Integration shortcut for Fourier series.

In computing the Fourier series of polynomials in x integrals of the form

$$\int x^2 \cos(\omega x)\, dx \quad , \quad \int x^2 \sin(\omega x)\, dx \tag{1}$$

arise. One way to evaluate these is to use integration by parts. For quadratic functions, as above, this can be tedious since one has to integrate by parts *twice*. An alternative way to determine (1) is to observe that

$$\frac{d^2}{d\omega^2} \sin(\omega x) = - x^2 \sin(\omega x) \quad , \quad \frac{d^2}{d\omega^2} \cos(\omega x) = - x^2 \cos(\omega x) \ . \tag{2}$$

Hence one can rewrite (1) to give

$$\begin{aligned} \int x^2 \cos(\omega x)\, dx &= - \frac{d^2}{d\omega^2} \int \cos(\omega x)\, dx \\ \int x^2 \sin(\omega x)\, dx &= - \frac{d^2}{d\omega^2} \int \sin(\omega x)\, dx \ . \end{aligned} \tag{3}$$

Then

$$\begin{aligned} \int x^2 \cos(\omega x)\, dx &= - \frac{d^2}{d\omega^2} \left[\frac{1}{\omega} \sin(\omega x) \right] \\ &= - \frac{2}{\omega^3} \sin(\omega x) + \frac{2x}{\omega^2} \cos(\omega x) + \frac{x^2}{\omega} \sin(\omega x) \\ \int x^2 \sin(\omega x)\, dx &= \frac{d^2}{d\omega^2} \left[\frac{1}{\omega} \cos(\omega x) \right] \\ &= \frac{2}{\omega^3} \cos(\omega x) + \frac{2x}{\omega^2} \sin(\omega x) - \frac{x^2}{\omega} \cos(\omega x) \end{aligned} \tag{4}$$

which provides a quicker method for evaluating a definite integral.