MATH224: Introduction to the Methods of Applied Mathematics

2nd semester 2012, edited version of Pieter Collins' script,
based on original notes by Nigel Wilding

Contents

Chapter 1

Introduction

This course is mostly concerned with the subject of ordinary and partial differential equations. Many physical laws and relations appear mathematically in the form of such equations, prime examples being Laplace's equation, the wave equation and the diffusion equation. The ability to express ("model") physical problems in terms of differential equations and to find the corresponding solutions is therefore a necessary and important skill for the applied mathematician.

During the course you will be introduced to several techniques of fundamental significance in applied mathematics. For example, the method of separation of variables applied to partial differential equations such as the one-dimensional wave equation, and the Laplace transform method for changing differential equations into algebraic equations. We shall also study Fourier series and look at the solution of partial differential equations by the method of characteristics. Whenever possible, we shall endeavour to illustrate the relevance of the differential equations we study to the corresponding physical situation.

1.1 Ordinary differential equations

An *ordinary differential equation* (ODE) is a relation which involves one or more derivatives of an unspecified function y of another variable, usually taken to be x (though note that t is often used instead for functions of time). Formally, this is expressed as

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0 \tag{1}$$

The *order* n of an ordinary differential equation is the highest derivative $d^n y/dx^n$ it contains.

An important second order ordinary differential equation is that describing the simple harmonic motion of a mass acted on by a spring that obeys Hooke's law.

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

where ω (the angular frequency) is a constant.

The main problem associated with such an equation is to find functions y in terms of x for which the derivatives

$$\frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}$$

exist and satisfy equation (1).

The *general solution* of a differential equation is the set of all functions satisfying it, usually expressed as some kind of general formula involving a number of parameters. It can be proved that if F satisfies certain conditions, then the differential equation (1) has a unique general solution involving n parameters or *arbitrary constants.*

Rather than concentrate on finding the general solution of each equation individually, we look for solutions to particular *classes* of equations. We consider a class of equations solved if we can express the general solution for any equation in that class in terms of integrals. An important (and relatively simple) class of equations is the class of *linear equations.*

Example: The general solution of the first order ordinary differential equation

$$\frac{dy}{dx} = -2y$$

has the form

$$y = Ae^{-2x}$$

where A is an arbitrary constant.

Check:

$$y = Ae^{-2x} \quad \Rightarrow \quad \frac{dy}{dx} = -2Ae^{-2x} = -2y. \qquad \checkmark$$

A solution given by a particular set of values of the parameters is called a *particular solution* or *particular integral* of the given differential equation. A particular integral is usually specified by *initial conditions*, which give the values of

$$y(x_0),\ y'(x_0), y''(x_0), \ldots, y^{(n-1)}(x_0)$$

for some particular value x_0 of the independent variable x. The problem of finding the solution to an ordinary differential equation with given initial conditions is an *initial value problem.*

Example: The solution to the initial value problem

$$\frac{dy}{dx} = -2y; \qquad y(0) = 3;$$

can be found from the general solution by evaluating the arbitrary constant A. We have

$$3 = y(0) = Ae^0 = A$$

yielding the solution

$$y(x) = 3e^{-2x}$$

which is the unique solution to the initial value problem.

Often a particular solution of a second order ordinary differential equation is given by a pair of *boundary conditions*, specifying the value of y at two values of x,

$$y(x_0) = a_0 \quad \text{and} \quad y(x_1) = a_1.$$

Example: The boundary value problem

$$\frac{d^2y}{dx^2} + \omega^2 y = 0; \qquad y(0) = y(1) = 0;$$

clearly has the *trivial solution*, y(x)=0, for any value of ω. However, if $\omega = n\pi$ where n is an integer, then we also have a *nontrivial* solution,

$$y(x) = \sin(\omega x) = \sin(n\pi x).$$

Check:

$$y'(x) = \omega\cos(\omega x), \quad y''(x) = -\omega^2\sin(\omega x) \quad \Rightarrow \quad y''(x) + \omega^2 y(x) = 0. \quad \checkmark$$

$$y(0) = \sin(0) = 0, \quad y(1) = \sin(n\pi) = 0. \quad \checkmark$$

Sometimes we have several unspecified functions $y_1, y_2, y_3, \dots y_n$, each of which depend on x. This gives rise to a *system* of ordinary differential equations:

$$\begin{aligned} \frac{dy_1}{dx} &= F_1(x, y_1, y_2, \dots, y_n) \\ \frac{dy_2}{dx} &= F_2(x, y_2, y_2, \dots, y_n) \\ &\vdots \\ \frac{dy_n}{dx} &= F_n(x, y_2, y_2, \dots, y_n) \end{aligned} \tag{2}$$

where $F_1, F_2, \dots, F_n$ are specified. If $F_1, F_2, \dots F_n$ are all independent of x, then the system is said to be *autonomous.*

Example: In a model of a predator-prey system, in which one species (e.g. foxes) eats another (e.g. rabbits), we have two variables y_1 and y_2 giving the population of the predators and prey respectively. A simple model for the rates of change of these populations is:

$$\begin{aligned} \frac{dy_1}{dt} &= -ky_1 + \beta y_1 y_2 \\ \frac{dy_2}{dt} &= (b-d)y_2 - \alpha y_1 y_2 \end{aligned}$$

where b, d, α, k, β are constants which depend on the environment and the species involved. Note that the right hand side of these equations depends on both y_1 and y_2.

An n-th order ordinary differential equation can always be written as a system of n first order equations by setting

$$y_k = \frac{d^k y}{dx^k}$$

for $k = 0, 1, \dots, n-1$. Then the equation

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}}\right)$$

can be expressed

$$\frac{dy_0}{dx} = y_1, \quad \frac{dy_1}{dx} = y_2, \quad \dots, \quad \frac{dy_{n-1}}{dx} = f\left(x, y_0, y_1, \dots, y_{n-1}\right).$$

The general solution to both systems has n arbitrary coefficients.

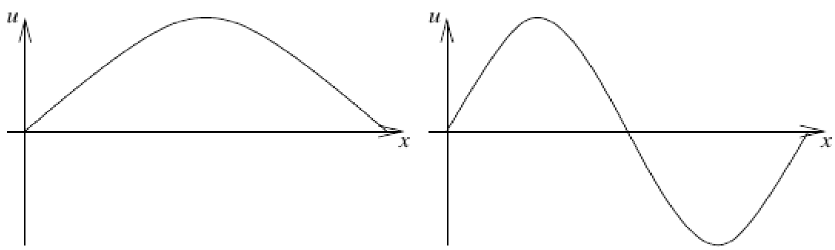
1.2 Partial differential equations

A *partial differential equation* (PDE) is an equation where some unknown function u, say, is a function of several variables. Partial differential equations thus contain *partial derivatives*, in which the function is differentiated with respect to one of the variables, all others being kept fixed.

In physics and engineering, the important partial differential equations are mainly of second order, such as the *one-dimensional wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3}$$

which describes the propagation of waves through a one-dimensional medium. For the case of a taut string $u(x, t)$ represents the displacement of the string from equilibrium as a function of the position along the string x and the time t. Solutions to the one-dimensional wave equation depend on the *boundary conditions* i.e. the value of $u(x, t)$ and its derivatives at the ends of the string. For a string tethered at both ends (e.g. on a violin), the solutions are the *standing wave harmonics*, the first two of which are depicted below:



The flow of heat in a two-dimensional conductor such as a metal sheet is described by the (two-dimensional) *heat equation*

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{4}$$

In this case $u(x, y, t)$ represents the temperature of the wire at position (x, y) at time t. Again the precise form of the solution will depend on the boundary conditions. If the temperature distribution on the boundary of the sheet is held constant, then as $t \to \infty$ the temperature can be shown approach an equilibrium state satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{5}$$

which is *Laplace's equation* in two dimensions.

The general solution of a partial differential equation typically depends on arbitrary *functions* of the variables, unlike the case of an ordinary differential equation, where the general solution only depends on finitely many arbitrary constants. A particular solution is specified by *boundary conditions*, which give the values of u and appropriate derivatives of u on the boundary of the domain of interest. Conditions which specify the values of u or its derivatives when $t = 0$ are called *initial conditions*.

Chapter 2

First order ordinary differential equations.

2.1 Separable and homogeneous equations

A general first order ordinary differential equation can be written in the form

$$\frac{dy}{dx} = F(x, y)$$

for some function F. We first consider two particularly simple forms of $F(x, y)$ where we can solve for $y(x)$ explicitly in terms of integrals.

Separable equations

A separable equation has the form:

$$\frac{dy}{dx} = f(x)\, g(y). \tag{1}$$

We can write this as

$$\frac{dy}{g(y)} = f(x)\; dx$$

giving the general solution:

$$\boxed{\int \frac{1}{g(y)}\; dy = \int f(x)\; dx + c.}$$

Notice that the constants of integration from the two integrals can be combined to give one arbitrary constant for the general solution.

Example: If

$$\frac{dy}{dx} = -\frac{4x}{9y}$$

then

$$\int 9y\ dy = -\int 4x\ dx$$

so

$$\frac{9y^2}{2} = -x^2 + c.$$

This solution constitutes a family of ellipses parameterised by the constant c.

Problem: (Newton's law of cooling) A copper ball is heated to $T = 100\,°\mathrm{C}$. Then at time $t = 0$ it is placed in air at $30\,°\mathrm{C}$. After 3 minutes, the temperature of the ball has fallen to $T = 70\,°\mathrm{C}$. Find an expression for the temperature of the ball at time t.

Solution: Newton's cooling law states that

$$\frac{dT}{dt} = -k(T - 30)$$

for some constant k. This equation is separable, and can be rearranged to give

$$\frac{dT}{T - 30} = -k\ dt.$$

Integrating the equation we find

$$\ln|T - 30| = -kt + c,$$

so the general solution is

$$T(t) = e^{-kt+c} + 30 = Ae^{-kt} + 30$$

where $A = e^c$. The initial condition requires $T = 100$ when $t = 0$, hence $A = 70$. So

$$T(t) = 70e^{-kt} + 30.$$

We can find the value of k using the second condition, i.e. $T(t = 3) = 70$:

$$70 - 30 = 70e^{-3k} \quad \Rightarrow \quad k = \frac{\ln(7/4)}{3}.$$

Problem: A mass m, initially at rest, falls vertically from rest from a distance h above the ground. If air resistance is proportional to the square of the velocity, determine the velocity with which it hits the ground.

Solution: Let x be the distance above the ground, and let

$$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = v\,\frac{dv}{dx}$$

be the velocity and acceleration respectively. From Newton's second law, $F = ma$, and so balancing forces gives:

$$mg - mkv^2 = mv\,\frac{dv}{dx}$$

where mk is the constant of proportionality for air resistance. Cancelling the m, we obtain

$$g - kv^2 = v\frac{dv}{dx},$$

which is a separable equation, so can be rearranged into the form

$$dx = \frac{v\;dv}{g - kv^2},$$

which integrates to give (here we can assume $g > kv^2$)

$$x = \int \frac{v}{g - kv^2}\,dv = \frac{1}{-2k}\ln(g - kv^2) + c.$$

The initial condition gives $v = 0$ when $x = 0$, so

$$c = \frac{\ln(g)}{2k},$$

and hence

$$x = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right).$$

Rearranging gives

$$e^{2kx} = \frac{g}{g - kv^2} \quad \text{or} \quad \frac{g}{k} - v^2 = \frac{g}{k}e^{-2kx},$$

and finally

$$v = \sqrt{\frac{g}{k}(1 - e^{-2kx})}\;.$$

This expression gives the velocity at distance x from the start. In particular,

$$v^2 \to g/k \text{ as } x \to \infty.$$

Homogeneous equations

Sometimes an ordinary differential equation may not appear to be separable, but can be made so via a change of variable. Specifically ordinary differential equations of the *homogeneous* form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{2}$$

may be made separable by writing $v = y/x$, where $v(x)$ is a new dependent variable. Differentiating $y = x\,v$, we obtain:

$$\frac{dy}{dx} = v + x\,\frac{dv}{dx},$$

and the differential equation becomes

$$v + x\,\frac{dv}{dx} = f(v)$$

or

$$\frac{dv}{dx} = \frac{f(v) - v}{x},$$

which is separable. Thus the solution is given by:

$$\boxed{\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + c.}$$

After integrating, we replace v by y/x.

Example: To find the general solution to

$$x\,\frac{dy}{dx} = x + 3y,$$

we first divide through by x to obtain the homogeneous equation

$$\frac{dy}{dx} = 1 + 3\,\frac{y}{x}.$$

Let $v = y/x$ so $dy/dx = v + x\,dv/dx$. The differential equation for v is

$$v + x\,\frac{dv}{dx} = 1 + 3v,$$

which simplifies to

$$\frac{dv}{dx} = \frac{1 + 2v}{x}.$$

Thus

$$\frac{\ln(1 + 2v)}{2} = \ln(x) + c \qquad \Rightarrow \qquad v = \frac{Cx^2 - 1}{2}$$

and so

$$y = x(Cx^2 - 1)/2.$$

2.2 Exact differential equations

The differential equation

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}$$

can be re-written as

$$P(x,y)\ dx + Q(x,y)\ dy = 0, \tag{3}$$

where the expression

$$P(x,y)\ dx + Q(x,y)\ dy$$

is called a *differential* in x and y.

Equation (3) is *exact* if there is a function $u(x,y)$ such that

$$du \equiv \frac{\partial u}{\partial x}\ dx + \frac{\partial u}{\partial y}\ dy = P(x,y)\ dx + Q(x,y)\ dy.$$

Since then $du = 0$, the solution is given by:

$$\boxed{u(x, y(x)) = c.}$$

For this to happen we must have:

$$P(x,y) = \frac{\partial u}{\partial x} \quad \text{and} \quad Q(x,y) = \frac{\partial u}{\partial y}.$$

Now assuming that the second partial derivatives of $u(x,y)$ are continuous, we can exploit the fact that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

to write

$$\boxed{\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}.}$$

This turns out to be a necessary and sufficient condition for the differential equation to be exact.

Now we must find u given P and Q. Integrating the equation for P gives

$$u(x,y) = \int P(x,y)\ dx,$$

but of course this integral is only defined up to an arbitrary constant which may here depend on y. Thus we can write

$$\boxed{u(x,y) = v(x,y) + k(y) \quad \text{where} \quad v(x,y) = \int P(x,y)\, dx}$$

and $k(y)$ is some unknown function of y. The equation for Q is then

$$Q(x,y) = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \frac{dk}{dy},$$

so

$$\boxed{\frac{dk}{dy} = Q(x,y) - \frac{\partial v}{\partial y}}$$

from which we can determine $k(y)$, then $u(x,y)$, and finally y as a function of x.

Problem: Solve:

$$2x \sin(3y)\, dx + \left(3x^2 \cos(3y) + 2y\right)\, dy = 0.$$

Solution: Let $P(x,y) = 2x\sin(3y)$ and $Q(x,y) = 3x^2\cos(3y) + 2y$. Then

$$\frac{\partial P(x,y)}{\partial y} = 6x\cos(3y) = \frac{\partial Q(x,y)}{\partial x},$$

so the equation is exact. Therefore

$$\begin{aligned} u(x,y) &= \int P(x,y)\, dx + k(y) \\ &= x^2 \sin(3y) + k(y), \end{aligned}$$

and the equation $Q(x,y) = \partial u/\partial y$ gives

$$3x^2\cos(3y) + 2y = 3x^2\cos(3y) + \frac{dk}{dy} \quad \Rightarrow \quad \frac{dk}{dy} = 2y.$$

Hence $k(y) = y^2$, and the general solution is:

$$x^2 \sin(3y) + y^2 = c.$$

Check by differentiating:

$$d\left(x^2\sin(3y) + y^2\right) = 2x\sin(3y)\, dx + \left(3x^2\cos(3y) + 2y\right)\, dy = 0. \quad \checkmark$$

2.3 First order linear differential equations.

A first order linear differential equation takes the form

$$\frac{dy}{dx} + p(x)y = q(x), \tag{4}$$

which is linear in y and dy/dx.

To solve this equation, we first try to find a function $\mu(x)$, known as an *integrating factor*, such that

$$\frac{d}{dx}(\mu y) = \mu\left(\frac{dy}{dx} + p(x)y\right),$$

which is a multiple of the left hand side of equation (4).

We therefore require

$$\frac{d\mu}{dx}\,y + \mu\,\frac{dy}{dx} = \mu\,\frac{dy}{dx} + p(x)\mu y,$$

which is satisfied if and only if

$$\frac{d\mu}{dx} = p(x)\mu,$$

a separable equation for μ in terms of p. Therefore

$$\ln \mu = \int \frac{d\mu}{\mu} = \int p(x)\;dx,$$

so

$$\boxed{\mu(x) = \exp\left(\int p(x)\;dx\right).}$$

(We ignore the constant of integration since *any* solution will do.)

We have now shown that

$$\frac{d}{dx}(\mu(x)y) = \mu(x)\left(\frac{dy}{dx} + p(x)y\right) = \mu(x)q(x),$$

which we can integrate to find the general solution:

$$\mu(x)y(x) = \int \mu(x)q(x)\;dx + c$$

or

$$\boxed{y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)\,q(x)\;dx + c\right).}$$

Example:

$$\frac{dy}{dx} + \frac{1}{x}y = x.$$

First find the integrating factor:

$$\mu(x) = e^{\int (1/x)dx} = e^{\ln(x)} = x.$$

Multiplying through by $\mu(x)$ gives:

$$x\frac{dy}{dx} + y = x^2,$$

so

$$xy = \int x^2 \, dx = \frac{x^3}{3} + c$$

yielding the general solution:

$$y = \frac{x^2}{3} + \frac{c}{x}.$$

2.4 Linear differential operators

A function takes a number to a number:

$$\text{e.g.} \quad f(x) = x^2; \qquad f(2) = 4.$$

An *operator* takes a function to a function. You are already familiar with the *derivative operator* D which takes a function y to $Dy = y'$ with

$$Dy\,(x) = y'(x) = \frac{dy(x)}{dx} = \frac{dy}{dx}(x).$$

The derivative is an example of a *linear* operator, which means that for any functions y_1 and y_2, and any constants c_1 and c_2,

$$D(c_1y_1 + c_2y_2) = c_1\,Dy_1 + c_2\,Dy_2.$$

This condition can be re-written as

$$\frac{d}{dx}\left(c_1y_1(x) + c_2y_2(x)\right) = c_1\frac{dy_1(x)}{dx} + c_2\frac{dy_2(x)}{dx},$$

and is equivalent to the two conditions

$$D(cy) = cDy \quad \text{and} \quad D(y_1 + y_2) = Dy_1 + Dy_2.$$

Note however, that if $f(x)$ is a function,

$$D(fy) = \frac{d}{dx}\left(f(x)y(x)\right) = f\,\frac{dy}{dx} + \frac{df}{dx}\,y = f\,Dy + Df\,y \neq f\,dy,$$

i.e. linearity only holds for *constant* multiples!

The second derivative is also a linear operator, written

$$D^2 = \frac{d^2}{dx^2},$$

so that

$$D^2 y\,(x) = \frac{d^2 y(x)}{dx^2}.$$

A *first-order linear differential operator* has the general form:

$$L = D + p$$

where p is a function, i.e.

$$Ly\,(x) = \frac{dy(x)}{dx} + p(x)y(x).$$

We should check that this is linear:

$$\begin{aligned}
L\left(c_1 y_1(x) + c_2 y_2(x)\right) &= \frac{d}{dx}\left(c_1 y_1(x) + c_2 y_2(x)\right) + p(x)\left(c_1 y_1(x) + c_2 y_2(x)\right) \\
&= c_1 \frac{dy_1(x)}{dx} + c_2 \frac{dy_2(x)}{dx} + c_1 p(x) y_1(x) + c_2 p(x) y_2(x) \\
&= c_1 \left(\frac{dy_1(x)}{dx} + p(x) y_1(x)\right) + c_2 \left(\frac{dy_2(x)}{dx} + p(x) y_2(x)\right) \\
&= c_1\, Ly_1(x) + c_2\, Ly_2(x),
\end{aligned}$$

as required. The differential equation

$$\frac{dy}{dx} + p(x)y = q(x)$$

can now be written in the operator form:

$$Ly = q. \qquad (5)$$

If $q = 0$, this equation is *homogeneous*, otherwise it is *non-homogeneous*.

Equation (5) has a number of properties which depend only on the *linearity* of the differential operator L, and not its particular form. Of particular importance is the *complementary equation*, which is the homogeneous equation

$$Ly = 0. \tag{6}$$

A solution of the homogeneous equation (6) is known as a *characteristic function* of the linear operator L. The general solution y_c of the complementary equation is the *complementary solution* of the non-homogeneous equation (5).

Suppose y_1 and y_2 are two solutions of $Ly = 0$, and c_1 and c_2 are constants. Then by linearity,

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2 = 0,$$

so $c_1y_1 + c_2y_2$ is also a solution. In fact, if $y_1, y_2, \ldots, y_n$ are solutions of $Ly = 0$ and and $c_1, c_2, \ldots, c_n$ are constants, then

$$L\left(c_1y_1 + c_2y_2 + \cdots + c_ny_n\right) = 0$$

so $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is also a solution of $Ly = 0$. This result is known as the *principle of superposition* for linear homogeneous equations. Note that $y = 0$ is a solution to *any* homogeneous equation.

Now consider the non-homogeneous equation (5). Solutions y_1 and y_2 now cannot in general be added, since

$$L(c_1y_1 + c_2y_2) = c_1\,Ly_1 + c_2\,Ly_2 = c_1q + c_2q = (c_1 + c_2)q$$

which is only a solution if $c_1 + c_2 = 1$. However, solutions for different right hand sides can be added; if y_1 is a solution of $Ly = q_1$ and y_2 is a solution of $Ly = q_2$, then

$$L(y_1 + y_2) = Ly_1 + Ly_2 = q_1 + q_2,$$

so $y_1 + y_2$ is a solution to the equation $Ly = q_1 + q_2$.

Now suppose y_p is a solution of $Ly = q$, and y_c is a solution of the complementary homogeneous equation $Ly = 0$. Then

$$L(y_c + y_p) = Ly_c + Ly_p = 0 + q = q,$$

so $y_c + y_p$ is also a solution of $Ly = q$. In fact, the general solution of $Ly = q$ has the form $y = y_c + y_p$, where the *complementary solution* y_c is the general solution to $Ly = 0$, and y_p, the *particular integral*, is a solution to $Ly = q$.

Example: Consider the differential equation

$$Ly(x) = f(x) \quad \text{where} \quad Ly = \frac{dy}{dx} + 2y.$$

The complementary homogeneous equation is

$$\frac{dy}{dx} + 2y = 0$$

which has the general solution

$$y_c(x) = ce^{-2x}.$$

Suppose $f(x) = 4x^2$. A particular integral is given by

$$y_p(x) = 2x^2 - 2x + 1,$$

since

$$\frac{dy_p}{dx} + 2y_p = (4x - 2) + 2(2x^2 - 2x + 1) = 4x^2.$$

(This solution could have been computed using the general form of the solution to a linear first order equation.) Hence the general solution is

$$y(x) = ce^{-2x} + 2x^2 - 2x + 1.$$

Suppose $f(x) = 5\sin(x)$. Then

$$y_p(x) = 2\sin(x) - \cos(x)$$

is a particular solution. Hence the general solution to

$$\frac{dy}{dx} + 2y = 4x^2 + 5\sin(x)$$

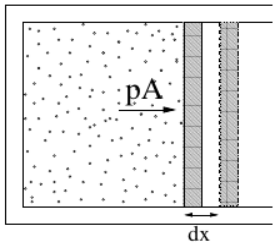
is

$$y(x) = \left(ce^{-2x}\right) + \left(2x^2 - 2x + 1\right) + \left(2\sin(x) - \cos(x)\right).$$

2.5* Case study: expansion of an ideal gas

The first law of thermodynamics states that the total energy U of a physical system can be changed by adding heat Q and/or performing work W:

$$dU = dQ + dW.$$



Consider an arbitrary quasi-static reversible process in which one mole of an ideal gas in a piston-cylinder arrangement at constant pressure p absorbs an amount of heat dQ: If the piston moves out a distance dx, then the work done is

$$-F\,dx = -pA\,dx = -p\,dV,$$

where F is the net force on the piston and V is the volume. Hence

$$dQ = dU + p\,dV.$$

Now $dU = C_v\,dT$, where the specific heat C_v is a function of T only. Invoking the ideal gas law $pV = RT$, with R constant,

$$dQ = C_v\,dT + \frac{RT}{V}\,dV.$$

Here,

$$\frac{\partial C_v}{\partial V} = 0 \quad \text{and} \quad \frac{\partial}{\partial T}\left(\frac{RT}{V}\right) = \frac{R}{V} \neq 0,$$

so the differential equation is not exact. However, there exists an integrating factor $1/T$;

$$\frac{\partial}{\partial V}\left(\frac{C_v}{T}\right) = 0 = \frac{\partial}{\partial T}\left(\frac{R}{V}\right),$$

implying that

$$\frac{dQ}{T} = ds$$

is exact. The quantity s is the *entropy*.

In thermodynamics the integrals of exact differentials are known as *functions of state*.

Chapter 3

Higher order linear ODEs

3.1 Homogeneous second order linear equations with constant coefficients

A linear second order ordinary differential equation can be written as

$$Ly\,(x) \equiv \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x). \tag{1}$$

If p, q and r are defined and continuous in some interval I, it can be shown that equation (1) has a twice-differentiable solution in I for any initial condition specifying $y(x_0)$ and $y'(x_0)$ with $x_0 \in I$.

The simplest case is that of a homogeneous equation with *constant coefficients*, which can be written as

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = 0 \tag{2}$$

where a, b and c are real numbers and $a > 0$.

For a solution, we try $y = e^{\lambda x}$ for some constant λ. Then

$$\frac{dy}{dx} = \lambda e^{\lambda x}, \qquad \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$$

which implies

$$(a\lambda^2 + b\lambda + c)\,e^{\lambda x} = 0.$$

Now since $e^{\lambda x} \neq 0$ for any x, we can write

$$a\lambda^2 + b\lambda + c = 0, \tag{3}$$

the *auxiliary* or *characteristic* equation for equation (2). The general solution depends on the nature of the roots of equation (3).

Real roots: If the roots λ_1 and λ_2 are real and different, the general solution is

$$\boxed{y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.}$$

A repeated root: If there is one repeated root λ, equation (2) is of the form

$$\frac{d^2y}{dx^2} - 2\lambda\frac{dy}{dx} + \lambda^2 y = 0$$

One solution of this is

$$y_1(x) = e^{\lambda x}.$$

A second is

$$y_2(x) = xe^{\lambda x},$$

which we can check:

$$\frac{dy_2}{dx} = (1+\lambda x)e^{\lambda x}, \qquad \frac{d^2y_2}{dx^2} = (2\lambda + \lambda^2 x)e^{\lambda x}. \quad \checkmark$$

Therefore

$$\frac{d^2y_2}{dx^2} - 2\lambda\frac{dy_2}{dx} + \lambda^2 y_2 = \left((2\lambda + \lambda^2 x) - 2\lambda(1+\lambda x) + \lambda^2 x\right) e^{\lambda x} = 0$$

as required. The general solution is then

$$\boxed{y(x) = (c_1 + c_2 x)e^{\lambda x}.}$$

Complex conjugate roots: If the roots of the auxiliary equation are complex, they must be complex conjugates, and we can write

$$\lambda_+ = \alpha + i\beta, \qquad \lambda_- = \alpha - i\beta.$$

Then

$$y_\pm(x) \equiv e^{\lambda_\pm x} = e^{(\alpha \pm i\beta)x} = e^{\alpha x}\left(\cos(\beta x) \pm i\sin(\beta x)\right)$$

are solutions, and so are

$$\begin{aligned} y_1(x) &= \frac{1}{2}\left(y_+(x) + y_-(x)\right) &= e^{\alpha x}\cos(\beta x), \\ y_2(x) &= \frac{1}{2i}\left(y_+(x) + y_-(x)\right) &= e^{\alpha x}\sin(\beta x). \end{aligned}$$

Hence the general solution is

$$\boxed{y(x) = e^{\alpha x}\left(c_1\cos(\beta x) + c_2\sin(\beta x)\right).}$$

Example:

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 0$$

Try a solution of the form

$$y = e^{\lambda x}.$$

Then

$$2\lambda^2 + 3\lambda - 2 = 0,$$

which factorises to

$$(2\lambda - 1)(\lambda + 2) = 0.$$

Therefore

$$\lambda_1 = 1/2 \text{ and } \lambda_2 = -2,$$

and hence the general solution is

$$y = c_1 e^{x/2} + c_2 e^{-2x}$$

and we have

$$\frac{dy}{dx} = \frac{c_1}{2}e^{x/2} - 2c_2 e^{-2x}.$$

Now suppose the initial conditions are $y(0) = 4$, $y'(0) = 1$. Then

$$\begin{aligned} c_1 + c_2 &= 4, \\ \frac{1}{2}c_1 - 2c_2 &= 1 \end{aligned}$$

which has the solution

$$c_1 = \frac{18}{5}, \qquad c_2 = \frac{2}{5}.$$

Thus the particular solution is

$$y = \frac{18}{5}e^{x/2} + \frac{2}{5}e^{-2x}.$$

Example:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

Try

$$y = e^{\lambda x}.$$

Then

$$\lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \quad \lambda_1 = -1 + 2i \text{ and } \lambda_1 = -1 - 2i.$$

Hence the general solution is

$$y(x) = e^{-x}\left(c_1 \cos(2x) + c_2 \sin(2x)\right).$$

We can check the solution $y(x) = e^{-x}\cos(2x)$ by computing

$$\begin{aligned} \frac{dy}{dx} &= -e^{-x}\cos(2x) - 2e^{-x}\sin(2x), \\ \frac{d^2y}{dx^2} &= -3e^{-x}\cos(2x) + 4e^{-x}\sin(2x), \end{aligned}$$

so

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = (-3 - 2 + 5)e^{-x}\cos(2x) + (4 - 2 \times 2)e^{-x}\sin(2x) = 0. \qquad \checkmark$$

Example:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0; \qquad y(0) = 1, \quad y'(0) = 3.$$

The auxiliary equation is

$$\lambda^2 + 4\lambda + 4 = 0$$

which has a repeated root $\lambda = -2$, so the general solution is

$$y(x) = (c_1 x + c_2)e^{-2x}.$$

We compute

$$y'(x) = \left(-2c_1 x + c_1 - 2c_2\right)e^{-2x}; \qquad y(0) = c_2, \quad y'(0) = c_1 - 2c_2,$$

so $c_2 = 1$ and then $c_1 = 5$. Hence the particular solution is

$$y(x) = (5x + 1)e^{-2x}.$$

3.2 Inhomogeneous second order linear equations

These take the form

$$Ly\,(x) \equiv a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x). \tag{4}$$

We know from the general theory of linear differential equations that the general solution can be written as a sum of the complementary solution y_c and a particular integral y_p. Since we already know how to find the complementary solution, we need only consider how to find a particular integral.

Method of undetermined coefficients

To solve non-homogeneous linear equations we can use the *method of undetermined coefficients* to find y_p, which amounts to little more than guessing.

Example:

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 4$$

The auxiliary (characteristic) equation is

$$2\lambda^2 + 3\lambda - 2 = 0 \quad \Rightarrow \quad (2\lambda - 1)(\lambda + 2) = 0.$$

To find $y_p(x)$, we use a trial particular integral of a form suggested by $r(x)$. In this case, we take $y_p(x) = \alpha$, a constant, since $r(x) = 4$ is constant.

Substitute in the differential equation:

$$y_p(x) = \alpha, \quad y_p'(x) = y_p''(x) = 0.$$

Hence $-2y_p = 4$, so $y_p(x) = -2$. Therefore, a particular integral is

$$y_p(x) = -2,$$

while, from a previous example, the complimentary function is

$$y_c(x) = Ae^{x/2} + Be^{-2x}\ .$$

Hence the general solution is

$$y(x) = y_c(x) + y_p(x) = Ae^{x/2} + Be^{-2x} - 2.$$

Example:

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 6x$$

For the trial particular integral take

$$y_p(x) = \alpha x + \beta$$

since $r(x) = 6x$.

Substitute in the differential equation:

$$y_p(x) = \alpha x + \beta, \quad y_p'(x) = \alpha, \quad y_p''(x) = 0.$$

This gives

$$3\alpha - 2(\alpha x + \beta) = 6x$$

for all values of x. Equating coefficients of powers of x:

$$\begin{aligned} 3\alpha - 2\beta &= 0, \\ -2\alpha &= 6. \end{aligned}$$

Hence $\alpha = -3$ and $\beta = -9/2$, so

$$y_p(x) = -3x - 9/2.$$

The general solution is

$$y(x) = Ae^{x/2} + Be^{-2x} - 3x - 9/2.$$

Example:

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 2e^{3x}$$

For the trial particular integral take

$$y_p(x) = \alpha e^{3x}.$$

Substitute in the differential equation:

$$y_p(x) = \alpha e^{3x}, \quad y_p'(x) = 3\alpha e^{3x}, \quad y_p''(x) = 9\alpha e^{3x}.$$

Hence

$$(18 + 9 - 2)\alpha e^{3x} = 2e^{3x},$$

so $\alpha = 2/25$ and we have a particular integral

$$y_p(x) = \frac{2}{25}e^{3x}.$$

Example:

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 68\cos(2x)$$

For the trial particular integral take

$$\begin{aligned} y_p(x) &= \alpha\cos(2x) + \beta\sin(2x), \\ y_p'(x) &= -2\alpha\sin(2x) + 2\beta\cos(2x), \\ y_p''(x) &= -4\alpha\cos(2x) - 4\beta\sin(2x). \end{aligned}$$

Therefore

$$2\frac{d^2y_p}{dx^2} + 3\frac{dy_p}{dx} - 2y_p = (-8\alpha + 6\beta - 2\alpha)\cos(2x) + (-8\beta - 6\alpha - 2\beta)\sin(2x).$$

Equating coefficients of $\cos(2x)$ and $\sin(2x)$ gives

$$\begin{aligned} -10\alpha + 6\beta &= 68, \\ -6\alpha - 10\beta &= 0 \end{aligned}$$

which has the solution $\alpha = -5$ and $\beta = 3$. Hence

$$y_p(x) = 3\sin(2x) - 5\cos(2x).$$

In general, we can use the following form for the trial solution $y_p(x)$ depending on the form of $r(x)$:

1. If $r(x)$ is a polynomial of degree n, then try a polynomial of degree n:

$$y_p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots \alpha_n x^n.$$

2. If $r(x) = e^{kx}$, then try

$$y_p(x) = \alpha e^{kx}.$$

3. If $r(x) = \sin(\omega x)$ or $r(x) = \cos(\omega x)$, then try

$$y_p(x) = \alpha\sin(\omega x) + \beta\cos(\omega x).$$

4. If $r(x)$ is a product of functions of the types given above, try the corresponding product for y_p. E.g., if $r(x) = xe^{2x}$, try $y_p(x) = (\alpha_0 + \alpha_1 x)e^{2x}$.

5. If the trial function given above contains a solution of the homogeneous equation, multiply the trial function by x.

Example:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x + e^{-x}$$

The complementary solution is

$$y_c(x) = c_1 + c_2 e^{-x}.$$

For $r(x) = 2x + e^{-x}$, we would usually try $y_p(x) = \alpha_0 + \alpha_1 x + \beta e^{-x}$, but $\alpha_0 + \beta e^{-x}$ solves the homogeneous equation, so is no good. Instead we try

$$y_p = \alpha_1 x + \alpha_2 x^2 + \beta x e^{-x}$$

which gives

$$\frac{dy_p}{dx} = \alpha_1 + 2\alpha_2 x + \beta(1-x)e^{-x} \quad \text{and} \quad \frac{d^2y_p}{dx^2} = 2\alpha_2 + \beta(x-2)e^{-x}.$$

Then

$$\begin{aligned} \frac{d^2y_p}{dx^2} + \frac{dy_p}{dx} &= \left(2\alpha_2 + \beta(x-2)e^{-x}\right) + \left(\alpha_1 + 2\alpha_2 x + \beta(1-x)e^{-x}\right) \\ &= (2\alpha_2 + \alpha_1) + 2\alpha_2 x - \beta e^{-x}. \end{aligned}$$

Notice that the terms in x^2 and xe^{-x} have cancelled each other out. We find $\alpha_2 = 1$, $\alpha_1 = -2$ and $\beta = -1$, giving particular solution

$$y_p(x) = x^2 - 2x - xe^{-x}.$$

Variation of parameters

When the method of undetermined coefficients fails, there is another more general approach to finding y_p, called *variation of parameters*. Suppose the complementary solution to the homogeneous equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

is $c_1y_1(x) + c_2y_2(x)$. To solve the non-homogeneous equation, we replace the constants by functions:

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Since we now have two unknown functions, we must impose an extra condition, which we take to be

$$u_1'y_1 + u_2'y_2 = 0.$$

Then

$$\begin{aligned} y' &= (u_1y_1' + u_2y_2') + (u_1'y_1 + u_2'y_2) = u_1y_1' + u_2y_2', \\ y'' &= (u_1y_1'' + u_2y_2'') + (u_1'y_1' + u_2'y_2'). \end{aligned}$$

Substituting into the differential equation and rearranging gives

$$\begin{aligned} \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y &= u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2) + (u_1'y_1' + u_2'y_2') \\ &= u_1'y_1' + u_2'y_2'. \end{aligned}$$

since y_1 and y_2 solve the homogeneous equation. Thus we obtain two equations for u_1' and u_2',

$$\begin{aligned} u_1'y_1' + u_2'y_2' &= r, \\ u_1'y_1 + u_2'y_2 &= 0 \end{aligned}$$

which have the solution

$$\boxed{u_1' = \frac{-y_2r}{y_1y_2' - y_1'y_2}, \qquad u_2' = \frac{y_1r}{y_1y_2' - y_1'y_2}.}$$

The function

$$W(y_1, y_2) = y_1y_2' - y_1'y_2$$

is called the *Wronskian* of y_1 and y_2.

Example:

$$\frac{d^2y}{dx^2} + y = \sec(x)$$

The complementary homogeneous equation has solutions

$$y_1(x) = \cos(x) \quad \text{and} \quad y_2(x) = \sin(x),$$

so we try

$$y(x) = u_1(x)\cos(x) + u_2(x)\sin(x).$$

The equations for u_1' and u_2' are then

$$\begin{aligned} -u_1' \sin(x) + u_2' \cos(x) &= \sec(x), \\ u_1' \cos(x) + u_1' \sin(x) &= 0. \end{aligned}$$

Solving for u_1' and u_2' gives

$$u_1'(x) = -\tan(x) \quad \text{and} \quad u_2'(x) = 1.$$

Integrating gives

$$u_1(x) = \ln|\cos(x)| + c_1 \quad \text{and} \quad u_2'(x) = x + c_2,$$

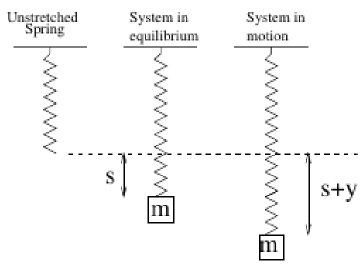
so the general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x) + \ln|\cos(x)| \cos(x) + x \sin(x).$$

3.3 Case study: oscillations and resonance

Free oscillations

Consider the motion of a mass m suspended from a spring. The weight of the mass (mg) extends the spring an amount s. If the extension is small, the restoring force exerted by the spring will be proportional to s (Hooke's law).



Hence in equilibrium

$$mg = ks$$

where k is the spring (stiffness) constant. Now suppose we extend the spring a further distance y. The acceleration $\ddot{y}$ is given in terms of the upwards force

$$F = k(s + y).$$

Applying Newton's second law (balancing net forces and resulting accelerations), we have that

$$m\ddot{y} = -k(s+y) + mg = -ky,$$

giving rise to the second order ODE

$$\ddot{y} + \left(\frac{k}{m}\right) y = 0.$$

Usually one defines $\omega_o^2 = k/m$, the square of the angular frequency.

The general solution is

$$y(t) = A\cos(\omega_0)t + C\sin(\omega_0 t),$$

known as *harmonic oscillation*.

Now suppose the spring is damped, i.e. there is a frictional force opposing the motion, whose magnitude is proportional to the velocity. Then the differential equation of motion becomes:

$$\ddot{y} + \left(\frac{c}{m}\right)\dot{y} + \left(\frac{k}{m}\right) y = 0.$$

The auxiliary equation is

$$r^2 + \frac{c}{m}r + \frac{k}{m} = 0,$$

so

$$r = \frac{-c}{2m} \pm \frac{1}{2}\sqrt{\frac{c^2 - 4km}{m^2}} = \frac{1}{2m}\left(-c \pm \sqrt{c^2 - 4km}\right).$$

Now write

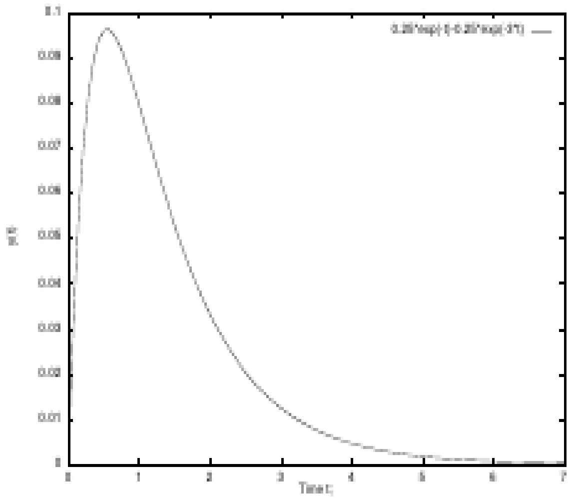
$$\alpha = c/2m > 0, \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4km}.$$

If $c^2 > 4km$, we have two distinct real roots, and the system is said to be *overdamped.* If $c^2 < 4km$, we have complex conjugate roots, and the system is *umderdamped.* Finally, if $c^2 = 4km$, we have a a double real root, which gives *critical damping.* We examine each of these cases in turn:

Overdamping: This corresponds to the case of two real roots, for which the solution is

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$$

This depicts the typical behaviour if $y(0) = 0$. There is no oscillation: both terms in the above equation tend to zero as $t \to \infty$, the 2nd at a faster rate than the first.



Underdamping: This corresponds to the case of complex conjugate roots, for which the solution is

$$y(t) = e^{-\alpha t}[A\cos(\omega_1 t) + B\sin(\omega_1 t)],$$

where $\omega_1 = \sqrt{4km - c^2}\,/\,2m$.

Note that if c is small, we can expand the square root using the binomial theorem

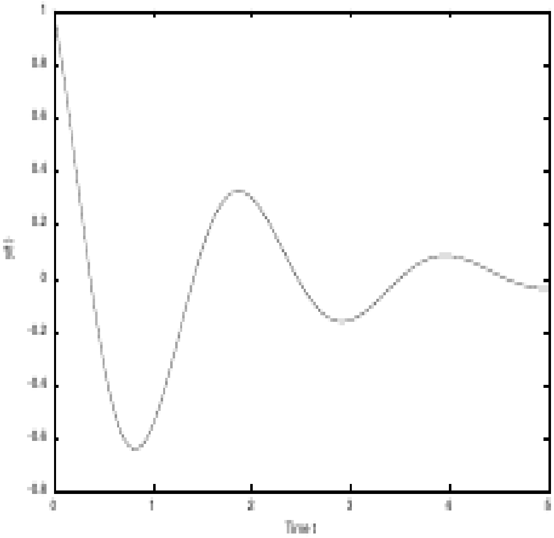
$$\omega_1 = \frac{\sqrt{4km}}{2m}\left[\sqrt{1 - \frac{c^2}{4km}}\right] = \sqrt{\frac{k}{m}}\left[1 - \frac{c^2}{8km} + \cdots\right] = \omega_0\left[1 - \frac{c^2}{8km} + \cdots\right].$$

Thus the frequency of oscillations is reduced from the underdamped case.

Critical damping: This corresponds to a double (repeated) real root, so the general solution is

$$y(t) = (c_1 t + c_2)e^{-\alpha t}.$$

Since $e^{-\alpha t} > 0$ and because (owing to its linearity) $c_1 t + c2$ can have at most one zero for positive t, the motion can have at most one passage through the equilibrium position ($y = 0$). If the initial conditions are such that c_1 and c_2 have the same sign, there is no such passage.



Forced oscillations

We look at the equation

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \alpha\omega_0^2 y = r(t),$$

where $r(t)$ is an *input* (usually oscillatory) of this system. Suppose that $r(t)$ has the form

$$r(t) = R\sin(\omega t),$$

where R is a constant. Then

$$y_p = A\cos(\omega t) + B\sin(\omega t).$$

Substituting into the equation we find

$$\begin{aligned} \dot{y_p} &= \omega B\cos(\omega t) - \omega A\sin(\omega t), \\ \ddot{y_p} &= -\omega^2\left[B\sin(\omega t) + A\cos(\omega t)\right]. \end{aligned}$$

Hence

$$\begin{aligned} &-\omega^2\left[B\sin(\omega t) + A\cos(\omega t)\right] \\ &\quad + 2\alpha\left[\omega B\cos(\omega t) - \omega A\sin(\omega t)\right] + \omega_0^2(A\cos(\omega t) + B\sin(\omega t)) = R\sin(\omega t). \end{aligned}$$

Equating coefficients gives:

$$\begin{aligned} -\omega^2 A + 2\alpha\omega B + \omega_0^2 A &= 0, \\ -\omega^2 B - 2\alpha\omega A + \omega_0^2 B &= R. \end{aligned}$$

Solving yields:

$$\begin{aligned} A &= \frac{-2\alpha\omega R}{(\omega_0^2-\omega^2)^2+(2\alpha\omega)^2} \\ B &= \frac{R(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+(2\alpha\omega)^2}. \end{aligned}$$

So the general solution is:

$$y(t) = e^{-\alpha t}\left[c_1\cos(\omega_1 t)+c_2\sin(\omega_1 t)\right] + \frac{-2\alpha\omega R\cos(\omega t)}{(\omega_0^2-\omega^2)^2+(2\alpha\omega)^2} + \frac{R(\omega_0^2-\omega^2)\sin(\omega t)}{(\omega_0^2-\omega^2)^2+(2\alpha\omega)^2}.$$

The first term quickly goes to zero as $t\to\infty$, so for large t, the solution is (approximately)

$$y(t) = \frac{-2\alpha\omega R\cos(\omega t)}{(\omega_0^2-\omega^2)^2+(2\alpha\omega)^2} + \frac{R(\omega_0^2-\omega^2)\sin(\omega t)}{(\omega_0^2-\omega^2)^2+(2\alpha\omega)^2}.$$

Now let

$$\sin\phi = \frac{2\alpha\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(2\alpha\omega)^2}}$$

and

$$\cos\phi = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+(2\alpha\omega)^2}}.$$

Then

$$y(t) = \frac{R\sin(\omega t-\phi)}{\sqrt{(\omega_0^2-\omega^2)^2+(2\alpha\omega)^2}}.$$

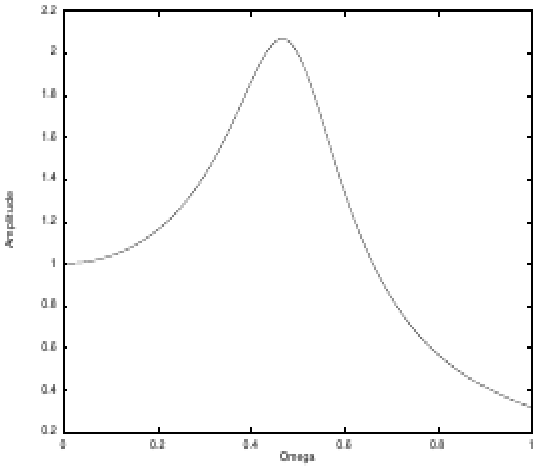
Thus the response to the input $\sin(\omega t)$ is an oscillatory wave with amplitude

$$A = \frac{R}{\sqrt{(\omega_0^2-\omega^2)^2+(2\alpha\omega)^2}}$$

and the same frequency as the input, but out of phase by the angle ϕ. It is instructive to plot the amplitude of the motion (output) as a function of the frequency ω of the input:

The output has a peak at $\omega=\sqrt{\omega_0^2-2\alpha^2}$ which is termed the *resonance frequency*. For small α this is ω_0. Note that at $\omega=\omega_0$ the phase lag is $\phi=\pi/2$, i.e. the output motion is 90° out of phase with the input.

The above exposition applies both to mechanical and electrical resonance. In the latter case, it explains how radios and televisions are tuned.



3.4 The Euler equation

The Euler equation is:

$$x^2\,\frac{d^2y}{dx^2} + ax\,\frac{dy}{dx} + by = r(x).$$

Notice that the coefficient of d^2y/dx^2 has a zero at $x = 0$, which means that the Euler equation is an example of a *singular differential equation.* However it can be solved in a very similar way to the equation with constant coefficients.

The complementary homogeneous equation is

$$x^2\,\frac{d^2y}{dx^2} + ax\,\frac{dy}{dx} + by = 0.$$

Try a solution $y(x) = x^m$ where m is unknown. Then

$$\begin{aligned} y'(x) &= mx^{m-1}, \\ y''(x) &= m(m-1)x^{m-2}. \end{aligned}$$

Substituting into the equation gives

$$(m(m-1) + am + b)\,x^m = 0.$$

This has a solution if

$$m(m-1) + am + b = m^2 + m(a-1) + b = 0.$$

We therefore obtain

$$m = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}.$$

The form of the solution is dictated by the nature of the roots. We consider two cases:

Two distinct roots: Here the solution is straightforward:

$$\boxed{y(x) = c_1 x^{m_1} + c_2 x^{m_2},}$$

where m_1 and m_2 are the roots. Notice that if m_1 and m_2 are positive, then *both* solutions are zero at $x = 0$.

A double root: Again, $y = x^m$ is a root. We can express the coefficients a and b in terms of m, and re-write the equation

$$x^2 \frac{d^2y}{dx^2} + (1-2m)x \frac{dy}{dx} + m^2 y = 0.$$

The second solution is given by

$$y(x) = \ln(x) x^m.$$

We check this by substituting into the original equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x} x^m + m \ln(x) x^{m-1} = (1 + m \ln x) x^{m-1}, \\ \frac{d^2y}{dx^2} &= (2m - 1 + m(m-1)\ln(x))\, x^{m-2}, \end{aligned}$$

so

$$\begin{aligned} & x^2 \frac{d^2y}{dx^2} + (1-2m)x \frac{dy}{dx} + m^2 y \\ &= (2m - 1 + m(m-1)\ln(x))\, x^m + (1-2m)(1 + m \ln x) x^m + m^2 \ln(x) x^m \\ &= \big((2m-1) + (1-2m) + ((m^2 - m) + (m - 2m^2) + m^2)\ln(x)\big)\, x^m \ = \ 0 \end{aligned}$$

as required. Hence the general solution is

$$\boxed{y(x) = (c_1 + c_2 \ln(x))\, x^m.}$$

Example:

$$x^2\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + 2y = 0; \qquad y(1) = 1, \quad y'(1) = 0.$$

The auxiliary equation is

$$m(m-1) - 2m + 2 = 0.$$

The left hand side simplifies to

$$m(m-1) - 2m + 2 = m^2 - 3m + 2 = (m-2)(m-1)$$

which has roots $m = 1$ and $m = 2$. Thus the general solution is

$$y(x) = c_1 x + c_2 x^2.$$

The initial conditions give

$$c_1 + c_2 = 1 \quad \text{and} \quad c_1 + 2c_2 = 0,$$

so $c_1 = 2$ and $c_2 = -1$, giving the solution

$$y(x) = 2x - x^2.$$

Note that $y = 0$ at $x = 0$ automatically. Initial conditions cannot violate this condition, which is not surprising, as the original equation requires it.

Example:

$$x^2\,\frac{d^2y}{dx^2} + 3x\,\frac{dy}{dx} + y = 0$$

The auxiliary equation is

$$m(m-1) + 3m + 1 = 0,$$

and since

$$m(m-1) + 3m + 1 = m^2 + 2m + 1 = (m+1)^2,$$

we have a double root $m = -1$. Thus the general solution is

$$y(x) = \frac{c_1 + c_2 \ln(x)}{x}.$$

The Euler equation can be transformed into an equation with constant coefficients by writing $x = e^t$, so $t = \ln(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt},$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x}\frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dt}\right) - \frac{1}{x^2}\frac{dy}{dt}\\ &= \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt},\end{aligned}$$

so

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = \frac{d^2y}{dt^2} - \frac{dy}{dt} + a\frac{dy}{dt} + by = \frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by.$$

Thus the equation

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = f(x)$$

becomes

$$\boxed{\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = f(e^t),}$$

a linear second order ordinary differential equation with constant coefficients.

If $f(e^t)$ is one of the forms we found previously for linear equations with constant coefficients, we should solve it in terms of t and then transform it back in terms of x.

Example:

$$x^2\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + 2y = \ln(2x)$$

Since

$$g(t) = f(e^t) = \ln(2e^t) = \ln(2) + t$$

the equation can be transformed to

$$\frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 2y = \ln(2) + t.$$

Using the trial solution $y_p(t) = \alpha t + \beta$, we find

$$-3\alpha + 2(\alpha t + \beta) = \ln(2) + t,$$

implying

$$2\alpha = 1 \quad \text{and} \quad 2\beta - 3\alpha = \ln(2).$$

Then $\alpha = 1/2$ and $\beta = (\ln(2) + 3/2)/2$, so

$$y_p(t) = \frac{t}{2} + \frac{1}{2}\left(\ln(2) + \frac{3}{2}\right); \qquad y_p(x) = \frac{\ln(2x)}{2} + \frac{3}{4}.$$

3.5 Homogeneous systems of ODEs

A homogeneous system of n first order linear ordinary differential equations with constant coefficients has the form

$$\frac{dy_i}{dx} = \sum_{j=1}^{n} a_{ij} y_i \tag{5}$$

for unknowns $y_1, y_2, \ldots, y_n$.

Example:

$$\begin{aligned} \frac{dy_1}{dx} &= -4y_1 - 2y_2, \\ \frac{dy_2}{dx} &= \quad y_1 - \ y_2. \end{aligned}$$

To solve this, we can eliminate one of the variables, say y_1, by expressing y_1 and its derivative in terms of y_2.

$$\frac{dy_2}{dx} = y_1 - y_2 \quad \Rightarrow \quad y_1 = \frac{dy_2}{dx} + y_2 \quad \text{and} \quad \frac{dy_1}{dx} = \frac{d^2y_2}{dx^2} + \frac{dy_2}{dx}.$$

Substituting into the equation for dy_1/dx, we obtain

$$\frac{d^2y_2}{dx^2} + \frac{dy_2}{dx} = -4\left(\frac{dy_2}{dx} + y_2\right) - 2y_2 \quad \Rightarrow \quad \frac{d^2y_2}{dx^2} + 5\frac{dy_2}{dx} + 6y_2 = 0,$$

which has the solution

$$y_2(x) = c_1 e^{-2x} + c_2 e^{-3x}.$$

From this we can show

$$y_1(x) = -c_1 e^{-2x} - 2c_2 e^{-3x}.$$

Matrix methods

While this method can always be used to compute the solution, there is a more elegant method based on computing *eigenvalues* and *eigenvectors* of matrices. Equation (5) can be written in vector form:

$$\frac{d\mathbf{y}}{dx} = \mathrm{A}\mathbf{y}, \tag{6}$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad \mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

We look for solutions of equation (6) of the form

$$\mathbf{y}(x) = e^{\lambda x}\mathbf{v},$$

where λ is a constant and $\mathbf{v}$ is a nonzero constant vector. Equation (6) then becomes

$$\lambda e^{\lambda x}\mathbf{v} = e^{\lambda x}\mathrm{A}\mathbf{v},$$

from which we can cancel the $e^{\lambda x}$ to deduce that we must have

$$\lambda\mathbf{v} = \mathrm{A}\mathbf{v}. \tag{7}$$

Equation (7) is the *eigenvalue* equation for A. If λ and $\mathbf{v}$ solve this equation, we say λ is an *eigenvalue* of A, and $\mathbf{v}$ is the corresponding *eigenvector*. An $n \times n$ matrix A has at least one and at most n eigenvalues.

The eigenvalue equation can be re-written as

$$(\mathrm{A} - \lambda\mathrm{I})\mathbf{v} = 0,$$

which has a non-trivial solution if and only if the determinant of the matrix $\mathrm{A} - \lambda\mathrm{I}$ is equal to zero. Therefore, we can compute the eigenvalues of A by solving the *characteristic equation* of A,

$$\det(\mathrm{A} - \lambda\mathrm{I}) = 0. \tag{8}$$

If A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, then, by linearity,

$$\boxed{y(x) = c_1 e^{\lambda_1 x}\mathbf{v}_1 + c_2 e^{\lambda_2 x}\mathbf{v}_2 + \cdots + c_m e^{\lambda_m x}\mathbf{v}_m}$$

is a solution of the homogeneous equation.

Example: The system

$$\begin{aligned}\frac{dy_1}{dx} &= -4y_1 - 2y_2,\\ \frac{dy_2}{dx} &= \quad y_1 - \ y_2\end{aligned}$$

can be written in vector form as

$$\frac{d\mathbf{y}}{dx} = \mathrm{A}\mathbf{y} \quad \text{where} \quad \mathrm{A} = \begin{pmatrix} -4 & -2 \\ 1 & -1 \end{pmatrix}.$$

The characteristic equation for A is

$$\det\begin{pmatrix} -4-\lambda & -2 \\ 1 & -1-\lambda \end{pmatrix} = (-4-\lambda)(-1-\lambda) - 1(-2) = \lambda^2 + 5\lambda + 6 = 0,$$

which gives the eigenvalues

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = -3.$$

The eigenvector $\mathbf{v}_1$ satisfies $(\mathrm{A} - \lambda_1 \mathrm{I})\mathbf{v}_1 = \mathbf{0}$, so

$$\begin{aligned} -2\alpha - 2\beta &= 0 \\ \alpha + \beta &= 0 \end{aligned} \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Both these equations *must* have the same solutions. Here we find $\alpha = -\beta$, so we can choose $\alpha = 1$ and $\beta = -1$ to give

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(or any nonzero multiple of this). Similarly, we can show that

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

so the general solution is

$$\mathbf{y} = c_1 e^{-2x} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-3x} \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

or explicitly for y_1 and y_2,

$$\begin{aligned} y_1 &= c_1 e^{-2x} + 2c_2 e^{-3x}, \\ y_2 &= -c_1 e^{-2x} - c_2 e^{-3x}. \end{aligned}$$

Complex eigenvalues

Suppose A has a complex eigenvalue λ with corresponding eigenvector $\mathbf{v}$. Write $\lambda = \alpha + i\omega$ and $\mathbf{v} = \mathbf{u} + i\mathbf{w}$. Then we have a solution

$$\begin{aligned} \mathbf{y}(x) &= e^{\lambda x}\mathbf{v} = e^{\alpha x}\left(\cos(\omega x) + i\sin(\omega x)\right)(\mathbf{u} + i\mathbf{w}) \\ &= e^{\alpha x}\left(\mathbf{u}\cos(\omega x) - \mathbf{w}\sin(\omega x)\right) + ie^{\alpha x}\left(\mathbf{u}\sin(\omega x) + \mathbf{w}\cos(\omega x)\right). \end{aligned}$$

Since the equation is a real equation, the complex conjugate function $\overline{y}(x)$ must also be a solution. Then, by linearity, we deduce that the real and imaginary parts of $y(x)$ are solutions to the system. Therefore, we obtain the solutions

$$\begin{aligned} \mathbf{y}_{Re}(x) &= e^{\alpha x}\left(\cos(\omega x)\mathbf{u} - \sin(\omega x)\mathbf{w}\right), \\ \mathbf{y}_{Im}(x) &= e^{\alpha x}\left(\sin(\omega x)\mathbf{u} + \cos(\omega x)\mathbf{w}\right), \end{aligned}$$

which combine to give the general solution:

$$\boxed{\mathbf{y}(x) = e^{\alpha x}\Big(\left(c_1\cos(\omega x) + c_2\sin(\omega x)\right)\mathbf{u} + \left(-c_1\sin(\omega x) + c_2\cos(\omega x)\right)\mathbf{w}\Big).}$$

Repeated eigenvalues

Suppose λ is a double root of the characteristic polynomial, but has only one eigenvector $\mathbf{v}$ (up to constant multiples). We can then find a vector $\mathbf{u}$ such that

$$\mathrm{A}\mathbf{u} = \lambda\mathbf{u} + \mathbf{v}.$$

Note that $\mathbf{u}$ is only defined up to addition of a scalar multiple of $\mathbf{v}$. A second solution to the system of equations is then

$$\boxed{\mathbf{y}(x) = e^{\lambda x}\left(\mathbf{u} + x\mathbf{v}\right).}$$

Check:

$$\frac{d\mathbf{y}}{dx} = \lambda e^{\lambda x}\left(\mathbf{u} + x\mathbf{v}\right) + e^{\lambda x}\mathbf{v}; \qquad \mathrm{A}\mathbf{y} = e^{\lambda x}\left(\lambda\mathbf{u} + \mathbf{v} + \lambda x\mathbf{v}\right). \quad \checkmark$$

3.6 Non-homogeneous systems of ODEs

A non-homogeneous system of linear equations has the general form

$$\frac{d\mathbf{y}}{dx} = \mathrm{A}\mathbf{y} + \mathbf{f}(x), \tag{9}$$

where $\mathbf{f}$ is a vector-valued function,

$$\mathbf{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}.$$

The complementary homogeneous equation is

$$\frac{d\mathbf{y}}{dx} = \mathrm{A}\mathbf{y}.$$

Just as for a single higher-order ordinary differential equation, we use the method of undetermined coefficients for certain functions $\mathbf{f}(x)$. We can use the following forms for the trial solution $\mathbf{y}_p$:

1. If $\mathbf{f}(x)$ is a polynomial of degree n, then try a polynomial of degree n,

$$\mathbf{y}_p(x) = \mathbf{v}_0 + x\mathbf{v}_1 + x^2\mathbf{v}_2 + \cdots x^n\mathbf{v}_n.$$

2. If $\mathbf{f}(x) = e^{kx}\mathbf{u}$, then try

$$\mathbf{y}_p(x) = e^{kx}\mathbf{v}.$$

3. If $\mathbf{f}(x) = \sin(\omega x)\mathbf{u}$ or $\mathbf{f}(x) = \cos(\omega x)\mathbf{u}$, then try

$$\mathbf{y}_p(x) = \sin(\omega x)\mathbf{v} + \cos(\omega x)\mathbf{w}.$$

The method of variation of parameters can also be used to solve non-homogeneous linear systems.

Example:

$$\begin{aligned} \frac{dy_1}{dx} &= -4y_1 - 2y_2 - 2e^{2x}, \\ \frac{dy_2}{dx} &= y_1 - y_2 + 7e^{2x}. \end{aligned}$$

To find a particular solution, we use a trial solution

$$\mathbf{y}_p(x) = e^{2x}\mathbf{v}.$$

The system of equations then becomes

$$2e^{2x}\mathbf{v} = e^{2x}\mathrm{A}\mathbf{v} + e^{2x}\mathbf{b},$$

where

$$\mathrm{A} = \begin{pmatrix} -4 & -2 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}.$$

We can cancel out the factor e^{2x} and rearrange to obtain

$$(2\mathrm{I} - \mathrm{A})\,\mathbf{v} = \mathbf{b}.$$

Writing the matrix equation in full, we find

$$\begin{pmatrix} 6 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix},$$

which we can solve to obtain $v_1 = -1$ and $v_2 = 2$. A particular solution is therefore given by

$$\mathbf{y}_p(x) = e^{2x}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

or, in components, as

$$y_1(x) = -e^{2x}, \qquad y_2(x) = 2e^{2x}.$$

Example:

$$\begin{aligned} \frac{dy_1}{dx} &= -y_1 - 2y_2 + 2e^{-x}, \\ \frac{dy_2}{dx} &= 2y_1 - y_2. \end{aligned}$$

We can write this system in vector form as

$$\frac{d\mathbf{y}}{dx} = \mathrm{A}\mathbf{y} + e^{-x}\mathbf{b}\,, \quad \text{where} \quad \mathrm{A} = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

To find the general solution, we first try to solve the corresponding homogeneous system $d\mathbf{y}/dx = \mathrm{A}\mathbf{y}$. The characteristic polynomial of A is

$$(\lambda + 1)^2 + 4,$$

which has roots $\lambda = -1 \pm 2i$. Taking the eigenvalue $\lambda = -1 + 2i$, we find

$$\mathrm{A} - \lambda \mathrm{I} = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix},$$

so the corresponding eigenvector $\mathbf{v}_\lambda$ is

$$\mathbf{v}_\lambda = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Hence the general solution of the homogeneous equation is

$$\mathbf{y_c}(x) = e^{-x} \begin{pmatrix} c_1 \cos(2x) + c_2 \sin(2x) \\ c_1 \sin(2x) - c_2 \cos(2x) \end{pmatrix}.$$

We now try for a particular solution of the form

$$\mathbf{y_p} = e^{-x}\mathbf{v},$$

where $\mathbf{v}$ is a constant vector. Substituting into the equation gives

$$-e^{-x}\mathbf{v} = e^{-x}\mathrm{A}\mathbf{v} = e^{-x}\mathbf{b} \quad \Rightarrow (\mathrm{A} + \mathrm{I})\mathbf{v} = -\mathbf{b}.$$

Now

$$\mathrm{A} + \mathrm{I} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix},$$

so

$$(\mathrm{A} + \mathrm{I})^{-1} = \frac{1}{4} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

and hence

$$\mathbf{v} = -(\mathrm{A} + \mathrm{I})^{-1}\mathbf{b} = -\frac{1}{4} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the general solution is

$$\begin{aligned} y_1 &= e^{-x} \left(c_1 \cos(2x) + c_2 \sin(2x) \right), \\ y_2 &= e^{-x} \left(c_1 \sin(2x) - c_2 \cos(2x) + 1 \right). \end{aligned}$$

Example: Consider the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= x + 4y - t - 5, \qquad x(0) = 1; \\ \frac{dy}{dt} &= -x + 5y + t - 3, \qquad y(0) = 0. \end{aligned}$$

We can write the system in vector form with

$$\mathrm{A} = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{f}(t) = t\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -5 \\ -3 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(\mathrm{A} - \lambda\mathrm{I}) \;=\; (1-\lambda)\,(5-\lambda) - 4\,(-1) \;=\; \lambda^2 - 6\lambda + 9 \;=\; (\lambda-3)^2,$$

so $\lambda = 3$ is a double root. We then compute

$$\mathrm{A} - 3\mathrm{I} = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix},$$

so an eigenvector $\mathbf{v}$ satisfying $(\mathrm{A}-3\mathrm{I})\mathbf{v} = \mathbf{0}$ and a vector $\mathbf{u}$ satisfying $(\mathrm{A}-3\mathrm{I})\mathbf{u} = \mathbf{v}$ are given by

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The complementary solution is then

$$\mathbf{y}_c(t) = (c_1 + c_2 t)e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

For a trial particular solution, use

$$\mathbf{y}_p(t) = \mathbf{v}_0 + t\mathbf{v}_1.$$

Substituting into the system, we obtain

$$\mathbf{v}_1 = \mathrm{A}\mathbf{v}_0 + t\mathrm{A}\mathbf{v}_1 + \mathbf{u}_0 + t\mathbf{u}_1.$$

Equating coefficients of t gives

$$\mathbf{v}_1 = \mathrm{A}\mathbf{v}_0 + \mathbf{u}_0 \quad \text{and } \mathrm{A}\mathbf{v}_1 + \mathbf{u}_1 = 0.$$

The inverse of A is

$$\mathrm{A}^{-1} = \frac{1}{9}\begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix},$$

so

$$\begin{aligned} \mathbf{v}_1 &= -\mathrm{A}^{-1}\mathbf{u}_1 = -\frac{1}{9}\begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \mathbf{v}_0 &= \mathrm{A}^{-1}(\mathbf{v}_1 - \mathbf{u}_0) = \frac{1}{9}\begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \end{aligned}$$

The general solution is therefore

$$\mathbf{y}(t) = (c_1 + c_2 t)e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Setting $t = 0$, we find

$$\mathbf{y}(0) = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so $c_1 = c_2 = -1$, giving the particular solution

$$\mathbf{y}(t) = -(1+t)e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - e^{3t}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

or, in components,

$$\begin{aligned} x(t) &= -(2t+1)e^{3t} + t + 2, \\ y(t) &= -(t+1)e^{3t} + 1. \end{aligned}$$

Check:

$$\begin{aligned} x'(t) &= -(6t+5)e^{3t} + 1 \;=\; x + 4y - t - 5. \quad \checkmark \\ y'(t) &= -(3t+4)e^{3t} \;=\; -x + 5y + t - 3. \quad \checkmark \end{aligned}$$

Chapter 4

Fourier Series

4.1 Derivation of the Fourier series

A function $f(x)$ is said to be *periodic* if it is defined for all real x and there exists $T > 0$ such that

$$f(x+T) = f(x).$$

T is called *period* of f.

Obviously, if n is an integer, then $\cos(nx)$ and $\sin(nx)$ have period 2π. Therefore, the *trigonometric series*

$$\frac{a_0}{2} + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \dots$$

also has period 2π (assuming that it converges). If we can write $f(x)$ as a trigonometric series with period 2π, the series is called the *Fourier series* of $f(x)$.

Suppose $f(x)$ is periodic with period 2π. We assume that $f(x)$ can be expressed as a trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

and want to find the coefficients $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ known as the *Fourier coefficients*.

To find a_0, we integrate both sides:

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)\,dx \\
&= a_0 \int_{-\pi}^{\pi} \frac{1}{2}\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\,dx \\
&= \pi a_0\,,
\end{aligned}$$

since

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = \int_{-\pi}^{\pi} \sin(nx)\,dx = 0.$$

Therefore,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

To get a_n, we multiply through by $\cos(mx)$ and integrate term by term.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx &= \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)\right)\cos(mx)\,dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} a_0 \cos(mx)\,dx \\
&\quad + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx \\
&\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx.
\end{aligned}$$

We now use the trigonometric identities:

$$\begin{aligned}
\cos(mx)\cos(nx) &= \frac{\cos((m+n)x) + \cos((m-n)x)}{2}, \\
\sin(mx)\sin(nx) &= \frac{\cos((m-n)x) - \cos((m+n)x)}{2}, \\
\sin(mx)\cos(nx) &= \frac{\sin((m+n)x) + \sin((m-n)x)}{2}
\end{aligned}$$

to deduce that most of these integrals vanish:

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0 \quad \text{for} \quad m, n \in \mathbb{Z},$$

and

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = 0 \quad \text{if} \quad m \neq n.$$

These results mean that the functions $\cos(nx)$ and $\sin(nx)$ are mutually *orthogonal*. Orthogonality implies that all integrals in the expression for a_m are zero except for the integral of $\cos(nx)\cos(mx)$ for $m = n$, which gives:

$$\begin{aligned}\int_{\pi}^{\pi} f(x)\cos(mx)\,dx &= \int_{\pi}^{\pi} a_m \cos^2(mx)\,dx \\ &= a_m \int_{-\pi}^{\pi} \frac{1-\sin(2mx)}{2}\,dx \;=\; a_m \int_{-\pi}^{\pi} \frac{1}{2}\,dx \;=\; a_m\pi.\end{aligned}$$

Thus

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx.$$

Similarly, by multiplying the series by $\sin(mx)$, one finds that

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx.$$

Let us combine these results:

$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\,, \\ a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx\,, \\ b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx\,. \end{aligned}}$$

Collectively these are known as Euler's formulae.

The series

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos(mx) + b_m\sin(mx)\right)}$$

is called the *Fourier series* (FS) of $f(x)$. The coefficients a_m and b_m are called the *Fourier coefficients* of $f(x)$. Note that the coefficients are linear in f.

Note: Because of the periodicity, $\int_{-\pi}^{\pi}$ can be replaced by $\int_0^{2\pi}$.

4.2 Even and odd functions

Recall that a function f is *even* if $f(-x) = f(x)$ for all x, and *odd* if $f(-x) = -f(x)$.

If f is 2π-periodic, then in the Fourier series for f,

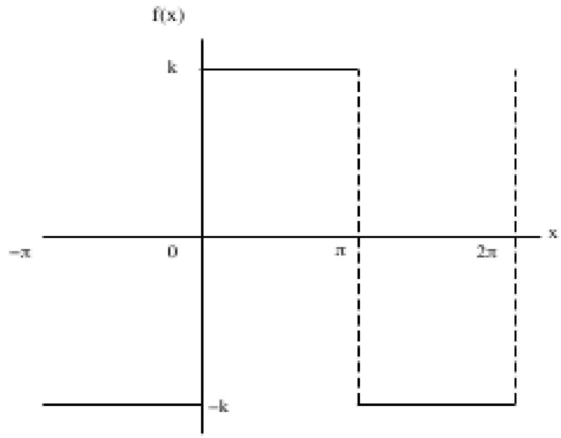
$$a_n = \frac{1}{\pi} \int_{x=-\pi}^{\pi} f(x) \cos(nx)\, dx.$$

If f is an odd function, we can set $y = -x$, and find

$$a_n = \frac{1}{\pi} \int_{y=\pi}^{-\pi} f(-y) \cos(-ny)\ (-1) dy = -\frac{1}{\pi} \int_{y=-\pi}^{\pi} f(y) \cos(ny)\, dy = -a_n,$$

which can only be true if $a_n = 0$.

Example: Consider the *square wave* function shown below, with period 2π.



$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0\,, \\ k & \text{if} \quad 0 < x < \pi\,. \end{cases}$$

Then

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = 0$$

for $n = 0, 1, 2, \ldots$ since $f(x)$ is an odd function.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\sin(nx)\;dx + \int_{0}^{\pi} f(x)\sin(nx)\;dx\right) \\
&= \frac{2k}{\pi}\int_{0}^{\pi}\sin(nx)\;dx \;=\; \frac{-2k}{\pi n}\left[\cos(nx)\right]_0^{\pi} \\
&= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \dfrac{4k}{\pi(2m+1)} & \text{if } n = 2m+1. \end{cases}
\end{aligned}
$$

Therefore

$$
f(x) = \sum_{m=0}^{\infty} \frac{4k}{\pi(2m+1)} \sin((2m+1)x)\,.
$$

Aside: Note that at $x = \pi/2$ we have $f(x) = k$. Dividing through by k then yields

$$
1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) \quad \Rightarrow \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\,,
$$

which is a famous series expansion due to Leibniz.

If $f(x)$ is an odd function, then again setting $y = -x$, we find

$$
\frac{1}{\pi}\int_{x=-\pi}^{0} f(x)\sin(nx)\;dx = \frac{1}{\pi}\int_{y=\pi}^{0} f(-y)\sin(-ny)(-1)\;dy = \frac{1}{\pi}\int_{0}^{\pi} f(y)\sin(y)\;dy,
$$

since we have a total of four minus signs, one due to f being an odd function, one due to sin being an odd function, one since $dx = (-1)\;dy$ and a final one due to reversing the order of integration. Therefore

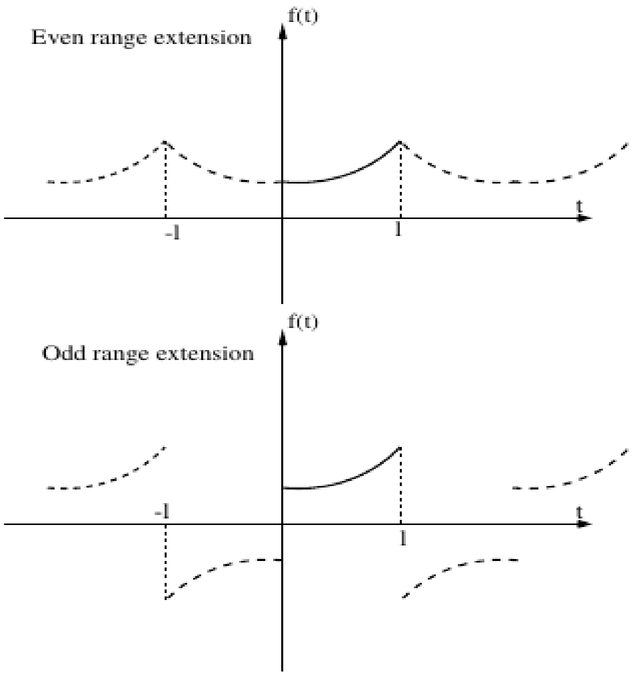
$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\;dx \;=\; \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(nx)\;dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\;dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\;dx.
\end{aligned}
$$

We can find similar formulae if f is an even function. Therefore,

> If f is an odd function, then $a_n = 0$ and $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\sin(nx)\;dx\,.$
>
> If f is an even function, then $b_n = 0$, and $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos(nx)\;dx\,.$

Half range expansions

Suppose $f(x)$ is defined for $0 < x < \pi$ and we wish to extend f to a periodic function with period 2π. There are many ways one could do this, but two obvious choices are to extend $f(x)$ such that it is either odd or even.



Even extension

If we choose the even extension, then $b_n = 0$,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dt$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx))\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx.$$

We have now represented the function $f(x)$ in the range $0 < x < \pi$ by a *Fourier cosine series.*

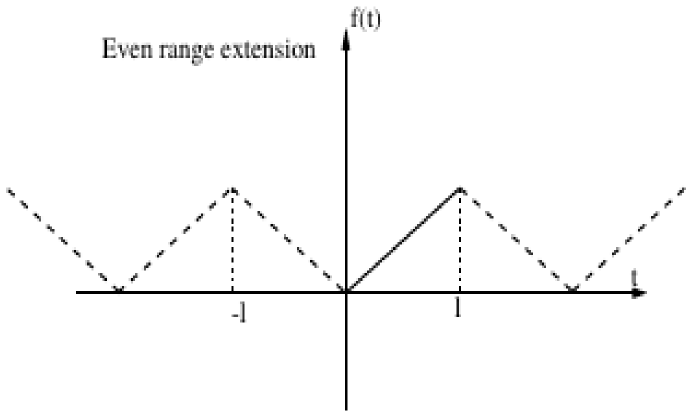
Odd extension

If we choose the odd extension, then $a_n = 0$, and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

We have represented the function $f(x)$ in the range $0 < x < \pi$ by a *Fourier sine series.*

Example: Let $f(s) = x$ for $0 < x < \pi$. Consider first the *even periodic extension.* Since we are considering an even extension, $b_n = 0$. The coefficients a_n are given



by:

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi\,, \\
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx \;=\; \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dt \\
&= \frac{2}{\pi}\left(\left[\frac{x\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin(nx)\,dx\right) \\
&= \frac{2}{\pi}\left[\frac{\cos(nx)}{n^2}\right]_0^{\pi} \;=\; \frac{2\left((-1)^n - 1\right)}{n^2\pi}.
\end{aligned}$$

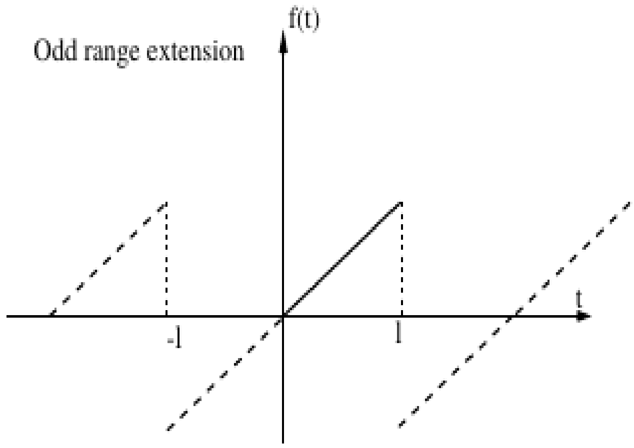
Hence $a_n = 0$ if n is even, and if $n = 2m + 1$ is odd,

$$a_n = \frac{-4}{(2m+1)^2\pi}.$$

Therefore we have the Fourier cosine series for $0 < x < \pi$:

$$x = f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos((2m+1)x)}{(2m+1)^2}.$$

Now consider the *odd periodic extension.* Here we have $a_n = 0$, and



$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx \\ &= \frac{2}{\pi}\left(\left[\frac{-x\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos(nx)\,dx\right) = \frac{-2(-1)^n}{n}. \end{aligned}$$

Hence for $0 < x < \pi$

$$x = f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx).$$

4.3 Functions having an arbitrary period

Suppose (as is often the case) that the period differs from 2π, so f is periodic with period $T > 0$; that is, $f(t+T) = f(x)$.

A simple scaling (stretching or shrinking the period to 2π) allows us to work out the Fourier series for such a function. To see this, define a new function

$$g(x) = f\left(\frac{Tx}{2\pi}\right).$$

Then g has period 2π since

$$g(x+2\pi) = f\left(\frac{T(x+2\pi)}{2\pi}\right) = f\left(\frac{Tx}{2\pi}+T\right) = f\left(\frac{Tx}{2\pi}\right) = g(x),$$

hence its Fourier series is

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big)$$

with the usual formulae for the a_n and b_n. Now setting $x = 2\pi t/T$, we have $f(t) = g(2\pi t/T)$, so the Fourier series of f is

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2n\pi t}{T}\right) + b_n\sin\left(\frac{2n\pi t}{T}\right)\right)}$$

with

$$\boxed{\begin{aligned} a_0 &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt\,, \\ a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2n\pi t}{T}\right)\,dt\,, \\ b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2n\pi t}{T}\right)\,dt\,, \end{aligned}}$$

for $n = 1, 2, 3, \ldots$. Again, because of the periodicity, $\int_{-T/2}^{T/2}$ can be replaced by $\int_0^T$.

Example:

$$f(t) = \begin{cases} 0 & \text{if} \quad -2 < t < -1\,, \\ 1 & \text{if} \quad -1 < t < 1\,, \\ 0 & \text{if} \quad 1 < t < 2\,. \end{cases}$$

First note that f is even, so $b_n = 0$ for all n. Now $T = 4$, so

$$\begin{aligned} a_0 &= \frac{2}{4}\int_{-2}^{2} f(t)dt \;=\; \frac{1}{2}\int_{-1}^{1} dt \;=\; 1\,, \\ a_n &= \frac{2}{4}\int_{-2}^{2} f(t)\cos\left(\frac{2n\pi t}{T}\right)\,dt \;=\; \frac{1}{2}\left[\frac{T}{2n\pi}\sin\left(\frac{2n\pi t}{T}\right)\right]_{-1}^{1} \\ &= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\,. \end{aligned}$$

We note further that $a_n = 0$ when n is even. Otherwise, we have

$$a_1 = \frac{2}{\pi}\,, \quad a_3 = \frac{-2}{3\pi}\,, \quad a_5 = \frac{2}{5\pi}\,, \quad a_7 = \frac{-2}{7\pi}\,, \quad \ldots .$$

Hence

$$\begin{aligned} f(t) &= \frac{1}{2} + \frac{2}{\pi}\left(\cos\left(\frac{\pi t}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi t}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi t}{2}\right) - \frac{1}{7}\cos\left(\frac{7\pi t}{2}\right) + \cdots\right) \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\cos\left(\frac{(2m+1)\pi t}{2}\right). \end{aligned}$$

Problem: In electronics, a half-wave rectifier clips the negative portion of any input voltage applied to it. If a sinusoidal input voltage of amplitude V and angular frequency ω is applied to the rectifier, calculate the Fourier series of the resulting output voltage.

Solution: The output has the form

$$f(t) = \begin{cases} 0 & \text{if} \quad -T/2 < t < 0\,, \\ V\sin(\omega T) & \text{if} \quad 0 < t < T/2\,, \end{cases}$$

where $T = 2\pi/\omega$ or $\omega = 2\pi/T$.

Since $f(t) = 0$ when $-T/2 < t < 0$, we have

$$\begin{aligned} a_0 &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{\omega}{\pi}\int_0^{\pi/\omega} V\sin(\omega t)\,dt = \frac{2V}{\pi}\,, \\ a_n &= \frac{2}{T}\int_0^{T/2} f(t)\cos(n\omega t)\,dt = \frac{\omega V}{\pi}\int_0^{\pi/\omega}\sin(\omega t)\cos(n\omega t)\,dt \\ &= \frac{\omega V}{2\pi}\int_0^{\pi/\omega}\left(\sin\left((n+1)\omega t\right) + \sin\left((1-n)\omega t\right)\right)\,dt\,. \end{aligned}$$

Taking $n = 1$ we find

$$a_1 = \frac{\omega V}{2\pi}\int_0^{\pi/\omega}\sin(2\omega t)\,dt = \frac{\omega V}{2\pi}\left[\frac{-\cos(2\omega t)}{2\omega}\right]_0^{\pi/\omega} = 0.$$

For $n = 2, 3, 4, \ldots$ we have

$$\begin{aligned} a_n &= \frac{\omega V}{2\pi}\left[-\frac{\cos((1+n)\omega t)}{(1+n)\omega} - \frac{\cos((1-n)\omega t)}{(1-n)\omega}\right]_0^{\pi/\omega} \\ &= \frac{V}{2\pi}\left(\frac{1-\cos((1+n)\pi)}{1+n} + \frac{1-\cos((1-n)\pi)}{1-n}\right) \end{aligned}$$

which is zero for n odd. For n even,

$$a_n = \frac{V}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = \frac{-2V}{(n-1)(n+1)\pi}.$$

Similar manipulations yield $b_n = 0$ unless $n = 1$, for which

$$b_1 = \frac{V}{2}.$$

Hence finally the Fourier series can be written as

$$f(t) = \frac{V}{\pi} + \frac{V}{2}\sin(\omega t) - \frac{2V}{\pi}\sum_{m=1}^{\infty}\frac{\cos(2m\omega t)}{(2m+1)(2m-1)},$$

where we have set $n = 2m$ in the sum.

4.4 Representation by Fourier series

Now we can compute Fourier series, it is useful to know how the series approximates the function. We consider the *partial sums*

$$\widetilde{f}_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n\cos(nx) + b_n\sin(nx)\right),$$

and how $\widetilde{f}(x) = \lim_{N\to\infty}\widetilde{f}_N(x)$ approximates $f(x)$.

Pointwise convergence

Suppose $f(x)$ is 2π periodic and piecewise continuous, and that $f(x)$ has a right and a left hand derivative at each point. Then

1. The Fourier series of $f(x)$ converges for all values of x.
2. If f is continuous at x_0, then the Fourier series at x_0 converges to $f(x_0)$.
3. If f is discontinuous at x_0, the sum of the series is the average of the left and the right hand limits of $f(x)$ at the discontinuity:

$$\widetilde{f}(x_0) = \frac{1}{2}\left(\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right).$$

Example:

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x < 0\,, \\ 1 & \text{if} \quad 0 < x < \pi\,. \end{cases}$$

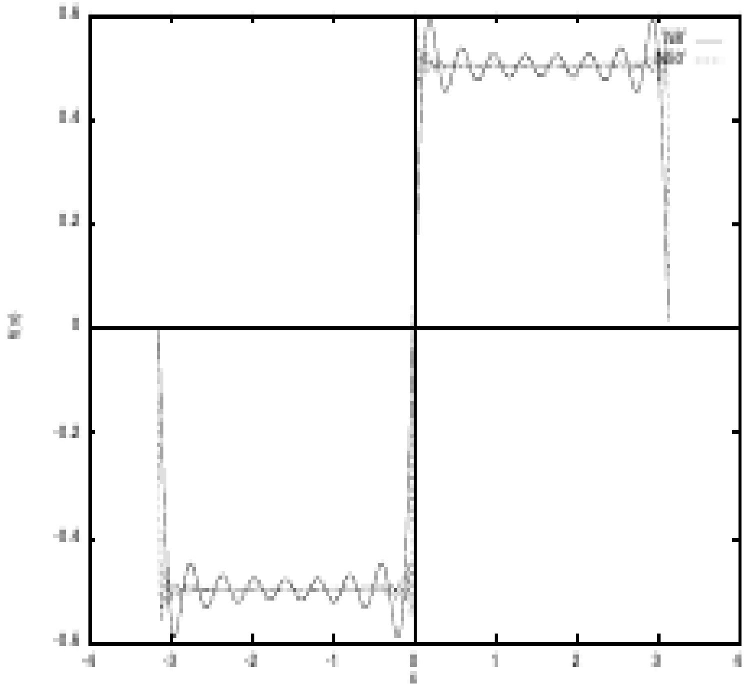
Utilising the results from an earlier example, f has the Fourier series

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{m=0}^{\infty} \frac{\sin((2m+1)x)}{2m+1}\,.$$

If $x = 0$, the sum of the series is $1/2$.

The Gibbs phenomenon

Below we plot the form of the first 8 and the first 50 terms of the Fourier series of the square wave computed previously. Clearly there is a fairly rapid convergence to the limiting form, but notice the spikes at the discontinuities.



Although the Fourier series for f converges to f pointwise, the partial sums overshoot the function values at the discontinuity by a *uniformly* large amount, even when considering a large number of terms ($N = 50$). This is known as the *Gibbs phenomenon.*

The *uniform* or *maximum* error between a function f and an approximation $\widehat{f}$ is

$$\max_{-\infty<x<\infty} \left| f(x) - \widehat{f}(x) \right|.$$

The following results hold for the maximum error of the Fourier approximations:

1. If f is continuous, then the Fourier series of f approximates f uniformly:

$$\max_{-\infty<x<\infty} \left| f(x) - \widetilde{f}_N(x) \right| \to 0 \quad \text{as} \quad N \to \infty.$$

2. If f is discontinuous at x_0 with a jump

$$\delta f(x_0) = \lim_{x \to x_0^+} f(x) - \lim_{x \to x_0^-} f(x),$$

then for all sufficiently large N, there is a value x_1 slightly greater than x_0 such that

$$\frac{\widetilde{f}_N(x_1) - f(x_1)}{\delta f(x_0)} \geqslant G,$$

where G is a constant with approximate value 0.09. In other words, the approximation $\widetilde{f}_N(x_1)$ overshoots the exact value $f(x_1)$ near the discontinuity. The Fourier approximations do not converge uniformly in this case.

The total square error

Let $f(x)$ be a given function of period 2π. The *total square error* when approximating f by a function $\widehat{f}$ is

$$\int_{-\pi}^{\pi} \left(f(x) - \widehat{f}(x) \right)^2 dx.$$

If f is a 2π-periodic piecewise-continuous function, then:

1. The Fourier approximations $\widetilde{f}_N$ converge to f, i.e. the total square error

$$\int_{-\pi}^{\pi} \left(f(x) - \widetilde{f}_N(x) \right)^2 dx \to 0 \qquad \text{as } N \to \infty.$$

2. The total square error on approximating f by $\widetilde{f}_N$ is

$$\int_{-\pi}^{\pi} \left(f(x) - \widetilde{f}_N(x)\right)^2 dx \;=\; \pi \sum_{n=N+1}^{\infty} \left(a_n^2 + b_n^2\right) .$$

Thus the Fourier series provides a good approximation in terms of the total square error, and we can estimate the error of the approximation by the Fourier coefficients. Moreover, we can compute the *square integral* of f in terms of the Fourier coefficients:

$$\begin{aligned}\int_{-\pi}^{\pi} \left(f(x)\right)^2 dx &= \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)\right)^2 dx \\ &= \int_{-\pi}^{\pi} \frac{a_0^2}{4} + \sum_{n=1}^{\infty} a_n^2 \cos^2(nx) + \sum_{n=1}^{\infty} b_n^2 \sin^2(nx)\, dx \\ &= \pi \left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)\right) ,\end{aligned}$$

since most of the terms integrate to zero by the orthogonality relations.

4.5 Forced oscillations revisited

Fourier series find applications in solving many types of differential equations. A prime example is the differential equation for forced oscillations of a damped system,

$$m\ddot{y} + c\dot{y} + ky = r(t)$$

where k is the spring constant, m the mass and $r(t)$ is the forcing term which we take to be a periodic function with period T. If we further assume that $c \ll km$, we have under-damping.

Consider first the homogeneous equation corresponding to the unforced system,

$$\ddot{y} + \frac{c}{m}\dot{y} + \omega^2 y = 0$$

where $\omega^2 = k/m$ is the natural frequency of the undamped system. This has solutions

$$y = e^{-ct/2m} \left(A \cos(\omega_1 t) + B \sin(\omega_1 t)\right)$$

where A and B are arbitrary coefficients and $\omega_1 < \omega$ is the oscillation frequency of the damped system. Since this decays to zero at large t, the motion at late times is controlled by the particular integral, which depends on the forcing term $r(t)$. We call this the *steady state solution*. As we shall now see, it can be written as a superposition of harmonic oscillations having the frequency of the external force and multiples of this frequency.

To calculate the particular integral we write $r(t)$ as a Fourier series,

$$r(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right) \right)$$

and write the trial particular integral as

$$y(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \left(c_n \cos\left(\frac{2n\pi t}{T}\right) + d_n \sin\left(\frac{2n\pi t}{T}\right) \right) .$$

To find the undetermined coefficients c_n and d_n we calculate term by term:

$$\begin{aligned}
\ddot{y}_n &= \left(\frac{2\pi n}{T}\right)^2 \left(-c_n \cos\left(\frac{2n\pi t}{T}\right) - d_n \sin\left(\frac{2n\pi t}{T}\right) \right) , \\
\frac{c}{m}\dot{y}_n &= \frac{2\pi nc}{Tm} \left(-c_n \sin\left(\frac{2n\pi t}{T}\right) + d_n \cos\left(\frac{2n\pi t}{T}\right) \right) , \\
\omega^2 y_n(t) &= \omega^2 \left(c_n \cos\left(\frac{2n\pi t}{T}\right) + d_n \sin\left(\frac{2n\pi t}{T}\right) \right) , \\
r_n(t) &= \left(a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right) \right) .
\end{aligned}$$

Matching coefficients gives

$$\omega^2 c_0 = a_0$$

and

$$\begin{aligned}
a_n &= \left(\omega^2 - \left(\frac{2\pi n}{T}\right)^2 \right) c_n + \frac{2\pi n}{T}\frac{c}{m} d_n , \\
b_n &= -\frac{2\pi n}{T}\frac{c}{m} c_n + \left(\omega^2 - \left(\frac{2\pi n}{T}\right)^2 \right) d_n .
\end{aligned}$$

This we can solve to find c_n and d_n in terms of a_n and b_n.

Note that in the above, if c/m is small (light damping) and $\omega^2 \approx \left(\frac{2\pi n}{T}\right)^2$, then the coefficients of c_n and d_n are small, and so c_n and d_n will be large. This phenomenon is called *resonant* behaviour, where the response of the system to the input force is very large.

Example: Consider the equation

$$\ddot{y} + 0.02\dot{y} + 25y = r(t)$$

with the forcing term

$$r(t) = \begin{cases} 1 & \text{if} \quad -\pi \leqslant t < 0\,, \\ -1 & \text{if} \quad 0 < t < \pi\,. \end{cases}$$

The function $r(t)$ is odd and can be represented by a Fourier series with $a_n = 0$ and

$$b_n = \frac{2}{n\pi}\left(1 + (-1)^{n+1}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 4/(2m+1)\pi & \text{if } n = 2m+1, \end{cases}$$

so the right hand side of the differential equation is

$$\sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin((2k+1)t)\,.$$

Now write for the particular integral

$$\begin{aligned} y(t) &= \sum_{n=1}^{\infty} \left(c_n \cos(nt) + d_n \sin(nt)\right), \\ \dot{y}(t) &= \sum_{n=1}^{\infty} n\left(-c_n \sin(nt) + d_n \cos(nt)\right), \\ \ddot{y}(t) &= \sum_{n=1}^{\infty} -n^2\left(c_n \cos(nt) + d_n \sin(nt)\right), \end{aligned}$$

so the left hand side of the differential equation becomes

$$\sum_{n=1}^{\infty} \left((25 - n^2)\ \left(c_n \cos(nt) + d_n \sin(nt)\right) \ + \ 0.02n\left(-c_n \sin(nt) + d_n \cos(nt)\right)\right)\ .$$

Comparing like terms gives $c_n = d_n = 0$ if n is even, and

$$\begin{aligned} (25 - n^2)\, c_n + 0.02n\, d_n &= 0\,, \\ (25 - n^2)\, d_n - 0.02n\, c_n &= \frac{4}{n\pi} \end{aligned}$$

if n is odd. In matrix notation:

$$\begin{pmatrix} 25 - n^2 & 0.02n \\ -0.02n & 25 - n^2 \end{pmatrix} \begin{pmatrix} c_n \\ d_n \end{pmatrix} = \begin{pmatrix} 0 \\ 4/n\pi \end{pmatrix}.$$

Inverting the matrix gives

$$\begin{pmatrix} c_n \\ d_n \end{pmatrix} = \frac{\begin{pmatrix} 25-n^2 & -0.02n \\ 0.02n & 25-n^2 \end{pmatrix}\begin{pmatrix} 0 \\ 4/n\pi \end{pmatrix}}{(25-n^2)^2+0.0004n^2},$$

so for n odd,

$$\begin{aligned} d_n &= \frac{4(25-n^2)/n\pi}{(25-n^2)^2+0.0004n^2}, \\ c_n &= \frac{-0.08/\pi}{(25-n^2)^2+0.0004n^2}. \end{aligned}$$

Evaluating the first few terms we obtain

$$d_1 \approx \frac{1}{6\pi}, \quad d_3 \approx \frac{1}{12\pi}, \quad d_5 = 0, \quad d_7 \approx -\frac{1}{42\pi}, \quad \ldots$$

and

$$c_1 \approx -\frac{0.08}{(24)^2\pi}, \qquad c_3 \approx -\frac{0.08}{(12)^2\pi}, \quad c_5 = -\frac{0.08}{0.01\pi} = -\frac{8}{\pi}, \quad c_7 \approx -\frac{0.08}{(24)^2\pi}, \ \ldots \ .$$

We see that all terms are small except c_5, so y_5 is the dominating term in the oscillation. So the steady state motion is approximately a harmonic oscillation with frequency five times that of the applied force.

4.6* Approximation by trigonometric polynomials

The partial sums of the Fourier series provide one approximation to f. It is natural to ask whether this represents the best approximation that can be obtained using trigonometric functions i.e. if there exists another trigonometric polynomial $\widehat{f}_N(x)$ of N terms,

$$\widehat{f}(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{N} (\alpha_n \cos(nx) + \beta_n \sin(nx)),$$

for which the approximation is better.

The total square error for $\widehat{f}$ is

$$e(\widehat{f}_N) = \int_{-\pi}^{\pi} (f(x))^2 \, dx + \int_{-\pi}^{\pi} \left(\widehat{f}_N(x)\right)^2 \, dx - \int_{-\pi}^{\pi} 2f(x)\widehat{f}_N(x) \, dx.$$

Now

$$\begin{aligned}\int_{-\pi}^{\pi} \left(\widehat{f}_N(x)\right)^2 \, dx &= \int_{-\pi}^{\pi} \left(\frac{\alpha_0}{2} + \sum_{n=1}^{N} (\alpha_n \cos(nx) + \beta_n \sin(nx))\right) \\ &\qquad \times \left(\frac{\alpha_0}{2} + \sum_{n=1}^{N} (\alpha_n \cos(nx) + \beta_n \sin(nx))\right) \, dx \\ &= \pi \left(\frac{\alpha_0^2}{2} + \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_N^2 + \beta_1^2 + \beta_2^2 + \cdots + \beta_N^2\right),\end{aligned}$$

and

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\widehat{f}_N(x) \, dx &= \int_{-\pi}^{\pi} f(x) \left[\frac{\alpha_0}{2} + \sum_{n=1}^{N} (\alpha_n \cos(nx) + \beta_n \sin(nx))\right] \\ &= \pi \left(\frac{\alpha_0 a_0}{2} + \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_N a_N \right. \\ &\qquad \left. + \beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_N b_N\right).\end{aligned}$$

Hence

$$\begin{aligned}e(\widehat{f}_N) &= \int_{-\pi}^{\pi} (f(x))^2 \, dx + \pi \left(\frac{\alpha_0^2}{2} + \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_N^2 + \beta_1^2 + \beta_2^2 + \cdots + \beta_N^2\right) \\ &\quad - 2\pi \left(\frac{\alpha_0}{2} a_0 + \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_N a_N + \beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_N b_N\right).\end{aligned}$$

Now let us compare this error with the error for the N-th term of the Fourier series.

$$\begin{aligned}&e(\widehat{f}_N) - e(\widetilde{f}_N) \\ &\quad = -2\pi \left(\frac{\alpha_0}{2} a_0 + \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_N a_N + \beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_N b_N\right) \\ &\qquad + \pi \left(\frac{\alpha_0^2}{2} + \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_N^2 + \beta_1^2 + \beta_2^2 + \cdots + \beta_N^2\right) \\ &\qquad + 2\pi \left(\frac{a_0^2}{2} + a_1^2 + a_2^2 + \cdots + a_N^2 + b_1^2 + b_2^2 + \cdots + b_N^2\right) \\ &\qquad - \pi \left(\frac{a_0^2}{2} + a_1^2 + a_2^2 + \cdots + a_N^2 + b_1^2 + b_2^2 + \cdots + b_N^2\right) \\ &\quad = \pi \left(\tfrac{1}{2}(a_0 - \alpha_0)^2 + (a_1 - \alpha_1)^2 + \cdots + (a_N - \alpha_N)^2 \right. \\ &\qquad \left. + (b_N - \beta_N)^2 + \cdots + (b_N - \beta_N)^2 \right).\end{aligned}$$

Therefore $e(\widehat{f}_N) - e(\widetilde{f}_N) \geqslant 0$, with equality if and only if

$$\alpha_0 = a_0,\ \alpha_1 = a_1,\ \ldots,\ \alpha_N = a_N,\ \beta_1 = b_1,\ \beta_1 = b_1,\ \ldots,\ \beta_N = b_N.$$

Hence the Fourier coefficients *minimise* the total square error.

Note: We have also shown that

$$e(\widetilde{f}_N) = \int_{-\pi}^{\pi} f^2(x)\,dx - \pi\left(\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right),$$

which means that as N increases, the error $e(\widetilde{f})$ must decrease. Thus with increasing N, the partial Fourier sum yields a better and better approximation to f. We have already seen the stronger result that the total square error actually converges to 0, though to show this is much harder.

Problem: Let $f(x) = x^2$ for $-\pi < x < \pi$, $f(x + 2\pi) = f(x)$. Find the function

$$\widehat{f}_N(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{N}\left(\alpha_n \cos(nx) + \beta_n \sin(nx)\right),$$

so that the mean square error $e(\widehat{f}_N) = \int_{-\pi}^{\pi}(f(x) - \widehat{f}_N(x))^2\,dx$ is a minimum for the cases $N = 1, 2, 3, 4$.

Solution: The Fourier coefficients minimise e, i.e. $\alpha_n = a_n$ and $\beta_n = b_n$. The function $f(x)$ is even, hence $b_n = 0$.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3},\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx = \frac{1}{\pi}\left[\frac{1}{n}x^2\sin(nx) + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin(nx)\right]_{-\pi}^{\pi}\\
&= \frac{4(-1)^n}{n^2}.
\end{aligned}$$

Now the error is given by

$$e(\widehat{f}_N) = \int_{-\pi}^{\pi} f^2(x)\,dx - \pi\left[\frac{\alpha_0^2}{2} + \sum_{n=1}^{N}(\alpha_n^2 + \beta_n^2)\right],$$

and the square integral of f is

$$\int_{-\pi}^{\pi} f^2(x)\,dx = \int_{-\pi}^{\pi} x^4\,dx = \frac{2\pi^5}{5}.$$

Thus

$$e(\widehat{f}_1) = \frac{2\pi^5}{5} - \pi\left(\frac{4\pi^4}{18} + 16\right) = 4.138,$$

$$
\begin{aligned}
e(\widehat{f_2}) &= \frac{2\pi^5}{5} - \pi\left(\frac{4\pi^4}{18} + 16 + 1\right) = 0.9964\,,\\
e(\widehat{f_3}) &= \frac{2\pi^5}{5} - \pi\left(\frac{4\pi^4}{18} + 16 + 1 + \frac{16}{81}\right) = 0.3758\,,\\
e(\widehat{f_4}) &= \frac{2\pi^5}{5} - \pi\left(\frac{4\pi^4}{18} + 16 + 1 + \frac{16}{81} + \frac{16}{256}\right) = 0.1795\,,
\end{aligned}
$$

giving the results to 4 significant figures.

Chapter 5

First Order Partial Differential Equations

5.1 The method of characteristics

A first order partial differential equation for a function of two variables $u(x, y)$ has the general form

$$F(x, y, u, u_x, u_y) = 0.$$

A solution represents a surface above the (x, y)-plane, $z = u(x, y)$. The particular surface is determined by the *boundary conditions.*

In many cases, we have one time variable t and one space variable x, and an equation for $u(x, t)$. If, in addition, $u(x, 0) = f(x)$ is specified, we have an *initial value* problem.

Example: The *first-order wave equation* is

$$\frac{\partial u}{\partial t} = c \frac{\partial u}{\partial x}.$$

If we take

$$u(x, t) = f(x + ct)$$

for a differentiable function f, we find

$$\frac{\partial u}{\partial t} = cf'(x + ct) \quad \text{and} \quad \frac{\partial u}{\partial x} = f'(x + ct),$$

so u is a solution, regardless of the choice of f. If we now impose the initial conditions

$$u(x,0) = \sin(x),$$

the solution of the initial value problem is

$$u(x,t) = \sin(x+ct),$$

which describes a sine wave moving to the left with speed c.

Looking at the solution to the first order wave equation, we notice the following:

1. The general solution depends on an arbitrary *function*, rather than a finite number of arbitrary constants.
2. The boundary condition specified u on a line ($t = 0$) in the (x,t)-plane.
3. Any solution depends only on $x+ct$, and so is constant on lines $x+ct = \text{const}$.

These features are typical of many first order partial differential equations. The lines $x + ct = \text{const}$ are a special case of *characteristic curves*. The characteristic curves depend only on the partial differential equation itself, and not on the boundary conditions. Characteristic curves are important in finding solutions, and also in determining appropriate boundary conditions. The solutions to the first order wave equation are constant on the characteristic curves.

Computing characteristic curves

Consider the following linear first-order partial differential equation:

$$p(x,y)u_x + q(x,y)u_y = 0\,. \tag{1}$$

We look for curves on which u is constant. We specify a curve by a *parameterisation*, in which x and y are given as functions of t,

$$x = x(t)\,, \quad y = y(t)\,.$$

To differentiate u along the curve, we take u as a function of t, $u = u(x(t), y(t))$. Then

$$\frac{d}{dt}u(x(t),y(t)) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}\,.$$

For u to be constant along the curve, we want $du/dt = 0$. However, since u also solves the partial differential equation, if we set

$$\frac{dx}{dt} = p(x,y) \quad \text{and} \quad \frac{dy}{dt} = q(x,y)$$

then

$$\frac{d}{dt}u(x(t), y(t)) \;=\; \frac{\partial u}{\partial x}\,p(x,y) + \frac{\partial u}{\partial y}\,q(x,y) \;=\; 0$$

as required. Hence the characteristic curves are given by the solutions of the system of equations

$$\boxed{\frac{dx}{dt} = p(x,y)\,, \quad \frac{dy}{dt} = q(x,y)\,.} \tag{2}$$

Solving the system of equations (2) gives an expression for the characteristic curves in terms of a parameter t and the two constants of integration, c_1 and c_2. Eliminating t, we find an expression for the characteristic curves in terms of x, y and the two constants, which we rearrange into an expression of the form

$$G(x,y) = \text{const.}$$

The general solution of equation (1) is then

$$u(x,y) = f(G(x,y))$$

for an arbitrary function f.

Example:

$$\frac{\partial u}{\partial x} \;+\; \sqrt{1+y^2}\,\frac{\partial u}{\partial y} \;=\; 0\,.$$

The differential equations for the characteristic curves are

$$\frac{dx}{dt} = 1\,, \qquad \frac{dy}{dt} = \sqrt{1+y^2}\,.$$

Clearly,

$$x = t + c_1,$$

and solving the equation for y, we obtain

$$\int \frac{dy}{\sqrt{1+y^2}} = \int dt\,,$$

so

$$\sinh^{-1}(y) = t + c_2 \quad \Rightarrow \quad y = \sinh(t + c_2).$$

Eliminating t, we find

$$y = \sinh(x - c_1 + c_2) = \sinh(x + c)\,,$$

where $c = c_2 - c_1$ is another constant. Rearranging, we find

$$\sinh^{-1}(y) - x = c\,.$$

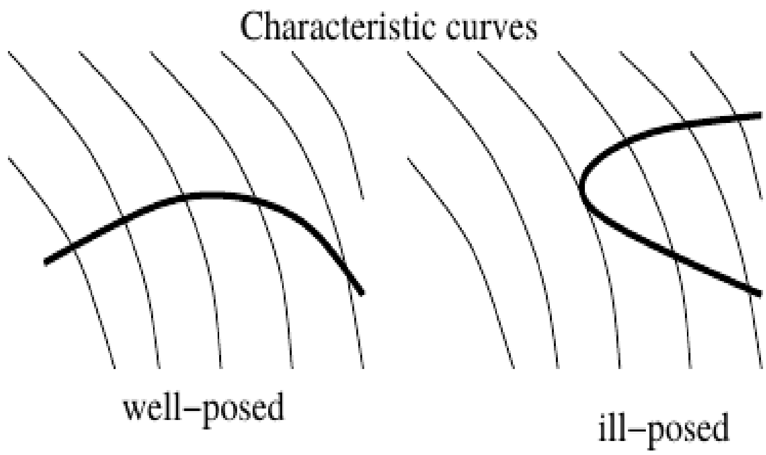
Therefore the general solution is

$$u(x, y) = f(\sinh^{-1}(y) - x)$$

for an arbitrary function f.

5.2 Characteristic curves and boundary conditions

Clearly, if we know the value of $u(x, t)$ at one point on each characteristic curve, we can determine $u(x, t)$ at any point. Therefore, if the boundary conditions specify $u(x, t)$ at exactly one point on every characteristic curve, we can determine a unique particular solution. The problem is then said to be *well-posed.* Conversely, if the boundary conditions do not specify u on some of the characteristic curves, then the solution is not uniquely specified, and if the boundary conditions specify two values of u on some characteristic curve, the problem is inconsistent, and there is no solution. We then say the boundary value problem is *ill-posed.*



Note: Here, we have assumed the solution is constant on the characteristic curves. Later, we shall consider cases where the solution on a characteristic curve is determined by an ordinary differential equation.

Suppose we now have boundary conditions specified on a curve B. We can parameterise B with a parameter s, and set $x = x_0(s)$ and $y = y_0(s)$. We now use the boundary conditions for the partial differential equation as initial conditions for the system of equations (2) determining the characteristic curves, so that

$$\boxed{x(0) = x_0(s), \quad y(0) = y_0(s) \quad \text{and} \quad u(x_0, y_0) = u_0(s).}$$

Solving the initial value problem for the characteristics gives x and y as functions of s and t. We then eliminate t and solve for s in terms of x and y, which allows us to compute u as a function of x and y.

Example:

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0\,, \qquad u(x,0) = 5x^2\,.$$

The characteristic curves are given by

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = -1,$$

so on integrating we get

$$x = t + x_0 \quad \text{and} \quad y = -t + y_0\,.$$

Thus the characteristic curves are $x + y = x_0 + y_0 = \text{const.}$ The boundary conditions can be expressed as

$$x_0 = s, \quad y_0 = 0, \quad u_0 = 5s^2.$$

Hence $x + y = s$ and $u = 5s^2$, so

$$u = 5(x+y)^2$$

is the solution of the problem.

Example: Suppose now that the boundary conditions in the previous problem had been $u = 5x^2$ along $y = x$. Then

$$x_0 = s, \quad y_0 = s, \quad u_0 = 5s^2\,.$$

The characteristic curves are the same, so from the boundary conditions we obtain $x + y = 2s$, so

$$u \;=\; 5s^2 \;=\; \frac{5(x+y)^2}{4}\,.$$

5.3 Quasi-linear equations

A *quasi-linear first order partial differential equation* has the form

$$p(x, y, u)u_x + q(x, y, u)u_y = r(x, y, u)\,.$$

Notice that the coefficients p and q can depend upon u as well as x and y.

These equations can also be tackled using the method of characteristics. However, to simplify matters, we will only consider the case where p and q only depend on x and y, though we still allow r to depend nonlinearly on u:

$$p(x, y)u_x + q(x, y)u_y = r(x, y, u)\,.$$

Let $u(t) = u(x(t), y(t))$, where $(x(t), y(t))$ is a characteristic curve defined, as before, by

$$\frac{dx}{dt} = p(x, y)\,, \qquad \frac{dy}{dt} = q(x, y)\,, \qquad \text{with} \qquad x(0) = x_0 \;\text{ and }\; y(0) = y_0\,.$$

Then

$$\frac{du}{dt} \;=\; \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \;=\; \frac{\partial u}{\partial x}p(x, y) + \frac{\partial u}{\partial y}q(x, y) \;=\; r(x, y, u)\,,$$

giving an ordinary differential equation for u. Applying the boundary conditions $u = u_0(s)$ on the curve $(x, y) = (x_0(s), y_0(s))$ gives the following system of ordinary differential equations and initial conditions:

$$\boxed{\begin{array}{c} \dfrac{dx}{dt} = p(x, y), \quad \dfrac{dy}{dt} = q(x, y), \quad \dfrac{du}{dt} = r(x, y, u); \\ x(0) = x_0(s), \quad y(0) = y_0(s), \quad u(0) = u(x_0, y_0) = u_0(s). \end{array}}$$

This allows us to find x, y and u as functions of s and t:

$$x = x(s, t), \quad y = y(s, t), \quad u = u(s, t).$$

Finally, we can express s and t in terms of x and y to find $u(x, y)$.

Example: Consider the partial differential equation

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x\,.$$

We already know the characteristic curves have the form

$$x = t + x_0, \quad y = -t + y_0\,.$$

The differential equation for u along the characteristics is

$$\frac{du}{dt} \;=\; x \;=\; t + x_0\,,$$

so

$$u \;=\; \frac{1}{2}t^2 + x_0 t + u_0\,.$$

Now suppose we are given the boundary conditions $u = x^2$ for $y = 1$. We can express this parametrically as

$$x_0 = s, \quad y_0 = 1, \quad u_0 = s^2.$$

Then along the characteristic curves we have

$$x \;=\; s + t\,, \qquad y \;=\; 1 - t\,, \qquad u \;=\; \frac{1}{2}t^2 + st + s^2\,.$$

Solving for s and t in terms of x and y gives

$$t \;=\; 1 - y \quad \text{and} \quad s \;=\; x - t \;=\; x + y - 1\,,$$

so finally

$$u \;=\; \frac{1}{2}(y-1)^2 + x(x+y-1)\,.$$

Problem: The function $u(x,y)$ satisfies the first order partial differential equation

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} \;=\; xy$$

in the domain $x > 0$ and $y > 0$, and the boundary condition $u = \exp(x^2)$ along $y = 0$. Show that the family of characteristics of this equation is given by

$$x = s\cosh t\,, \qquad y = s\sinh t\,.$$

Hence determine the function $u(x,t)$.

Solution: We have

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x, \qquad \frac{du}{dt} = xy\,,$$

and the boundary conditions for the partial differential equation give initial conditions for this system of

$$x(0) = x_0 = s, \quad y(0) = y_0 = 0, \qquad u(0) = u(x_0, y_0) = \exp(s^2).$$

Differentiating the equation $dx/dt = y$, we find

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = x$$

and hence

$$\begin{aligned} x &= Ae^t + Be^{-t}\,, \\ y &= Ae^t - Be^{-t}\,, \end{aligned}$$

where A and B are arbitrary constants. Substituting the boundary conditions at $t = 0$ gives

$$\begin{aligned} s &= A + B\,, \\ 0 &= A - B\,. \end{aligned}$$

Hence $A = B = s/2$, so

$$\begin{aligned} x &= s\left(e^t + e^{-t}\right)/2 \;=\; s\cosh t\,, \\ y &= s\left(e^t - e^{-t}\right)/2 \;=\; s\sinh t\,. \end{aligned}$$

Now

$$\frac{du}{dt} = xy = s^2 \sinh t\, \cosh t = \frac{s^2}{2}\sinh 2t\,,$$

so

$$u = \frac{s^2}{4}\cosh 2t + c = \frac{s^2}{4}\left(\cosh^2 t + \sinh^2 t\right) + c\,,$$

where $c = c(s)$ is a constant of integration which may depend on s. When $t = 0$, $u = \exp(s^2)$, so

$$c = \exp(s^2) - \frac{s^2}{4}$$

which means (since $\cosh^2(t) - 1 = \sinh^2(t)$)

$$u = \frac{s^2}{4}\left(\cosh^2 t + \sinh^2 t - 1\right) + \exp(s^2) = \frac{s^2\sinh^2 t}{2} + \exp(s^2)\,.$$

Now, $s\sinh t = y$ and $s^2 = s^2\cosh^2 t - s^2\sinh^2 t = x^2 - y^2$, so the solution is

$$u(x, y) = \frac{y^2}{2} + \exp\left(x^2 - y^2\right).$$

Chapter 6

Second Order Partial Differential Equations

6.1 Important PDEs of mathematical physics

Some second order partial differential equations are of particular importance in science and engineering:

- The *one dimensional wave equation* has the form:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

  where $u = u(x, t)$

- The *two dimensional wave equation* has the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \qquad u = u(x, y, t),$$

  and the *three dimensional wave equation* has the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \qquad u = u(x, y, z, t).$$

- The *one dimensional heat equation* is

$$\boxed{\frac{\partial u}{\partial t} = \kappa \, \frac{\partial^2 u}{\partial x^2} \, .}$$

Note the difference between this and the 1D wave equation. Similarly, the two and three dimensional versions are

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad \text{and} \qquad \frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) ,$$

respectively.

- The *Laplacian* $\nabla^2 u$ of a function u of space variables $x_1, x_2, \ldots x_n$ is defined by

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \cdots + \frac{\partial^2 u}{\partial {x_n}^2} \, ,}$$

e.g. the Laplacian of a function $u = u(x, y, z, t)$, where x, y and z are space variables and t is time, is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \, .$$

In terms of the Laplacian, the wave and heat equations are

$$\frac{\partial^2 u}{\partial t^2} = c^2 \, \nabla^2 u \qquad \text{and} \qquad \frac{\partial u}{\partial t} = \kappa \, \nabla^2 u \, ,$$

respectively.

- *Laplace's equation* is obtained from the heat equation if u is independent of t, and is written

$$\nabla^2 u = 0.$$

In two dimensions, Laplace's equation is therefore

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \, .}$$

This equation is important in fluid mechanics.

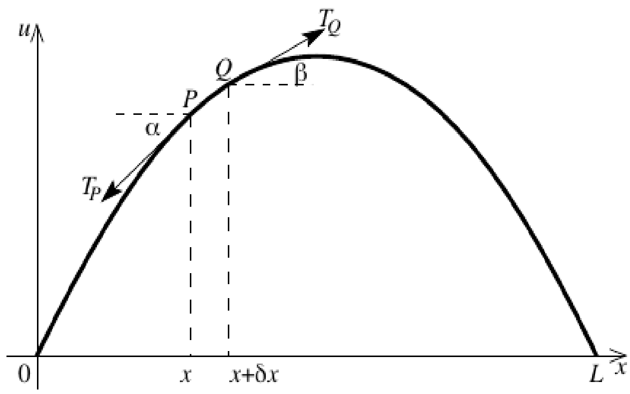
- *Poisson's equation* is
$$\nabla^2 u = f\,,$$
where f is some function of the independent variables. This equation is important in electrostatics.

These equations are important, and we will spend the rest of the course finding and discussing their solutions.

Derivation of the one dimensional wave equation

Imagine we take a string and stretch it between two points. We can derive the equation governing the waves' motions along the string subject to the following assumptions:

1. The mass per unit length ρ is constant. The string is perfectly elastic and offers no resistance to bending.
2. The force of tension T caused by stretching the string before it is fixed at the endpoints is so large that gravitational forces can be neglected.
3. The string performs a small transverse motion in a vertical plane containing the line at equilibrium.



A segment PQ does not move horizontally (in the x direction), so the net horizontal force (tension) on its ends must cancel, i.e.

$$T_P \cos(\alpha) = T_Q \cos(\beta)\,.$$

But since there is nothing special about the choice of segment PQ, the same must be true everywhere, including the fixed endpoints of the string. Thus

$$T_P \cos(\alpha) = T_Q \cos(\beta) = T = \text{const.}$$

Now consider the motion of the segment PQ in the vertical direction. This arises due to an unbalanced (non zero net) force $T_Q \sin(\beta) - T_P \sin(\alpha)$, which by Newton's second law is equal to the mass of the segment $\rho\,\Delta x$ times its acceleration:

$$\begin{aligned} \rho\,\Delta x\,\frac{\partial^2 u}{\partial t^2} &= T_Q \sin(\beta) - T_P \sin(\alpha) \\ &= T\left(\tan(\beta) - \tan(\alpha)\right)\,. \end{aligned}$$

We next note that

$$\tan(\alpha) = \frac{\partial u}{\partial x}(x) \quad \text{and} \quad \tan(\beta) = \frac{\partial u}{\partial x}(x + \Delta x)\,.$$

Therefore

$$\tan(\beta) - \tan(\alpha) = \frac{\partial^2 u}{\partial x^2}\,\Delta x$$

and so

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\,\frac{\partial^2 u}{\partial x^2}\,.$$

Setting $c^2 = T/\rho$, we obtain the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}\,.$$

Note that c has dimensions of velocity and defines the *wave speed*.

6.2 The wave equation

D'Alembert's solution of the wave equation

This elegant method for solving the wave equation starts by introducing new coordinates

$$\xi = x + ct, \quad \eta = x - ct$$

so that

$$x = \frac{\xi + \eta}{2} \quad \text{and} \quad t = \frac{\xi - \eta}{2c}\,.$$

Then, assuming u is a solution, we find that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta},$$

so

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta\,\partial \xi} + \frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\
&= \frac{\partial^2 u}{\partial \xi^2} + 2\,\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{\partial^2 u}{\partial \eta^2}.
\end{aligned}$$

Similarly,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = c\,\frac{\partial u}{\partial \xi} - c\,\frac{\partial u}{\partial \eta},$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial t}\right)\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial t}\right)\frac{\partial \eta}{\partial t} \\
&= c^2\,\frac{\partial^2 u}{\partial \xi^2} - c^2\,\frac{\partial^2 u}{\partial \eta\,\partial \xi} - c^2\,\frac{\partial^2 u}{\partial \xi\,\partial \eta} + c^2\,\frac{\partial^2 u}{\partial \eta^2} \\
&= c^2\,\frac{\partial^2 u}{\partial \xi^2} - 2c^2\,\frac{\partial^2 u}{\partial \xi\,\partial \eta} + c^2\,\frac{\partial^2 u}{\partial \eta^2}.
\end{aligned}$$

Therefore

$$\frac{\partial^2 u}{\partial t^2} - c^2\,\frac{\partial^2 u}{\partial x^2} = -4c^2\,\frac{\partial^2 u}{\partial \xi\,\partial \eta},$$

so

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0$$

from which u can be found in terms of ξ and η via straightforward integration:

$$\frac{\partial u}{\partial \xi} = \int \frac{\partial^2 u}{\partial \xi\,\partial \eta}\,d\eta = f(\xi)$$

for some function f, so

$$\begin{aligned}
u &= \int f(\xi)\,d\xi + G(\eta) \\
&= F(\xi) + G(\eta) \\
&= F(x+ct) + G(x-ct).
\end{aligned}$$

This is known as d'Alembert's solution of the wave equation.

We check this by substituting back into the wave equation itself:

$$\frac{\partial u}{\partial t} = c\left(F'(x+ct) - G'(x-ct)\right), \quad \frac{\partial^2 u}{\partial t^2} = c^2\left(F''(x+ct) + G''(x-ct)\right)$$

and

$$\frac{\partial^2 u}{\partial x^2} = F''(x+ct) + G''(x-ct)\,. \qquad \checkmark$$

Physical interpretation of D'Alembert's solution

We can interpret d'Alembert's solution as the superposition of a right-going and a left-going travelling wave, both having a constant shape and moving with velocity c, so that they cover a distance ct in time t. This is a complete general solution for waves moving in an *infinite* string. For a finite string, the boundary conditions also have to be taken into account.

The functions F and G can be determined from the initial conditions. Let us assume a given initial displacement $u_0(x)$ and initial velocity $v_0(x) = 0$. Then

$$\begin{aligned} F(x) + G(x) &= u_0(x)\,, \\ cF'(x) - cG'(x) &= 0\,, \end{aligned}$$

so

$$F(x) = G(x) = \frac{u_0(x)}{2}\,.$$

Thus

$$u(x,t) = \frac{u_0(x-ct) + u_0(x+ct)}{2}\,.$$

Solution of the wave equation by separation of variables

One of the most powerful techniques for solving a partial differential equation is the *separation of variables*. Given a partial differential equation in two variables (x,t), we can look for a solution of the form

$$u(x,t) = F(x)G(t)$$

for functions F and G of one variable. Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2F(x)}{dx^2}\,G(t) \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = F(x)\,\frac{d^2G(t)}{dt^2}\,,$$

so the wave equation becomes

$$F(x)\,\frac{d^2G(t)}{dt^2} = c^2\,\frac{d^2F(x)}{dx^2}\,G(t)\,.$$

Separating the variables we have

$$\frac{c^2}{F}\,\frac{d^2F}{dx^2} = \frac{1}{G}\,\frac{d^2G}{dt^2}\,.$$

Now note that the left hand side is independent of t, while the right hand side is independent of x. Since they are equal, they must both be independent of both x and t. Hence there is a constant λ such that

$$\frac{c^2}{F}\,\frac{d^2F}{dx^2} = \frac{1}{G}\,\frac{d^2G}{dt^2} = \lambda\,.$$

We therefore have two ordinary differential equations, one for F and one for G:

$$\boxed{\frac{d^2F}{dx^2} - \frac{\lambda}{c^2}F \;=\; 0 \quad \text{and} \quad \frac{d^2G}{dt^2} - \lambda G \;=\; 0\,.}$$

To proceed further we need to look at the initial and boundary conditions. If we are considering a stretched string fixed at both ends, which we take to be $x = 0$ and $x = L$, then the boundary conditions are

$$u(0,t) = u(L,t) = 0,$$

so

$$F(0) = 0 \quad \text{and} \quad F(L) = 0$$

since $u(x,t) = F(x)\,G(t)$.

Now the equation

$$\frac{d^2F}{dx^2} - \frac{\lambda}{c^2}F \;=\; 0$$

has the general solution

$$F(x) \;=\; Ae^{x\sqrt{\lambda/c^2}} + Be^{-x\sqrt{\lambda/c^2}}\,.$$

The boundary conditions give

$$\begin{aligned} F(0) &= A + B \;=\; 0\,, \\ F(L) &= Ae^{L\sqrt{\lambda/c^2}} + Be^{-L\sqrt{\lambda/c^2}} \;=\; 0\,. \end{aligned}$$

Clearly, if A and B are real and $e^{L\sqrt{\lambda/c^2}}$ is real, then $e^{L\sqrt{\lambda/c^2}}$ and $e^{-L\sqrt{\lambda/c^2}}$ are different, so A and B have to be zero, which doesn't correspond to a physically interesting situation. We can remedy this by instead requiring $\sqrt{\lambda/c^2}$ to be imaginary and writing

$$\lambda = -\omega^2 = -k^2c^2 \quad \Rightarrow \quad \sqrt{\lambda/c^2} = ik\,,$$

with k a real constant called the *wave number*. The general solution of the equation for $F(x)$ is then

$$F(x) = C\cos(kx) + D\sin(kx)\,.$$

Appealing again to the boundary conditions, we have

$$\begin{aligned} F(0) &= 0 \quad \Rightarrow \quad C = 0\,, \\ F(L) &= 0 \quad \Rightarrow \quad D\sin(kL) = 0\,. \end{aligned}$$

But $D \neq 0$ otherwise $F = 0$ which would again be uninteresting. Therefore

$$\sin(kL) = 0\,, \qquad \text{i.e.} \qquad kL = n\pi$$

which *is* a physically interesting solution. Thus a nontrivial solution which satisfies the boundary conditions is

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right)\,.$$

Turning now to the equation for G, which with $\lambda = -k^2c^2 = -n^2\pi^2c^2/L^2$ is

$$\frac{d^2G_n}{dt^2} + \left(\frac{n\pi c}{L}\right)^2 G_n = 0\,,$$

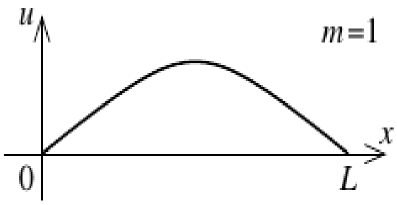
we find

$$G(t) = A\cos\left(\frac{n\pi ct}{L}\right) + B\sin\left(\frac{n\pi ct}{L}\right)\,.$$

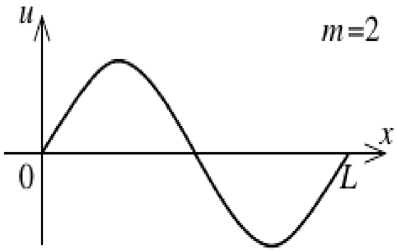
Thus a solution for $u(x,t)$ is

$$u_n = \sin\left(\frac{n\pi x}{L}\right)\left(A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right)\,,$$

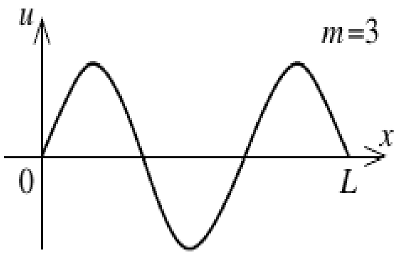
where n is any integer.



$n = 1$ corresponds to the fundamental mode



$n = 2$ corresponds to the first overtone



$n = 3$ corresponds to the second overtone

6.3 Normal modes and superposition

The solution u_n of the wave equation is called the n-th *normal mode*. It is a harmonic motion with wavelength L/n and frequency $cn/(2L)$; note that the frequencies are all multiples of $c/(2L)$, the *fundamental* frequency.

For a given n, a snapshot of u_n at some time t might look as follows: The nth overtone has n *nodes*. A node is a point on the string (other than an end point) which does not move.

Any arbitrary sum of normal modes is a solution of the wave equation, so the general solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right).}$$

This is an example of the *principle of superposition*, which allows us to obtain new solutions of a homogeneous linear equation by adding multiples of known solutions.

To find the coefficients in the general solution, we need the initial conditions. These are usually that the initial position $u(x,0)$ and the initial velocity $\dot{u}(x,0)$ are known. From the general solution, we have that

$$u(x,0) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)$$

and

$$\dot{u}(x,0) = \sum_n \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right).$$

If we suppose that $\dot{u}(x,0) = 0$ (i.e. that the string is initially at rest) then $B_n = 0$ and the general solution becomes

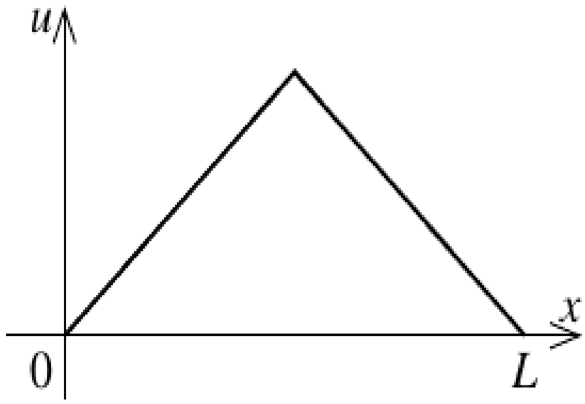
$$u(x,t) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right).$$

To find the solution for any time, we need to find the coefficients A_n which we do by solving the equation

$$u(x,0) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right).$$

This can be done by expressing $u(x,0)$ in terms of its Fourier sine series.

Example: Suppose we have a vibrating string in which the initial deflection $u(x,0)$ is triangular:



$$u(x,0) = u_0(x) = \begin{cases} 2kx/L & \text{if} \quad 0 \leqslant x < L/2\,, \\ 2k(L-x)/L & \text{if} \quad L/2 \leqslant x < L\,. \end{cases}$$

If we suppose that $\dot{u}(x,0)=0$, the solution is

$$u(x,t)=\sum_n A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)\,,$$

where

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\left(\int_0^{L/2}\frac{2kx}{L}\sin\left(\frac{n\pi x}{L}\right)\,dx+\int_{L/2}^{L}\frac{2k(L-x)}{L}\sin\left(\frac{n\pi x}{L}\right)\,dx\right) \\
&= \frac{4k}{L^2}\left(\int_0^{L/2}x\sin\left(\frac{n\pi x}{L}\right)\,dx+\int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right)\,.
\end{aligned}$$

Now integration by parts gives

$$\begin{aligned}
\int_0^{L/2}x\sin\left(\frac{n\pi x}{L}\right)\,dx &= \left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^{L/2}+\int_0^{L/2}\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&= -\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)+\left[\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2} \\
&= -\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\,,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&\quad= \left[-\frac{L(L-x)}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L}-\int_{L/2}^{L}\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&\quad= \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)-\left[\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L} \\
&\quad= \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)-\frac{L^2}{n^2\pi^2}\sin\left(n\pi\right)+\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\,.
\end{aligned}$$

Therefore

$$A_n = \frac{4k}{L^2}\left(\frac{2L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right) = \frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\,.$$

This gives

$$A_n=\begin{cases} 0 & \text{if} \quad n=2m\,, \\ \dfrac{(-1)^m 8k}{(2m+1)^2\pi^2} & \text{if} \quad n=2m+1\,. \end{cases}$$

The solution is therefore

$$u(x,t) \;=\; 8k\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2\pi^2}\sin\left(\frac{(2m+1)\pi x}{L}\right)\cos\left(\frac{(2m+1)\pi ct}{L}\right).$$

6.4 The heat equation

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \kappa\,\frac{\partial^2 u}{\partial x^2}\,,$$

where κ is the *conductivity*, a constant. It describes the temperature $u(x,t)$ in a long thin bar of length L which is insulated along its sides (so that heat can only flow along the bar). Various boundary conditions may be imposed, but we shall only consider the case where the ends of the bar are held constant:

$$u(0,t) = T_0, \qquad u(L,t) = T_1 \quad \text{for all } t.$$

One immediate solution is the *equilibrium solution*

$$u_e(x,t) \;=\; \frac{T_0(L-x)+T_1 x}{L} \;=\; T_0 + \frac{T_1 - T_0}{L}\,x$$

in which the temperature varies uniformly with gradient $(T_1 - T_0)/L$, and heat flows from the hotter end of the bar to the colder.

Now, if u is any solution with the given boundary conditions, then $v = u - u_e$ is also a solution of the heat equation, since

$$\frac{\partial v}{\partial t} \;=\; \frac{\partial u}{\partial t} - \frac{\partial u_e}{\partial t} \;=\; \kappa\,\frac{\partial^2 u}{\partial x^2} \pm \kappa\,\frac{\partial^2 u_e}{\partial x^2} \;=\; \kappa\,\frac{\partial^2 v}{\partial x^2}\,,$$

but has boundary conditions

$$v(0,t) = v(L,t) = 0 \quad \text{for all } t,$$

which is easier to solve for.

To find the general solution for v, we again separate the variables and try

$$v(x,t) = F(x)G(t)\,.$$

Substituting into the heat equation gives

$$F(x)\,\frac{dG(t)}{dt} = \kappa\,\frac{d^2F(x)}{dx^2}\,G(t)\,,$$

and rearranging in a similar way to the solution of the wave equation gives the ordinary differential equations,

$$\frac{1}{\kappa G}\,\frac{dG}{dt} = \lambda = \frac{1}{F}\,\frac{d^2F}{dx^2}$$

where κ is a constant.

Now, if $\lambda > 0$, as before, the only solution satisfying the boundary condition is $F = 0$. Thus we write $\lambda = -k^2$ and find that to satisfy the boundary conditions we must have $k = n\pi/L$. Hence the solution for $F(x)$ has the form

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right)\,.$$

Similarly, the equation for $G(t)$ becomes

$$\frac{dG_n}{dt} = -\frac{\kappa n^2\pi^2}{L^2}\,G_n(t)$$

with solution

$$G_n(t) = \exp\left(\frac{-\kappa n^2\pi^2 t}{L^2}\right)\,.$$

Thus the general solution for v is

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-\kappa n^2\pi^2 t}{L^2}\right)\,.$$

Adding the equilibrium solution, we obtain the general solution for u,

$$\boxed{u(x,t) = \frac{T_0(L-x)+T_1 x}{L} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-\kappa n^2\pi^2 t}{L^2}\right)\,.}$$

The solution of the heat equation has a similar form to that we found for the wave equation, except that the function of t is a negative exponential rather than a trigonometric function. Consequently, each of the terms dies away as time increases, with the higher harmonics decreasing most rapidly. Hence the solution converges to the equilibrium solution as $t \to \infty$.

More generally, we can prove the following result, valid in any number of dimensions:

Convergence of solutions of the heat equation:
Suppose the heat equation in a finite n dimensional domain D has an equilibrium solution $u_e(x_1, \ldots, x_n)$ satisfying the boundary conditions. Then for any initial condition, the solution $u(x_1, \ldots, x_n, t)$ converges uniformly to the equilibrium solution u_e as $t \to \infty$.

Problem: Initially an insulated bar of length L is in thermal equilibrium with one end at $T_1\,°\mathrm{C} > 0$ and the other at $0\,°\mathrm{C}$. The hot end is suddenly immersed in freezing water. Find the temperature distribution at time t.

Solution: Since the system is initially in thermal equilibrium, the initial temperature distribution along the bar has the linear form

$$T(x,0) = T_1\left(1 - \frac{x}{L}\right).$$

Now the boundary conditions are that $T(0,0) = 0$ and $T(L,0) = 0$. Thus, at time t, the temperature is

$$T(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(\frac{-t\kappa n^2\pi^2}{L^2}\right),$$

where the coefficients A_n are given by the Fourier sine series of the initial ($t = 0$) temperature distribution:

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L T(x,0)\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{2T_1}{L}\int_0^L \left(1-\frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2T_1}{L}\left(\left[\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \frac{L}{n\pi}\left[\frac{x}{L}\cos\left(\frac{n\pi x}{L}\right)\right] - \frac{1}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\,dx\right) \\
&= \frac{2T_1}{L}\left(-\frac{L}{n\pi}\cos(n\pi) + \frac{L}{n\pi} + \frac{L}{n\pi}\cos(n\pi) - \frac{L}{(n\pi)^2}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_0^L\right) \\
&= \frac{2T_1}{L}\left(\frac{L}{n\pi}\right) = \frac{2T_1}{n\pi}.
\end{aligned}$$

Hence the solution is

$$T(x,t) = \frac{2T_1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-\kappa n^2\pi^2 t}{L^2}\right).$$

6.5 Laplace's equation

In two dimensions, Laplace's equation is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,.$$

Most commonly, the values of u are specified on the boundary, giving a boundary value problem known as *Dirichlet* problem. Sometimes, the *normal derivative* of u to the boundary may be specified instead. (The normal derivative measures the rate of change of u in the direction orthogonal to the boundary.) This gives a *Neumann* problem.

Laplace's equation in a rectangle

Suppose that the domain of x and y in which we wish to find the solution is a rectangle

$$0 \leqslant x \leqslant a\,, \quad 0 \leqslant y \leqslant b\,,$$

and that the boundary conditions are

$$u(0,y) = u(a,y) = u(x,0) = 0 \quad \text{and} \quad u(x,b) = g(x)\,.$$

Again, we can solve this problem by separation of variables. Set

$$u = F(x)G(y)$$

and substitute into Laplace's equation to get

$$F(x)\,\frac{d^2G(y)}{dy^2} + \frac{d^2F(x)}{dx^2}\,G(y) = 0\,.$$

Rearranging this equation to separate the variables gives

$$\frac{1}{F}\,\frac{d^2F}{dx^2} = -\frac{1}{G}\,\frac{d^2G}{dy^2} = \lambda$$

where λ is a constant.

If $\lambda = k^2$ is positive, we obtain the general solution for F:

$$F(x) = Ae^{kx} + Be^{-kx},$$

but the boundary conditions give $F(0) = F(a) = 0$, so we must have the trivial solution $F(x) = 0$. Therefore, we take λ to be negative, say $\lambda = -n^2\pi^2/a^2$, so

$$\frac{d^2F}{dx^2} = -\frac{n^2\pi^2}{a^2}F\,.$$

This equation has a nontrivial solution

$$F_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

which satisfies the boundary conditions. The corresponding function $G_n(y)$ then satisfies the differential equation

$$\frac{d^2G_n}{dy^2} = \frac{n^2\pi^2}{a^2}G_n\,.$$

Since $u(x,0) = 0$ for every x, we require $G_n(0) = 0$. We therefore deduce that

$$G_n(y) = A_n \sinh\left(n\pi y/a\right),$$

hence the general solution is

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right).}$$

Finally, we consider the boundary condition at $y = b$ which gives

$$\boxed{\sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right) = u_1(x)\,.}$$

We can then deduce the coefficients A_n by computing the Fourier sine series of $u_1(x)$.

Note: A similar analysis allows us to find other solutions which are zero except on one side of the rectangle. For example, if we require $u(x,0) = u_0(x)$ and $u(x,b) = 0$, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right),$$

with

$$\sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right) = u_0(x)\,.$$

Example: Suppose u satisfies Laplace's equation in the rectangle $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$, and the boundary conditions

$$u(0,y) = u(1,y) = 0, \quad u(x,0) = -1 \quad \text{and} \;\; u(x,2) = \sin(3\pi x)\,.$$

We first look for a solution u_A with $u_A(x,0) = 0$ and $u_A(x,2) = \sin(3\pi x)$. From the above,

$$u_A(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(n\pi x\right) \sinh\left(n\pi y\right),$$

with

$$\sum_{n=1}^{\infty} A_n \sinh\left(2n\pi\right) \sin\left(n\pi x\right) \;=\; \sin(3\pi x)\,.$$

Then clearly $A_3 = 1/\sinh(6\pi)$ and $A_n = 0$ for $n \neq 3$, so

$$u_A(x,y) \;=\; \frac{\sin(3\pi x)\sinh(3\pi y)}{\sinh(6\pi)}\,.$$

Next we look for a solution u_B with $u_B(x,0) = -1$ and $u_B(x,2) = 0$. The coefficients B_n are given by

$$\sum_{n=1}^{\infty} B_n \sinh(2n\pi)\sin(n\pi x) \;=\; -1\,.$$

We find

$$B_n \sinh(2n\pi) \;=\; \frac{2(\cos(n\pi) - 1)}{n\pi}$$

which gives $B_n = 0$ if n is even, and $B_n = -4/n\pi$ if n is odd. The solution is therefore

$$u_B(x,y) \;=\; -\frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi x)\sinh((2m+1)\pi(2-y))}{(2m+1)\sinh(2(2m+1)\pi)}\,.$$

Adding the solutions for the two different boundary conditions gives

$$u(x,y) \;=\; \frac{\sin(3\pi x)\sinh(3\pi y)}{\sinh(6\pi)} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi x)\sinh((2m+1)\pi(2-y))}{(2m+1)\sinh(2(2m+1)\pi)}\,.$$

Laplace's equation on a disc

Consider the Dirichlet problem in the disc $x^2 + y^2 \leqslant a^2$, with boundary condition

$$u(a\cos\theta, a\sin\theta) = f(\theta),$$

where, for consistency, f is 2π-periodic.

It is convenient to use *polar coordinates* (r, θ) where

$$x = r\cos\theta, \quad y = r\sin\theta \qquad \Rightarrow \qquad r^2 = x^2 + y^2, \quad \tan\theta = y/x$$

and express u in terms of r and θ.

In polar coordinates, the Laplacian of $u(r, \theta)$ takes the form

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,.}$$

We use separation of variables, and set

$$u(r, \theta) = F(r)\,\Psi(\theta)$$

where $\Psi(\theta)$ is 2π-periodic, so $\Psi(2\pi) = \Psi(0)$. Substituting into Laplace's equation, we find

$$\left(\frac{d^2F(r)}{dr^2} + \frac{1}{r}\frac{dF(r)}{dr}\right)\Psi(\theta) + \frac{F(r)}{r^2}\frac{d^2\Psi(\theta)}{d\theta^2} = 0$$

and hence

$$\frac{r^2}{F}\left(\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr}\right) = -\frac{1}{\Psi}\frac{d^2\Psi}{d\theta^2} = \lambda\,,$$

which give the differential equations

$$\begin{aligned} r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} - \lambda F &= 0\,, \\ \frac{d^2\Psi}{d\theta^2} + \lambda\Psi &= 0\,. \end{aligned}$$

If $\lambda = -k^2 < 0$, we find

$$\Psi(\theta) = Ae^{k\theta} + Be^{-k\theta}$$

which cannot be periodic, so λ must be positive. If $\lambda = 0$, we have the constant solution $\Psi_0(\theta) = A_0$, whereas, if $\lambda = n^2$, we have

$$\Psi_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)\,,$$

which is 2π-periodic.

The equation for F is an Euler equation. If $\lambda = 0$ we have solutions $F(r) = 1$ and $F(r) = \ln r$, but since $F(r)$ must be bounded, we can only take the constant solution, $F_0(r) = 1$. If $\lambda = n^2$, we find solutions

$$F(r) = r^n \quad \text{and} \quad F(r) = r^{-n},$$

but again, to keep $F(r)$ bounded, we cannot have the solution r^{-n}, so we must have $F_n(r) = r^n$. We therefore have the general solution

$$\boxed{u(r,\theta) \;=\; A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)}$$

with boundary conditions

$$\boxed{A_0 + \sum_{n=1}^{\infty} A_n a^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n a^n \sin(n\theta) \;=\; f(\theta)}$$

from which we can compute the coefficients A_n and B_n.

Problem: Verify that the function defined in polar coordinates by

$$u(r,\theta) = r^3 \cos(3\theta)$$

is a solution of Laplace's equation.

Solution:

$$\frac{\partial u}{\partial r} = 3r^2 \cos(3\theta), \quad \frac{\partial^2 u}{\partial r^2} = 6r \cos(3\theta) \quad \text{and} \quad \frac{\partial^2 u}{\partial \theta^2} = -9r^3 \cos(3\theta)\,,$$

so

$$\begin{aligned} \nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\ &= 6r\cos(3\theta) + 3r\cos(3\theta) - 9r\cos(3\theta) \;=\; 0\,. \qquad \checkmark \end{aligned}$$

6.6 Harmonic functions and complex variables

A solution of Laplace's equation is called a *harmonic function*. Harmonic functions have the following properties.

Regularity: Suppose u satisfies Laplace's equation in the interior of a domain D. Then u is smooth in the interior of D, by which we mean all partial derivatives (of all orders) of u exist and are continuous in the interior of D.

Note that this condition holds even if u is *discontinuous* at the boundary!

The maximum principle: Suppose u satisfies Laplace's equation in the interior of a domain D. Then the maximum value of u occurs on the boundary of D. i.e. u has no local maximum in the interior of D.

The maximum principle has an intuitive interpretation. Recall that Laplace's equation describes the temperature inside a uniform body in thermal equilibrium. Clearly, there can be no local maximum of the temperature *inside* the body, or else heat would flow out from that point.

Solution of Laplace's equation using complex variables

In two dimensions there is a connection between the solution to Laplace's equation and *complex functions*, i.e. functions of a complex variable $z \equiv x + iy$.

Recall that the real and imaginary parts of a *differentiable* or *analytic* complex function satisfy the *Cauchy-Riemann* equations,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\,.}$$

Conversely, if all the partial derivatives of $f(z)$ exist and satisfy the Cauchy-Riemann equations, then f is analytic.

Example: Let

$$f(z) = z^2; \quad f(x + iy) = x^2 - y^2 + 2ixy.$$

Thus $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$ and

$$\begin{aligned}\frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y}\,,\\ \frac{\partial v}{\partial x} &= 2y = -\frac{\partial u}{\partial y}\,.\end{aligned}$$

Thus z^2 is an analytic function.

Now suppose $u(x,y)$ is the real part of an analytic function $f(x+iy)$, and $v(x,y)$ is the imaginary part. Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x\,\partial y}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y\,\partial x}\,.$$

Hence

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial^2 v}{\partial y\,\partial x} = 0\,,$$

so u is a solution of Laplace's equation. Similarly, we can also show that v is a solution of Laplace's equation.

$u(x,y)$ and $v(x,y)$ are known as *conjugate harmonic functions*. Given either u or v it is possible to find the other, up to an arbitrary constant. Suppose we know $u(x,y)$. Then

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad\Rightarrow\quad v(x,y) = -\int \frac{\partial u(x,y)}{\partial y}\,dx + f(y)$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad\Rightarrow\quad -\int \frac{\partial^2 u(x,y)}{\partial y^2}\,dx + \frac{df}{dy} = \frac{\partial u}{\partial x}\,,$$

so

$$\frac{df}{dy} = \frac{\partial u}{\partial x} + \int \frac{\partial^2 u(x,y)}{\partial y^2}\,dx\,.$$

Example: Let

$$u(x,y) = x^3 - 3xy^2\,.$$

Then

$$\frac{\partial^2 u(x,y)}{\partial x^2} = 6x \quad\text{and}\quad \frac{\partial^2 u(x,y)}{\partial y^2} = -6x\,,$$

so u satisfies Laplace's equation. Then $v(x, y)$ is given by

$$\begin{aligned} v(x,y) &= -\int \frac{\partial u(x,y)}{\partial y}\, dx + f(y) = \int 6xy\, dx + f(y) \\ &= 3x^2y + f(y)\,. \end{aligned}$$

Now

$$\frac{\partial v}{\partial y} = 3x^2y + \frac{df}{dy} \quad \text{and} \quad \frac{\partial u}{\partial x} = 3x^2 - 3y^2\,,$$

so

$$\frac{df}{dy} = -3y^2 \quad \Rightarrow \quad f(y) = -y^3 + c\,.$$

Therefore

$$v(x,y) = 3x^2y - y^3 + c$$

and the complex function $f(z)$ is

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) = x^3 + 3ix^2y - 3xy^2 - iy^3 + ic = (x+iy)^3 + ic \\ &= z^3 + ic\,. \end{aligned}$$

6.7* The Laplace transform method of solving partial differential equations

Recall that the Laplace transform turns an ordinary differential equation into an algebraic equation. It turns out that applying the Laplace transform to a partial differential equation turns it into an ordinary differential equation. This is because we transform the given equation with respect to *one* of the independent variables (usually t), so that only derivatives with respect to the other variable remain in the transformed equation.

Let us illustrate the method by means of an example:

Consider heat conduction along a semi-infinite bar. Then the heat equation is

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad \text{for} \quad x > 0, \quad t > 0\,.$$

Let us suppose the initial condition $u(x, 0) = A$, a uniform temperature distribution, and boundary conditions

$$u(0,t) = \begin{cases} B & \text{for} \quad 0 < t < t_0\,, \\ 0 & \text{for} \quad t_0 < t\,. \end{cases}$$

This corresponds to the temperature of the end of the bar ($x = 0$) being quenched from B to 0 at $t = t_0$.

To proceed, we employ the Laplace transform to obtain

$$s\mathcal{L}u - u(x,0) \;=\; \kappa \frac{d^2}{dx^2}\mathcal{L}u\,.$$

Writing $\mathcal{L}u$ as $\tilde{u}$, the equation is

$$\frac{d^2\tilde{u}}{dx^2} - \frac{s}{\kappa}\,\tilde{u} \;=\; -\frac{A}{\kappa}\,,$$

which has a particular integral

$$\tilde{u}_p = \frac{A}{s}\,.$$

Thus the general solution is

$$\tilde{u} \;=\; \alpha(s)\exp\left(-\sqrt{s/\kappa}\,x\right) + \beta(s)\exp\left(\sqrt{s/\kappa}\,x\right) + \frac{A}{s}\,,$$

where α and β are functions of s and are determined from the initial and boundary conditions. Clearly, however, we must take $\beta = 0$, otherwise $\tilde{u} \to \infty$ as $x \to \infty$, which would result in a physically unreasonable situation. To get α we recall that

$$u(0,t) = [1 - H(t - t_0)]B\,.$$

Then

$$\tilde{u}(0,s) \;=\; \alpha(s) + \frac{A}{s} \;=\; B\left(\frac{1}{s} - \frac{e^{-st_0}}{s}\right)\,,$$

so

$$\alpha(s) \;=\; \frac{B - A}{s} - \frac{Be^{-st_0}}{s}\,,$$

hence

$$\tilde{u} \;=\; \frac{B\left((1 - \exp(-st_0)\right)}{s}\exp\left(\sqrt{s/\kappa x}\right).$$

The inverse Laplace transform gives us the solution $u(x,t)$. In this case, this is not trivial to find, but it turns out that it can be expressed in terms of the *error function*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-y^2)\,dy\,.$$

6.8* Canonical forms

Aside from being important equations in their own right, the wave equation, heat equation and Laplace's equation are prototypes for all second order linear equations in two variables.

Consider the linear partial differential equation with constant coefficients,

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0\,.$$

A linear change of variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

transforms this equation into the equation

$$Pu_{\xi\xi} + 2Qu_{\xi\eta} + R_{\eta\eta} = 0\,.$$

where P, Q and R are given in terms of A, B and C by

$$\begin{aligned} P &= \alpha^2 A + 2\alpha\beta B + \beta^2 C\,, \\ Q &= \alpha\gamma A + (\alpha\delta + \beta\delta)B + \beta\delta C\,, \\ R &= \gamma^2 A + 2\gamma\delta B + \delta^2 C\,. \end{aligned}$$

By a judicious choice of α, β, γ and δ, we can transform the equation into one of the three forms:

Elliptic: If $AC - B^2 > 0$, we can set $P = R = 1$ and $Q = 0$, giving Laplace's equation, $u_{\xi\xi} + u_{\eta\eta} = 0$.

Hyperbolic: If $AC - B^2 < 0$, we can set $P = 1$, $R = -1$ and $Q = 0$, giving the wave equation $u_{\xi\xi} - u_{\eta\eta} = 0$. Alternatively, we can set $P = R = 0$ and $Q = 1/2$, giving d'Alembert's form of the wave equation, $u_{\xi\eta} = 0$.

Parabolic: If $AC - B^2 = 0$, we can set $P = 1$ and $Q = R = 0$, giving $u_{\xi\xi} = 0$.

For a general linear equation, the coefficients can depend on x and y,

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y) \;=\; 0\,.$$

However, assuming that the sign of $AC - B^2$ does not change, we can still apply a linear transformation at each point to reduce the terms in the second derivatives to canonical form.

In the parabolic case, with constant coefficients, we obtain

$$u_{\xi\xi} + D(\xi,\eta)u_\xi + E(\xi,\eta)u_\eta + F(\xi,\eta)u \;=\; 0\,.$$

Assuming the coefficient E of u_η never vanishes, we can write

$$u_\eta = f(\xi, \eta, u, u_\xi, u_{\xi\xi}),$$

which gives a class of equation including the heat equation.

The canonical form is useful since many of the properties of an equation depend only on whether the equation is elliptic, hyperbolic or parabolic. In particular:

1. The type of boundary condition depends only on the canonical form of the equation.
2. Essentially the same numerical methods can be used for all equations of the same canonical form. Hence we only need to develop computer programs to solve these three types.

Example: The partial differential equation

$$5u_{xx} + 2u_{xy} + u_{yy} + u_x = 0$$

is elliptic, since $5 \times 1 - 1^2 = 34 > 0$.

Example: The partial differential equation

$$u_{xx} + 2u_{xy} + u_{yy} + u_x = 0$$

is parabolic, since $1 \times 1 - 1^2 = 0$.