( O' \) will be
\[ v' = \frac{D r'}{D t} = \frac{D r}{D t} + \frac{D c}{D t} = \dot{r} + \omega \times r + v'_0 \]
where \( v'_0 = Dc/Dt \) is the velocity of \( O \) relative to \( O' \). The acceleration of \( P \) relative to \( O' \) will be
\[
a' = \frac{Dv'}{Dt} = \frac{Dr}{Dt} + \frac{D}{Dt}(\omega \times r) + a'_o
\]

where \(a'_o = Dv'_o/Dt\) is the acceleration of \(O\) relative to \(O'\). Since

\[
\frac{Dr}{Dt} = \ddot{r} + \omega \times \dot{r}
\]

and

\[
\frac{D\omega}{Dt} = \dot{\omega} + \omega \times \omega = \dot{\omega}
\]

whence

\[
\frac{D}{Dt}(\omega \times r) = \frac{D\omega}{Dt} \times r + \omega \times \frac{Dr}{Dt} = \dot{\omega} \times r + \omega \times (\ddot{r} + \omega \times r),
\]

we thus obtain the relationship

\[
a' = \ddot{r} + 2\omega \times \dot{r} + \omega \times r + \omega \times (\omega \times r) + a'_o
\]

between the acceleration \(a'\) of \(P\) relative to the inertial observer \(O'\) and its acceleration \(\ddot{r}\) relative to the rotating observer \(O\).

The position dependent term \(\omega \times (\omega \times r)\) is called the centripetal acceleration, and the velocity dependent term \(2\omega \times \dot{r}\) the Coriolis acceleration. The term \(-\omega \times (\omega \times r)\) is called the centrifugal acceleration.

**Equation of motion relative to a rotating observer**

From Newton’s second law, relative to an inertial observer the motion of a particle \(P\) of mass \(m\) acted on by a force \(F\) is governed by the equation

\[
F = ma'
\]

where \(a'\) is the acceleration of \(P\) relative to the inertial observer. From the above result, the motion relative to the rotating observer is governed by the equation

\[
F = m\ddot{r} + 2m\omega \times \dot{r} + m\dot{\omega} \times r + m\omega \times (\omega \times r) + ma'_o.
\]

Rewriting this equation in the form

\[
m\ddot{r} = F - 2m\omega \times \dot{r} - m\dot{\omega} \times r - m\omega \times (\omega \times r) - ma'_o,
\]

the non-physical terms on the right hand side of this equation are sometimes called ‘fictitious’ forces, though a better term is geometric forces. The term \(-m\omega \times (\omega \times r)\) is called the centrifugal force, and \(-2m\omega \times \dot{r}\) the Coriolis force.
Motion relative to a rotating observer with a constant angular velocity

Let us consider motion relative to an observer O with a constant angular velocity $\omega$ relative to an inertial frame. We also suppose that the origin of the rotating observer O is stationary relative to the inertial observer, so that there only relative motion is due to the rotation. (A scenario could be a space station in deep space!) We choose Cartesian axes $OXYZ$ for the rotating observer and $O'X'Y'Z'$ with a common $z$-axis parallel to the axis of rotation. Since the angular velocity is constant and the translational acceleration is zero, the equation of motion in this case reduces to

$$m \ddot{r} = F - 2m\omega \times \dot{r} - m\omega \times (\omega \times r).$$

Making use of the identity

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

we obtain

$$m \ddot{r} = F - 2m\omega \times \dot{r} - m(\omega \cdot r)\omega + m\omega^2 r$$

Since

$$\omega \times \dot{r} = \begin{bmatrix} i & j & k \\ 0 & 0 & \omega_x \\ \dot{x} & \dot{y} & \dot{z} \end{bmatrix} = \begin{bmatrix} -\omega y \\ \omega x \\ 0 \end{bmatrix}, \quad \omega \cdot r = \omega z$$

and $\omega \cdot r = \omega z$, the component equations are

$$m \ddot{x} = F_x + 2m\omega \dot{y} + m\omega^2 x,$$

$$m \ddot{y} = F_y - 2m\omega \dot{x} + m\omega^2 y,$$

$$m \ddot{z} = F_z.$$

The centrifugal and Coriolis forces thus only affect motion perpendicular to the axis of rotation.

If there are no external forces, these equations reduce to

$$\ddot{x} = 2\omega \dot{y} + \omega^2 x,$$

$$\ddot{y} = -2\omega \dot{x} + \omega^2 y,$$

$$\ddot{z} = 0.$$

In the $z$-direction there is at most a uniform motion. Relative to the inertial observer the particle will move along a straight line with uniform velocity. Relative to the rotating observer the motion will be more complicated. Introducing the complex variable $u = x + iy$ (where $i = \sqrt{-1}$), noting that

$$\dot{y} - i \dot{x} = -(\dot{x} + i \dot{y}) = -i \dot{u}$$

the transverse equations can be written as a single second order differential equation
\[ \dddot{u} + 2 \omega i \dot{u} - \omega^2 u = 0. \]

The auxiliary equation
\[ p^2 + 2 \omega i p - \omega^2 = 0 \]
has a double root \( p = -\omega \), whence
\[ u = (A + Bt) e^{-i\omega t}. \]

Solving this equation subject to the initial conditions
\[ x(0) = 0, \quad y(0) = 0, \quad \dot{x}(0) = V, \quad \dot{y}(0) = 0 \]

since \( u(0) = 0, \quad \dot{u}(0) = V \) it follows that \( A = 0, \quad B = V \), whence
\[ u = Vt e^{-i\omega t}, \]

that is,
\[ x = Vt \cos \omega t, \quad y = -Vt \sin \omega t. \]

The particle thus spirals out from the origin, its distance from the origin at time \( t \) being \( r = Vt. \)

If the initial conditions are
\[ x(0) = a, \quad y(0) = 0, \quad \dot{x}(0) = 0, \quad \dot{y}(0) = 0 \]

since \( u(0) = a, \quad \dot{u}(0) = 0 \) it follows that \( A = a, \quad B = i\omega a \), whence
\[ u = a(1 + i\omega t) e^{-i\omega t}, \]

that is,
\[ x = a(\cos \omega t + \omega t \sin \omega t), \quad y = a(-\sin \omega t + \omega t \cos \omega t). \]

The particle again spirals out from the origin.
Motion relative to an observer at a fixed point on the Earth’s surface

Consider the motion of a particle of mass \( m \) relative to an observer O at a fixed point on the Earth’s surface rotating with the Earth, with axes OXYZ such that OZ points vertically upwards (along the radial outwards vector from O’), OX points east along the line of latitude, and OY points north along the line of longitude. Let O’ be an inertial observer situated at the centre of the Earth, with axes O’X’Y’Z’ such that O’Z’ is along the axis of rotation (passing through the poles) and O’X’Y’ in the equatorial plane.

The coordinates of the rotating observer O at time \( t \) relative to the inertial observer O’ will be

\[
x' = a \cos \theta \cos \omega t, \\
y' = a \cos \theta \sin \omega t, \\
z' = a \sin \theta,
\]

where \( \theta \) is the latitude of O, \( a \) is the radius of the earth, and \( \omega \) is the angular speed of rotation of the earth. Relative to O’ the base vectors, \( \mathbf{i} \) (in the easterly direction at O), \( \mathbf{j} \) (in the northerly direction at O) and \( \mathbf{k} \) (in the vertical direction at O) will have components

\[
\begin{pmatrix}
-\sin \omega t \\
\cos \omega t \\
0
\end{pmatrix}, \\
\begin{pmatrix}
-\sin \theta \cos \omega t \\
-\sin \theta \sin \omega t \\
\cos \theta
\end{pmatrix}, \\
\begin{pmatrix}
\cos \theta \cos \omega t \\
\cos \theta \sin \omega t \\
\sin \theta
\end{pmatrix}.
\]

Hence the rotation matrix describing the rotation of the observer on the earth’s surface relative to the inertial observer at the centre of the earth is

\[
R = \begin{pmatrix}
-\sin \omega t & -\sin \theta \cos \omega t & \cos \theta \cos \omega t \\
\cos \omega t & -\sin \theta \sin \omega t & \cos \theta \sin \omega t \\
0 & \cos \theta & \sin \theta
\end{pmatrix}.
\]
The spin matrix is
\[ \Omega = \left( \frac{D \mathbf{R}}{Dt} \right) R^T = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]
and the angular velocity
\[ \omega = \omega k^\prime. \]

Since
\[ \frac{D \omega}{Dt} = \dot{\omega} + \omega \times \omega = \dot{\omega}, \]
assuming the earth to have a constant angular velocity, so that
\[ \frac{D \omega}{Dt} = 0 \]
it follows also that \( \dot{\omega} = 0 \) and hence that the equation of motion for a moving particle \( P \) relative to \( O \) reduces to
\[ m \ddot{r} = F - 2m \omega \times r - m \omega \times (\omega \times r) - ma^\prime_0, \]
where \( a^\prime_0 \) is the acceleration of \( O \) relative to \( O' \).

The position vector of \( O \) relative to \( O' \) is the vector \( c = a \mathbf{k} \), where \( a \) is the radius of the earth. Noting that \( \dot{c} = 0 \), the velocity of \( O \) relative to \( O' \) is
\[ v^\prime_0 = \frac{D c}{Dt} = \dot{c} + \omega \times c = \omega \times c. \]
of \( O \) relative to \( O' \) is also constant (relative to \( O \)). Hence the acceleration of \( O \) relative to \( O' \) is
\[ a^\prime_0 = \frac{D v^\prime_0}{Dt} = \frac{D}{Dt} (\omega \times c) = \omega \times \frac{D c}{Dt} = \omega \times (\omega \times c). \]
Thus the equation of motion reduces to
\[ m \ddot{r} = F - 2m \omega \times r - m \omega \times (\omega \times r) - ma^\prime_0. \]

Since the angular velocity of the Earth is
\[ \omega = 2\pi/86400 = 7.27 \times 10^{-5} \text{ rad/sec}, \]
centripetal accelerations near the Earth’s surface are of order
\[ a \omega^2 = 6.4 \times 10^6 \times (7.27 \times 10^{-5})^2 = 0.034 \text{ m s}^{-2} \]
(where \( a = 6400 \text{ km} \) is the radius of the Earth). Near the Earth’s surface, terms involving \( \omega^2 \) will thus negligible compared to gravity, and the equation of motion further reduces to
\[ m \ddot{r} = F - 2m \omega \times r. \]
The only significant additional term is thus the Coriolis force \( -2m \omega \times \dot{r} \).
Noting that
\[ k' \cdot i = 0, \quad k' \cdot j = \cos \theta, \quad k' \cdot k = \sin \theta, \]
whence
\[ k' = \cos \theta j + \sin \theta k, \]
relative to the base vectors of O the angular velocity \( \omega = \omega k' \) can be written
\[ \omega = \omega (\cos \theta j + \sin \theta k). \]
It follows that the Coriolis force can be written
\[-2m \omega \times \dot{r} = \begin{bmatrix} 0 & \omega \cos \theta & \omega \sin \theta \\ \dot{x} & \dot{y} & \dot{z} \end{bmatrix} = -2m \omega \begin{bmatrix} \cos \theta \dot{z} - \sin \theta \dot{y} \\ \sin \theta \dot{x} \\ -\cos \theta \dot{x} \end{bmatrix}.\]
The equations of motion for a particle near the Earth’s surface are thus
\[
m\ddot{x} = F_x + 2m \omega (\sin \theta \dot{y} - \cos \theta \dot{z}), \]
\[
m\ddot{y} = F_y - 2m \omega \sin \theta \dot{x}, \]
\[
m\ddot{z} = F_z + 2m \omega \cos \theta \dot{x}. \]

**Motion of a freely falling particle under gravity**

The equations of motion (relative to a rotating observer at a fixed point on the Earth’s surface) for a particle near the Earth’s surface falling freely under gravity are
\[
m\ddot{x} = 2m \omega (\sin \theta \dot{y} - \cos \theta \dot{z}), \]
\[
m\ddot{y} = 2m \omega \sin \theta \dot{x}, \]
\[
m\ddot{z} = -mg + 2m \omega \cos \theta \dot{x}. \]
Assuming that the particle, initially at rest, is dropped from a height \( H \), integrating the first two equations gives
\[
\dot{x} = 2\omega \sin \theta \ y - 2\omega \cos \theta (z - H), \quad \dot{y} = -2\omega \sin \theta \ x. \]
Substituting for \( \dot{x} \) and neglecting terms of order \( \omega^2 \), the third equation reduces to
\[
\ddot{z} = -\frac{g}{\omega^2}, \]
whence
\[
\dot{z} = -g t, \quad z = H - \frac{1}{2} gt^2. \]
Substituting for \( \dot{x}, \dot{y}, \dot{z} \) in the original two equations and again neglecting terms of order \( \omega^2 \) gives
\[
\ddot{x} = 2\omega \cos \theta \ gt, \quad \ddot{y} = 0. \]
Integrating we obtain
\[
x = \frac{1}{4} \omega \cos \theta \ gt^3. \]
The particle returns to earth when \( z = 0 \), that is at time \( t = \sqrt{(2H/g)} \), at which time it is at a distance

\[
d = \frac{\sqrt{2}}{3} \omega \cos \theta \left( \frac{H^{3/2}}{g^{1/2}} \right)
\]

to the east of the initial dropping point. If the observer is at the equator, so that \( \theta = 0 \), and \( H = 1000 \) m, since \( \omega = 2\pi/86400 \) sec\(^{-1} \) the distance is

\[
d = \frac{\sqrt{2}}{3} \frac{2\pi}{86400} \left( \frac{1000^3}{9.81} \right)^{0.5} = 0.346 \text{ m},
\]

and thus negligible for all practical purposes.

**Foucault’s Pendulum**

The Foucault pendulum, named after the French physicist Léon Foucault, was conceived as an experiment to demonstrate the rotation of the Earth.

The experimental apparatus consists of a tall pendulum free to oscillate in any vertical plane. The direction along which the pendulum swings rotates with time because of Earth's daily rotation. The first public exhibition of a Foucault pendulum took place in February 1851 in the Meridian Room of the Paris Observatory. A few weeks later, Foucault made his most famous pendulum when he suspended a 28-kg bob with a 67-metre wire from the dome of the Panthéon in Paris. The plane of the pendulum's swing rotated clockwise 11° per hour, making a full circle in 32.7 hours.
To model this, let us assume that the bob is attached to a fixed point A by a long light string of length L and is free to swing in any vertical plane through P.

The physical forces acting on the bob are the tension $T$ directed along the string towards the point of suspension A, and a gravitational force $mg$ acting vertically downward. Choosing Cartesian axes OXYZ with origin O at the equilibrium position of the bob and OZ vertically upwards, when the bob has coordinates $(x, y, z)$, noting that

\[
\begin{pmatrix}
0 \\
0 \\
L \\
y \\
y \\
z \\
\end{pmatrix} = \begin{pmatrix}
-x \\
-y \\
L-z \\
\end{pmatrix},
\]

so that

\[-\frac{x}{L} \hat{i} - \frac{y}{L} \hat{j} + \frac{L-z}{L} \hat{k}
\]
is a unit vector directed along the string from the bob to the point of suspension, the tensile force acting on the bob will be

\[
T = -T \frac{x}{L} \hat{i} - T \frac{y}{L} \hat{j} + T \frac{L-z}{L} \hat{k}.
\]

The equations of motion for the bob can thus be written

\[
m\ddot{x} = -T \frac{x}{L} + 2m\omega (\sin \theta \dot{y} - \cos \theta \dot{z}),
\]

\[
m\ddot{y} = -T \frac{y}{L} - 2m\omega \sin \theta \dot{x},
\]

\[
m\ddot{z} = T \frac{L-z}{L} - mg + 2m\omega \cos \theta \dot{x}.
\]

Since the string is long and the swing small, motion in the vertical direction will be negligible and we may suppose that $\dot{z} = 0$, $\ddot{z} = 0$, and $L-z \approx L$. It follows that

\[
T \approx mg - 2m\omega \cos \theta \dot{x},
\]
and hence that the equations of motion in the \(xy\)-plane reduce to

\[
\ddot{x} = -\frac{gx}{L} + \frac{2\omega \cos \theta \dot{x}}{L} + 2\omega \sin \theta \dot{y},
\]

\[
\ddot{y} = -\frac{gy}{L} + \frac{2\omega \cos \theta \dot{y}}{L} - 2\omega \sin \theta \dot{x}.
\]

Since \(x\) and \(y\) are small, the terms \(x \dot{x}\) and \(y \dot{y}\) are negligible, and these further reduce to

\[
\ddot{x} = -\frac{gx}{L} + 2\omega \sin \theta \dot{y}, \quad \ddot{y} = -\frac{gy}{L} - 2\omega \sin \theta \dot{x}.
\]

Introducing the complex variable \(u = x + iy\), this can be written

\[
i\ddot{u} + 2i\omega \sin \theta \dot{u} + \Omega^2 u = 0
\]

where \(\Omega = \sqrt{g/L}\). This second order differential equation with constant coefficients has auxiliary (characteristic) equation

\[
m^2 + 2i\omega \sin \theta m + \Omega^2 = 0,
\]

with roots

\[
m = i\omega \sin \theta \pm i\sqrt{\omega^2 \sin^2 \theta + \Omega^2}.
\]

Since the period of swing \((2\pi\Omega)\) is of the order of a few seconds whereas the period of the Earth’s rotation is 24 hours, the term \(\omega^2 \sin^2 \theta\) will be negligible compared to \(\Omega^2\), and this simplifies to

\[
m = i\omega \sin \theta \pm \Omega,
\]

whence

\[
u = e^{i\omega \sin \theta t} \left( A e^{i\Omega t} + B e^{-i\Omega t} \right) = e^{i\omega \sin \theta t} \left( C \cos \Omega t + D \sin \Omega t \right)
\]

where \(C\) and \(D\) are arbitrary constants. Assuming the initial conditions \(u = a, \dot{u} = 0\) at time \(t = 0\) we obtain

\[
C = a, \quad D = -\frac{i\omega \sin \theta}{\Omega} C.
\]

Since \(\omega/\Omega\) is small we may suppose that \(D = 0\). Separating the solution into its real and imaginary parts,

\[
x = a \cos \Omega t \cos(\omega \sin \theta t), \quad y = a \cos \Omega t \sin(\omega \sin \theta t).
\]

Writing

\[
n(t) = \cos(\omega \sin \theta t)i + \sin(\omega \sin \theta t)j
\]

the position vector of the bob is

\[
r(t) = a n(t) \cos \Omega t.
\]
The unit vector \( \mathbf{n} \) defining the direction of swing rotates with angular velocity \( \omega \sin \theta \) and so is essentially constant on the short timescales characterised by the period

\[
T = \frac{2\pi}{\Omega} = 2\pi \sqrt{\frac{L}{g}}
\]

of swing of the pendulum. On these short timescales the pendulum swings in the vertical plane making an angle

\[
\phi = \omega \sin \theta t
\]

with the \( x \)-axis. On longer time scales this plane rotates with period

\[
\tau = \frac{2\pi}{\omega \sin \theta} = \frac{24}{\sin \theta} \text{ hrs}.
\]

Since the latitude of the Panthéon in Paris is \( \theta = 48.85^\circ \), the period of rotation of Foucault’s pendulum 31.9 hours, in good agreement with the empirical value.

Although the effects of the Coriolis force are relatively small in the above examples, on large spatial scales they do have an important effect on oceanic and atmospheric circulation.