MATH224. Homework 10.

1. The longitudinal displacement $u(x,t)$ for longitudinal vibrations of a beam of uniform cross-section satisfies the wave equation

$$\epsilon \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = 0$$

where ϵ is the modulus of elasticity of the beam and ρ is the mass density. For a beam of length L with the end $x = 0$ fixed and the end $x = L$ free, the boundary conditions are

$$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0 .$$

Use the method of separation of variables to determine the natural angular frequencies of vibration of such a beam.

(Hints: This is just like the string problem in the lectures, the only difference is the changed boundary condition at $x = L$. A harmonic motion has the form $a\sin(\omega t) + b\cos(\omega t)$ where ω is the angular frequency.)

2. The solution to the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

for a bar of length L whose left and right ends are held at temperatures T_0 and T_1 respectively is

$$u(x,t) = T_0 + (T_1 - T_0)\frac{x}{L} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2} .$$

Find the particular solution of the heat equation in the bar when the initial temperature distribution in degrees Celsius is

$$u(x,0) = \begin{cases} 0 & \text{if } 0 < x < L/2 \\ 50^{\text{O}} & \text{if } L/2 < x < L \end{cases}$$

and the ends are held at 20^{O} C.

3. Show that the function $u(x,y)$ satisfies Laplace's equation, where

$$u(x,y) = x^3 - 3xy^2 + 2x$$

Write down the Cauchy-Riemann equations for u and its conjugate harmonic function $v(x,y)$ and hence or otherwise find $v(x,y)$.

4. Find a function u satisfying Laplace's equation in the rectangle $0 < x < \pi$, $0 < y < b$, with the given boundary conditions:

$$u(x,0) = 1, \quad u(x,b) = 0, \qquad u(0,y) = u(\pi,y) = 0 .$$