FOURIER SERIES HANDOUT

These are the Fourier series formulae you should know:

STANDARD CASE, PERIODIC FUNCTIONS

A function $f(x)$ with period T can be written as a Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right) \right\}$$

To find the a_n and b_n we do the integrals

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T f(x)\,dx \\ a_n &= \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi nx}{T}\right)\,dx \\ b_n &= \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi nx}{T}\right)\,dx\ . \end{aligned}$$

The integration has to run over a compete cycle, but we are free to choose the starting point. So instead of $\int_0^T$ we could do $\int_{-T/2}^{T/2}$, which is often easier. If $f(x)$ is even, then all $b_n = 0$. If $f(x)$ is odd, all $a_n = 0$.

HALF-RANGE EXTENSIONS

In problems like these we are given a function in the range $0 < x < L$, which is *half* the function's period (so the period $T = 2L$). We can then choose the function's extension in the region $-L < x < 0$ *either* to make the function even *or* odd.

Even extension

If we choose the even extension the function is represented as a pure cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right)$$

(remember that $T = 2L$). The a_n are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi nx}{T}\right)\,dx \end{aligned}$$

[PTO]

Odd extension

If we choose the odd extension the function is represented as a pure sine series

$$f(x) \;=\; \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{T}\right)$$

To find the b_n we must do the integrals

$$b_n \;=\; \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{2\pi n x}{T}\right)\,dx$$

(In both formulae, remember that $T = 2L$.)

For a half range expansion you will either want the even or the odd series (it is a common mistake to do both and add the results).

In all cases (both for the full series and for the half-range series) the factor in front of the integral is

$$\frac{2}{\text{integration interval}}$$

It is often useful to know how the trigonometrical functions behave when the angle is an integer multiple of π:

$$\begin{aligned} \sin n\pi &= 0 \\ \cos n\pi &= (-1)^n \end{aligned}$$

if n is an integer.