String Phenomenology:
Compactification on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

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Abstract

In this report, I shall give a brief introduction and motivation for string theory as well some of the important details regarding superstring theories and supersymmetry. I will then discuss compactification and the properties of compact spaces before discussing holonomy groups and their role in reducing the number of supersymmetries in a theory. Furthermore I will describe general orbifold construction before introducing the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, where I define the untwisted and twisted sectors, and then introduce the concept of Wilson lines. I shall then continue by formulating the free fermionic model, and discussing its correspondence to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, before finally reintroducing Wilson lines to produce a three generation, $SO(10)$ embedded, $\mathcal{N} = 1$ model.

Contents

1 Introduction

The nineteenth and twentieth centuries gave rise to two of the most fundamental areas of physics; Quantum Mechanics and Relativity [1]. The union of which, would result in the underlying Theory of Everything; a single, simplistic fundamental law of nature, governing both the microscopic and macroscopic domains respectively.

The world of Quantum Mechanics sought to describe the microscopic universe, the physics of the fundamental building blocks of nature. Relativity; Einstein's theory of gravity, described the macroscopic world, and helped explain many shortcomings of Newton's gravity with very high precision.

One can consider the four fundamental forces that govern our universe; the electromagnetic, the weak nuclear, the strong nuclear and gravity, all of which describe the interactions of elementary particles. While trying to construct this Theory of Everything, one tries to place these four interactions into a single framework, and herein lies the problem.

Gravity appears to be completely incompatible with the other three forces. To understand why this is the case, we must first discuss the models describing the electromagnetic, weak and strong, in order to fully comprehend why gravity can not be incorporated into these pre-existing models.

1.1 The Standard Model and G.U.Ts

The Standard Model is one such model, describing the three forces excluding gravity as well as classifying all known elementary particles.

Within the Standard Model, elementary particles are categorised into fermions (with half integer spin), bosons (with spin 1) and the Higgs Boson (with spin 0) [1].

We will first consider the fermions, they are further categorised into quarks and leptons:

Quarks
There are six "flavours" of quarks, namely; up, down, top, bottom, charm, strange.
They are also distinguished by charge and the three "colours"; red, green and blue.

Leptons
There are six leptons, namely; the electron, the muon, the tau, followed by their respective neutrino (electron neutrino, muon neutrino and tau neutrino).

We will now move on the spin 1 bosons. These are also known as exchange particles or force-mediating particles. When a force-mediating particle is exchanged between two other particles, the observed effect of this exchange is analogous to to force acting between them. They are named as follows [2]:

Photon - is responsible for the electromagnetic force.
Gluons - are responsible for the strong nuclear force.
W **and** Z **Bosons -** are responsible for the weak nuclear force.

Finally we reach the other category of boson:

Higgs Boson - is responsible for assigning mass to the other particles and has spin 0.

The interactions of these force-mediating particles can be described by gauge groups, namely; $SU(3)$ corresponding to the strong nuclear force, $SU(2)$ corresponding to the weak nuclear force and $U(1)$ corresponding to the electromagnetic force [3].
We can combine these three Lie groups into a single gauge group, called

$$SU(3) \times SU(2) \times U(1) \tag{1}$$

This new single gauge group leads nicely into the formulation of new models known as Grand Unified Theories (G.U.Ts), which aim to describe all interactions under a single framework.

The Standard Model accurately describes experimental observation for energies less than 100GeV, so any G.U.T at these low energies must match the Standard Model. However it must also overcome a shortcoming of the Standard Model; neutrino masses.

A huge problem of the Standard Model is that it was formulated at a time when neutrinos were believed to be massless, however this has since been disproven. Therefore in order to match experimental observation, these new G.U.Ts must include a right-handed neutrino, in order to give neutrinos their known mass.

The first such G.U.T we could consider is the $SU(5)$ group. While the gauge group (1) does fit nicely into $SU(5)$, it still does not include the right-handed neutrino, and thus does not account for neutrino mass. Therefore we require a bigger G.U.T.

This is where $SO(10)$ comes into the picture. This group incorporates all the interactions of the Standard Model plus the right-handed neutrino and so appears to be very promising [4]. Despite the illusive graviton (gravity boson) still being missing, establishing this framework was crucial, as our desired Theory of Everything must still accurately describe the other three interactions as well as gravity. Therefore it is important to check that $SO(10)$ can be nicely incorporated into this. Thus far, the best candidate for such a model is String Theory.

1.2 String Theory Motivation

A key source for this section are chapters 1 and 2 in the introduction to string theory written by U. Danielsson [5].

To understand why String Theory is such a good candidate for unifying gravity with the other forces, we must first investigate why it is necessary. What problems arise when we try to include the graviton into the Standard Model?

To answer this question, we need to know a little bit about how Quantum Field Theories work. In QFT, the strength of a force is described by a coupling constant, which varies with energy or distance. This is due to what's known as vacuum fluctuations. The observed charge in a laboratory can be thought of as the combination of these vacuum fluctuations and what is known as the bare charge of the particle. One assumes that this bare charge is fixed and so one can endeavour to calculate how this observed charge depends on the energy, by ascertaining the effect that the energy has on these vacuum fluctuations. However when this calculation is done, it is discovered that the contribution of these fluctuations is infinite. This sounds like a huge problem, but it is one that is soon rectified by a process called renormalisation.

In Perturbation Theory, the amplitude of a given process is proportional to the coupling constants raised to some power. However in four-dimensional gauge theory, these coupling constants have no dimension, and thus the number of divergent amplitudes is finite, thus making it a renormalisable theory. In contrast to this though, the coupling constant of gravity has dimension

$$G_N \sim 1/(m_p)^2 \tag{2}$$

and thus the amplitudes become increasingly divergent.

These infinities in theory could be adjusted but it turns out one would have to make infinite adjustments in order to renormalise gravity, thus making it a non-renormalisable theory. Therefore it would appear that gravity and the G.U.Ts cannot be combined into one theory, but this is where String Theory seems to hold the answer.

The first notion of strings came in the 1960s, on the topic of Regge trajectories; patterns in the masses of hadrons. The Bootstrap hypothesis suggested that this pattern could be explain if the hadrons were made out of strings. From this hypothesis, it seemed that the evidence of this string physics would come from scattering amplitudes, which Veneziano later found. He discovered a scattering amplitude that was able to reproduce all the requirements of the physics of hadrons [6], and it was later realised that his equations actually described strings [7].

Originally, however, it seemed there was a problem with this string theory, as it predicted the existence of massless spin-2 particles that were not present in the original picture of the hadrons. But actually this turned out to be the illusive graviton.

Now that we know that string theory can describe all other particle data, I will now discuss how it solves the problem of the renormalisation of gravity.

We have already established that quantum gravity at high energies problematically leads to infinite divergent integrals. String Theory however very subtly limits the number of integrals by placing a limit on the energy without violating General Relativity. This can be visually understood by the below diagram;

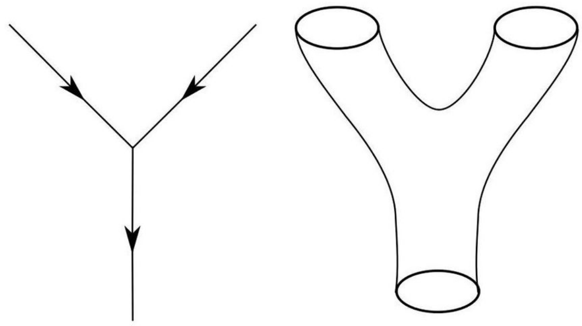

Figure 1: Graviton Interaction: Particle vs String

By limiting the energy, we lose some precision in determining the position of the particles and when we consider the particle picture (left), it is difficult to include this uncertainty in a consistent way. But with the string diagram (right), it is no longer obvious where the interaction actually occurs, so this uncertainty introduced by capping-off high energies is actually built into the model naturally.

Research over the last few decades has shown that string theory does consistently describe quantum gravity, and so far is the only theory we have to do so.

1.3 Bosonic String Theory

Chapter 2.1 in [5] provided many key insights in this section.

The first and simplest string theory was that of the Bosonic String. In order to formulate these strings, we need to derive the action integral.

For a particle, the action is given by

$$S = m \int ds \tag{3}$$

with m being the mass of the particle.

To generalise this to a string, instead of minimising the length of the world line, one minimises the area of the world sheet. This leads to the Nambu-Goto action:

$$S_{NG} = -\frac{T_0}{c} \int d^2\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \tag{4}$$

with T_0 being the tension in the string.

However, the square root in this integral makes using it quite complicated. Thankfully, there is a way of reformulating the action without a square root, this is known as the Polyakov action:

$$S_P = -\frac{T}{2} \int d^2\sigma (-\dot{X}^2 + X'^2) \tag{5}$$

which can be handled using normal field theory and is often given as the definition for bosonic String Theory.

In this report, I will not give any rigorous derivation of the dimensionality of the bosonic string, but will simply state it as $D = 26$. This result caused some concern as to the reliability of string theory to be an accurate physical theory since the number of spacetime dimensions far exceeds that of the $D = 4$ universe as observed. Another issue with the bosonic string theory is that it predicts the existence of tachyons; particles with imaginary mass that travel at superluminal speeds. Furthermore, as the name might suggest, the bosonic string theory does not describe the fermions. This leads us to Superstring Theories.

1.4 Superstring Theories

An important reference for this section was chapter 3 of [5].

In order for String Theory to also include the fermions, we introduce a powerful symmetry called supersymmetry.

Supersymmetry partners each boson to a corresponding fermion, and requires that the fundamental equations should be invariant under the exchange of bosons and fermions.

The addition of supersymmetry into the picture solves almost all the obvious problems of the bosonic string. Obviously it now incorporates the fermions and it additionally gets rid of the tachyons. While it still doesn't bring the dimensionality down to four, it does reduce it from 26 to 10, which is much more manageable.

The way in which the fermions can be integrated however is not unique, and leads to five different superstring theories. Below we shall consider all possible types of superstring theory.

> **Type I**, includes both open and closed strings and contains the symmetry group $SO(32)$.
>
> **Type IIa**, involves closed strings with symmetrical vibrational patterns, meaning it does not matter whether the string oscillation is travelling to the left (left-mover) or the right (right mover).
>
> **Type IIb**, also involves just closed strings but this time with asymmetrical vibrational patterns, so the boundary conditions do depend on whether the string is a left or right mover.

We are however not limited to just these three, as these are formed of just superstrings. We are actually allowed a combination of both $D = 10$ superstrings and $D = 26$ bosonic strings. Since superstring theories are 10-dimensional theories, the extra 16 bosonic coordinates are compactified on a torus, we shall see more of this in a moment. The left-moving vibrations mirror the bosonic strings, while the right-moving vibrations resemble the superstrings. This leads us on to our final two types; the heterotic theories.

Heterotic Theories, still only involve closed strings and appear to be the most promising theories as they cancel out all the anomalies of the previous theories. It turns out that only two symmetry groups are able to be applied to the 16-torus, namely; $SO(32)$ and $E_8 \times E_8$.

This leads to the two additional superstring theories; the $SO(32)$ Heterotic and the $E_8 \times E_8$ Heterotic.

The general action of this theory is given by

$$S = -\frac{T}{2}\int d^2\sigma \left(\sum_{\mu}^{9} (\partial_\alpha X_\mu \partial^\alpha X^\mu - 2\psi_+^\mu \partial_- \psi_{+\mu}) - 2\sum_{a=1}^{n} \lambda_-^a \partial_+ \lambda_-^a \right) \tag{6}$$

This is the bosonic action on the worldsheet incorporating the real Majorana fermions (fermions that are their own anti-particle) denoted by $\lambda_\pm^a$, where $\pm$ indicate right/left moving, and the spacetime fermions ψ^μ. Note that the spacetime fermions are only right moving while the left movers are the bosonic spacetime fields X_L^μ and the internal Majorana fermions λ_-^a.

We can now make use of the interchangeability of two Majorana fermions and a real boson in two dimensions. By this exchange, we can conclude that the theory contains $D + \frac{n}{2}$ bosons. Consistency requires that $D + \frac{n}{2} = 26$ since the bosonic string lives in 26 dimensions. By this condition, we obtain that $n = 32$. This defines our internal $SO(n)$ symmetry as $SO(32)$.

We are now ready to explicitly describe the difference between the $E_8 \times E_8$ theory and the $SO(32)$ theory. The different gauge symmetries come as a result of different boundary conditions on the Majorana fermions λ_-^a. For the $SO(32)$ theory, the set of internal fermions are all given the same boundary conditions, whereas in $E_8 \times E_8$ theory, the fermions are split into two groups and each group is assigned a different boundary condition. This difference in the boundary conditions decides which orbifolds can be constructed under each theory, the details of which will be discussed in section 3.

As was mentioned at the beginning of this section, these superstring theories are consistent for a 10-dimensional spacetime. Thus, in order to formulate a theory describing our 4-dimensional universe, we must apply a scheme called compactification. This was briefly mentioned in the construction of heterotic models and a more detailed explanation and formalism will be given.

Before we can do this however, we must first discuss the amount of supersymmetry relating to each theory. Within the theories we discussed previously, the supersymmetry varies. The Type I and both heterotic theories come with a single supersymmetry, classified as $\mathcal{N} = 1$ in 10-dimensions. The two Type II theories however, come with two supersymmetries; labelled as $\mathcal{N} = 2$.

To further complicate matters, a single supersymmetry in 10-dimensions actually leads to four supersymmetries in 4-dimensions. Therefore, Type I and the heterotic theories have $\mathcal{N} = 4$ in our $d = 4$ universe, and the Type II theories come equipped with $\mathcal{N} = 8$.

We will return to these supersymmetries in the context of compactification shortly, after I have provided the formalism for this crucial scheme.

2 Toroidal Compactification

The Introduction to String Compactification [8], specifically chapter 2.2, by Font and Theisen was a very helpful source for sections 2.1 and 2.2.

It is possible to assume that String Theory is just wrong, since it is only consistent with a 10-dimensional universe and not our 4-dimensional one. However this is not entirely true. When we describe our universe as 4-dimensional, what we should actually say is that it is 4-dimensional down to scales of about 10^{-18} metres. Meaning it is entirely possible for the extra six dimensions to just be *compactified* down to an incredibly small space. We can visualise this with a simple thought-experiment. Imagine you are looking at a telephone wire that is a reasonable distance away from you. For all intents and purposes, the wire appears one dimensional. Now imagine you are an ant walking along this wire. Suddenly, it looks like a three dimensional object; you can tranverse it in three dimensions. This is an example of when dimensions are not always visible due to their size, and this is what string theorists believe is happening with the extra six dimensions of spacetime. Rather that it being some arbitrary 10-dimensional space, it actually takes the form $\mathcal{M}_4 \times N^6$, where $\mathcal{M}_4$ is Minkowski spacetime and N^6 is some incredibly small, six dimensional manifold that has been curled up, the properties of which, we shall now discuss.

2.1 Properties of Compactification

We shall begin with a spacetime and internal manifold of arbitrary dimension to show the power of the result we are to prove in this section. Despite the general dimensionality, our overall aim is to describe this internal space such that it agrees with the results of superstring theories, hence we shall impose that some supersymmetry must be preserved. This condition actually hugely restricts the number of possibilities that these extra dimensions can live in.

In a D dimensional spacetime, with d-dimensions of Minkowski space, we are left with $D-d$ dimensions to compactify. Therefore, as above, we can impose that this spacetime has the product form

$$\mathcal{M}_D = \mathcal{M}_d \times N^{D-d} \tag{7}$$

where this N^{D-d} will be the subject of our analysis.

If (7) holds, with N^{D-d} compact, when a metric field G_{MN} is present, then we say the system undergoes spontaneous compactification. Here G_{MN} is the metric tensor from relativity.

We can now define the vacuum expectation value (VEV) as the expected value of a field equation in a vacuum. In the case of G_{MN}, we obtain its (VEV) $\langle G_{MN} \rangle$ as

$$\langle G_{MN}(x,y) \rangle = \begin{pmatrix} \bar{g}_{\mu\nu}(x) & 0 \\ 0 & \bar{g}_{mn}(y) \end{pmatrix} \tag{8}$$

where x_μ and y_m are coordinates of $\mathcal{M}_d$ and N^{D-d}.

We could check to see if the equations of motion have solutions of the form (8) but since they are non-linear we shall instead assume (8) and require that the symmetries in $\mathcal{M}_d$ are unbroken.

When we do this, it can be shown that, since fermionic fields are spinors that transform non-trivially under $SO(1, d-1)$;

$$\langle \Phi_{Fermi} \rangle = 0, \tag{9}$$

with Φ_{Fermi} describing a fermionic field.

In other words, the VEV of a fermionic field is zero. This is because if $\langle \Phi_{Fermi} \rangle \neq 0$, then Lorentz invariance would be spontaneously broken.

By supersymmetry, $\langle \delta_\epsilon \Phi_{Bose} \rangle \sim \langle \Phi_{Fermi} \rangle = 0$, meaning the VEV of the change in the bosonic field by ϵ is equivalent to the VEV of the fermionic field and thus is also zero. Note here, that ϵ is the spinor of $SO(1, d-1)$ parametrising the supersymmetry transformation.

Included in the Φ_{Fermi} in supergravity is the gravitino ψ_M, obviously also with a VEV of zero. This gravitino transforms as

$$\delta_\epsilon \psi_M = \nabla_M \epsilon + \dots \tag{10}$$

where ∇_M is the covariant derivative. All other terms in this expansion have been ommited since their VEV is zero.

Since $\langle \delta_\epsilon \psi_M \rangle = 0$, when we take the VEV of this expansion we conclude that

$$\langle \nabla_M \epsilon \rangle \equiv \bar{\nabla}_M \epsilon = 0 \qquad \Rightarrow \qquad \bar{\nabla}_m \epsilon = 0, \bar{\nabla}_\mu \epsilon = 0 \tag{11}$$

to put the covariant derivatives in the context of x, y as introduced in (8). Spinor fields ϵ that satisfy (11) are called Killing spinors. These Killing spinors significantly reduce that number of possible manifolds that can be used in compactification, since the existence of Killing spinors on a Riemannian manifold causes the Ricci tensor to vanish

$$\bar{R}_{\mu\nu} = \bar{R}_{mn} = 0 \tag{12}$$

therefore limiting this compact internal manifold N^{D-d} to be Ricci-flat.

Supersymmetry imposes that the internal manifold be Ricci-flat; this is an extremely powerful result that also leads to N^{D-d} being toroidal, since Ricci-flat compact manifolds do not admit Killing vectors other than those associated with tori. In the context of superstring theory, if the internal space $N^6 = T^6$ (a six-dimensional, compact, flat torus) then this would lead to $\mathcal{N} = 4$ supersymmetry in 4-dimensions. However it is believed that in our final string theory, the supersymmetries will reduce down to $\mathcal{N} = 1$. To obtain our desired supersymmetry, we must make some restrictions to our torus in the form of a holonomy group.

2.2 Holonomy Groups

Assume the manifold describing our internal space is orientable. A holonomy group, $\mathcal{H}$, describes a set of rotation matrices and forms a subgroup of $SO(m)$, $\mathcal{H} \subseteq SO(m)$, where $SO(m)$ is the tangent group of an m-dimensional manifold. The matrices $U_i \in \mathcal{H}$ describe the rotation of vectors upon parallel transport of a closed curve on the manifold.

The holonomy of a manifold can be described as trivial if the manifold is simply connected and has no curvature. When it is not simply connected however, the holonomy of a flat manifold could be trivial or non-trivial. In the example at the end of the previous section regarding T^6, this has trivial holonomy even though it is not simply connected.

I shall now discuss how the holonomy group of a manifold leads to the amount of supersymmetry. To do this, we shall return to our spinor, ϵ, that parametrises the supersymmetry transformation. By definition, spinors should rotate by elements of $\mathcal{H}$ too but since ϵ is convariantly constant, it is left unchanged. Therefore we must split $\mathcal{H}$ into spinors that do change and those that don't.

In the case of superstring theories where $N^{D-d} = N^6$, ϵ is an $SO(6)$ spinor that has left and right chirality pieces that transform as $\mathbf{4}$ and $\bar{\mathbf{4}}$ in $SO(6) \simeq SU(4)$, where $\mathbf{4}$ and $\bar{\mathbf{4}}$ are the Weyl spinors.

Under some $\mathcal{H}$, the $\mathbf{4}$ splits into some amount of triplets or doublets and some singlets. Its these singlets that lead to the supersymmetry number since they represent ϵ, the supersymmetry parameter. To make this definition more concrete, we shall examine the cases when $\mathcal{H} = SU(2)$ and $\mathcal{H} = SU(3)$.

For $\mathcal{H} = SU(2)$, the decomposition is

$$\mathbf{4}_{SU(4)} = (\mathbf{2} + \mathbf{2})_{SU(2)} \tag{13}$$

i.e. into a doublet and two singlets. The number of singlets correlates to the number of supersymmetries in $d = 4$ assuming that $\mathcal{N} = 1$ in $D = 10$. Therefore this example corresponds to $\mathcal{N} = 2$ in $d = 4$. It follows that if $\mathcal{N} = 2$ in $D = 10$ then the $\mathcal{N}$ value would double in $d = 4$ as well.

For $\mathcal{H} = SU(3)$, the $\mathbf{4}$ decomposes as follows

$$\mathbf{4}_{SU(4)} = (\mathbf{3} + \mathbf{1})_{SU(3)} \tag{14}$$

i.e. into a triplet and one singlet. Hence there is one covariantly constant spinor of each chirality, namely; $\epsilon_{\pm}$. Thus $\mathcal{H} = SU(3)$ leads to $\mathcal{N} = 1$.

In constructing these examples, we have actually illustrated the result we were hoping for; a holonomy group that leads to $\mathcal{N} = 1$. This result is why string theorists deem the heterotic string theories as more phenomenologically promising than the Type II theories since they have $\mathcal{N} = 1$ in $D = 10$, thus allowing us to reach $\mathcal{N} = 1$ in $d = 4$, unlike the Type II that have $\mathcal{N} = 2$ in $D = 10$.

These manifolds with $SU(3) \subset SO(6)$ or generally speaking, have $SU(n) \subset SO(2n)$ are called Calabi-Yau manifolds CY_n. These manifolds seem a promising area of research since they lead directly to the result we require. However they are not as ideal as they first seem. The reason for this is that there is not just one unique manifold satisfying these conditions, but thousands, and no metric on any CY_3 is known explicitly. This makes actually using them to extract any information near impossible. The interested reader may want to consult chapter 3 in [8] for more details on the Calabi-Yau. One way that string theory research can progress is to consider a class of toroidal orbifolds that are analogous to CY_3, in that their holonomy group is contained within $SU(3)$, but that give an exactly solvable theory, meaning we are actually able to extract information from them. This leads us on to the main focus of this report; the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold.

3 Construction of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

I would like to state that from this point onwards, I will only be referring to the $E_8 \times E_8$ heterotic string theory. This is due to the amount of supersymmetry the heterotic models present, making them phenomenologically appealing, but also so we can introduce and utilise the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold that is not compatible with the $SO(32)$ theory. The significance of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold will become clear in section 4.

3.1 General Orbifold Construction

Chapter 4 in [8] proved very valuable in this section.

I would first like to introduce the general notion of an orbifold before moving on to the $\mathbb{Z}_2 \times \mathbb{Z}_2$.

An orbifold $\mathcal{O}$ can be thought of as the quotient of a manifold $\mathcal{M}$ by a group action G that preserves $\mathcal{M}$. This $\mathcal{O}$ can be denoted as

$$\mathcal{O} = \mathcal{M}/G \tag{15}$$

By definition, every point x on $\mathcal{M}$ is associated with its orbit under G such that $x \sim gx$. This identification feature makes orbifolds a very viable option for N^6 as this curled up nature is a requirement of the internal space. Topologically, we can also define a torus using orbifold construction as

$$T^D = \mathbb{R}^D/\Lambda \tag{16}$$

where we have compactified $\mathbb{R}^D$ on a root lattice Λ to generate T^D. Obviously in the context of our heterotic string theory we can set $D = 6$.

However, as we previously saw, just compactifying on to a flat torus T^6 is not enough as it leaves us with $\mathcal{N} = 4$ supersymmetry as opposed to the desired $\mathcal{N} = 1$. Therefore we must generate a new orbifold, Ω, with this crucial property.

The first thing I should point out is, since we are working in $E_8 \times E_8$ heterotic theory, we must impose the $E_8 \times E_8$ symmetry on our Ω. This is done by compactifying the 16 bosonic left moving coordinates on to the root lattice of $E_8 \times E_8$; we can denote this as $T_{(E_8 \times E_8)}$, a 16-torus with $E_8 \times E_8$ symmetry. This was very briefly mentioned in the description of heterotic theory in section 1.4.

We now define the point group of an orbifold. Since points on a torus are identified to each other under rotation by some θ, we define the set of all possible θ^i as the point group, P. For example, if our $P = \mathbb{Z}_3$, then $P = \{\mathbb{I}, \theta, \theta^2\}$, with θ a rotation by $2\pi/3$. It is our group P that allows us to restrict the orbifold onto which we compactify while still maintaining a phenomenologically consistent model. This is because our point group $P \equiv \mathcal{H}$; our holonomy group.

The final piece we need before we can give our Ω in explicit product form is the gauge twisting group, G. This G is an automorphism of the $E_8 \times E_8$ Lie group and its action is needed to satisfy modular invariance; meaning that a rescaling or rotation of the torus should not change anything. We can now define our new and phenomenologically improved orbifold as

$$\Omega = T^6/P \times T_{(E_8 \times E_8)}/G \tag{17}$$

As we mentioned at the end of section 2, for this orbifold model to result in $N = 1$ in $d = 4$, it was required that $\mathcal{H} \subset SU(3)$, which as we have just stated, corresponds to $P \subset SU(3)$ in the context of orbifold compactification. We also restrict our choices such that P is abelian. This leaves us with only two options;

$$P \equiv \mathbb{Z}_N \tag{18}$$

$$P \equiv \mathbb{Z}_N \times \mathbb{Z}_M \tag{19}$$

these groups are referred to a twist of the orbifold.

3.2 Satisfying the Boundary Conditions

Chapter 4 of E. Manno's PhD thesis [9] was a key reference for the next three sections up to section 4.

As well as the point group, P, we also define the space group, S.

S defines both rotations and translations of the torus that lead to an identification of $\vec{x} \in \Omega$, unlike P which only describes rotations. S can be given by the set of elements

$$S = \{(\theta, \vec{l}) \mid \theta \in P, \;\; \vec{l} \in 2\pi\Lambda\} \tag{20}$$

where $\vec{l} = \vec{e_a} \cdot \vec{n_a}$ with $\vec{e_a}$ being a basis vector of the root lattice Λ and $\vec{n_a}$ being a basis vector of the $E_8 \times E_8$ gauge lattice.

We can therefore rewrite (17) using S as;

$$\Omega = \mathbb{R}^6/S \times T_{(E_8 \times E_8)}/G \tag{21}$$

Using this space group S, we can state the identification of $\vec{x} \in \Omega$ as:

$$\vec{x} \sim \theta\vec{x} + \vec{l} \tag{22}$$

Since the boundary conditions of the heterotic string action can be satisfied in different ways on an orbifold, we are left with the conclusion that there are different types of strings;

- The **untwisted** string is closed on the torus before the point symmetry group P is embedded.

- The **twisted** string is closed on Ω after P is imposed.

This can be realised by considering the identification of $\vec{x}$ using (22) in the context of the expression

$$X^{3,\ldots,8}(\tau,\sigma) = \theta^k X^{3,\ldots,8}(\tau,0) + n_a e_a \tag{23}$$

The untwisted sector, when $k = 0$, is clearly just the standard toroidal compactification. When $k > 0$, we are dealing with the twisted sector. This generates all the new string states under the twist action which are localised at the points unaltered by $(\theta, \vec{l}) \equiv (\theta^k, n_a e_a)$ of the space group S.

Since our orbifold in (21) is defined as taking a quotient by $S \otimes G$, we can think of our untwisted states as the conjugacy class containing the identity and the twisted states being all other conjugacy classes in the quotient group. For a string state to be physical, it must be invariant under the projection $S \otimes G$.

3.2.1 Untwisted States

As previously mentioned, the untwisted states correspond to simple toroidal compactifications, before any symmetry group action is imposed, which survive the $S \otimes G$ projections. In standard toroidal compactification to $d = 4$, the mass formula for the right and left movers respectively is:

$$\frac{1}{4}m_R^2 = N_R + \frac{1}{2}p_R^i p_R^i - a_R \tag{24}$$

$$\frac{1}{4}m_L^2 = \tilde{N}_L + \frac{1}{2}p_L^I p_L^I - a_L \tag{25}$$

where N_R is the right number operator; counting the bosonic and fermionic oscillators. $\tilde{N}_L$ is the left number operator accounting for the spatial operators $\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n$ and the left gauge contibutions $\tilde{\alpha}^I_{-n}\tilde{\alpha}^I_n$. Finally $a_{R,L}$ are the normal orderings for the Virasoro operators $\tilde{L}_0$ and L_0 in the momentum operator as $\hat{P} = L_0 - \tilde{L}_0$; corresponding to a point shift on the string. I also deem it important to state explicitly that in formulas (24) and (25) the i denotes the dimension of the right movers while the I denotes the dimension of the left movers.

I would now like to introduce weight vector notation for the right moving string states. This corresponds to the observation of the

$$b^i_{-\frac{1}{2}}|0\rangle_R \otimes \tilde{\alpha}^j_{-1}|0\rangle_L \tag{26}$$

state in the massless spectrum where $i = 1, 2$ and j takes any value from the compact space. This provides the spacetime vectors and is denoted by $q = (1, 0, 0, 0)$, where q includes all permuations of this vector.

We can now rewrite (24) and (25) by imposing the level matching condition, i.e. that the number of right moving states is equal to the number of left moving states. This new formula takes the form

$$\frac{1}{2}q^2 - \frac{1}{2} = \frac{1}{4}m_R^2 = \frac{1}{4}m_L^2 = \frac{1}{2}p^2 + N_L - 1 = 0 \tag{27}$$

Since the states must survive the $S \otimes G$ projections, the left and right states transform as

$$|p\rangle = e^{(2\pi i p\cdot V)}|p\rangle \quad ; \quad |q\rangle = e^{(2\pi i q\cdot v)}|q\rangle \tag{28}$$

with V describing a shift in the root lattice and $\vec{v}$ corresponding to the twist action. We can use these elements to explicitly state the orbifold and twist action. First, we can redefine an element of S in terms of V^I as

$$(\theta, n_a e_a) \to (\sigma V^I, n_a \sigma A_a^I) \tag{29}$$

with A_a^I are the gauge transformations called Wilson Lines. These will be discussed further in the section 3.4.

We can now describe the orbifold action as

$$X^i \to (\theta X)^i + n_a e_a^i \quad , \quad X_L^I \to X_L^I + V^I + n_a A_a^I \tag{30}$$

The twist action is simply:

$$\theta^k = k\vec{v} \tag{31}$$

Returning to the states in (28), the invariant states correspond to the product of the eigenvalues equalling 1. From this condition we can obtain two types of solutions. The first; when the right movers are invariant under S, giving

$$p \cdot V \equiv 0 \pmod 1 \quad , \quad p \cdot A_a \equiv 0 \pmod 1 \tag{32}$$

and the second; obtained when non-trivially transforming right movers are tensored with left moving states, are

$$p \cdot V \equiv \frac{k}{N} \pmod 1, k = 1, \ldots, N-1 \tag{33}$$

$$p \cdot A_a \equiv 0 \pmod 1 \tag{34}$$

This is useful in the case of the gravitino, since we can see only (33) is not projected out, giving us an $\mathcal{N} = 1$ in $d = 4$.

3.2.2 Twisted States

The twisted sector are the states for which $k \neq 0$ in (23), and their mass formula is as follows

$$\frac{1}{2}(q+v_i)^2 - \frac{1}{2} + \delta_c = \frac{1}{4}m_R^2 = \frac{1}{4}M_L^2 = \frac{1}{2}(p^I + V^I + n_a A_a)^2 + N_L - 1 + \delta_c = 0 \tag{35}$$

with δ_c being the zero point energy by the modded oscillators.

We can consider a twisted state $h = (\theta, n_a e_a; V, n_a A_a) \in S \otimes G$. All states in the same congruency class as h, form the Hilbert space H_h.

Now consider another element $g = (\bar{\theta}, \bar{n}_a e_a; V, \bar{n}_a A_a) \in S \otimes G$. If $[h, g] = 0$, then all states congruent to g are also in H_h. Meaning all states in H_h that are not congruent to g have to be projected out.

All elements in $S \otimes G$, such as g, that commute with h, form the centralizer of h

$$Z_h = \{g \in S \otimes G \mid [h, g] = 0\} \tag{36}$$

All states $\tilde{g} \in S \otimes G$ that do not commute with h, form other congruency classes that are constructed using linear combinations of H_h, i.e. $H_{\tilde{g}h\tilde{g}^{-1}}$.

3.3 $\mathbb{Z}_2 \times \mathbb{Z}_2$ Discrete Symmetry

It may come as no surprise that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, is simply our orbifold Ω, from (17), with $\mathbb{Z}_2 \times \mathbb{Z}_2$ discrete symmetry as our point group, P. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model has lead to some fascinating results and has progressed the field considerably. One of the factors that changes the phenomenology of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model is the root lattice that T^6 is compactified on. When we gave the product form of our orbifold in (17), the root lattice Λ was hidden in T^6. While it may be perfectly clear to the reader how Λ fits into Ω, I would nevertheless like to explicitly state it just for completion.

Ω is defined as $T^6/P \times T_{(E_8 \times E_8)}/G$. Our root lattice, Λ is incorporated into our torus T^6 via compactification of $\mathbb{R}^6$ on Λ, i.e. $T^6 = \mathbb{R}^6/\Lambda$. Hence we can see that Λ helps to define Ω. This is why different lattice choices lead to phenomenologically different orbifold models.

Root lattices can be characterised into two groups; factorisable and non-factorisable. In this section, we will only state what the difference is, the effect the difference has on the properties of the model will be discussed in section 4.4.

The basic notion is fairly simple to understand. A factorisable lattice is one that can be decomposed into a product of smaller lattices, for example $T^6 \cong T^2 \times T^2 \times T^2$. A non-factorisable lattice is one that cannot be written as the product of smaller lattices, for example $SO(6)^2$.

I would like to apply some of the general properties introduced in sections 3.1 and 3.2 to our chosen orbifold; the $\mathbb{Z}_2 \times \mathbb{Z}_2$. I will begin with the point group, P. $P_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \{\mathbb{I}, \theta_1, \theta_2, \theta_3\}$ where $\mathbb{I}$ corresponds to the untwisted sector and $\theta_1, \theta_2, \theta_3$ correspond to the twisted sectors. Each of these represented as a twist vector is given below

$$\begin{aligned}
\mathbb{I} &\rightarrow \vec{v}_0 = (0,0,0,0) \\
\theta_1 &\rightarrow \vec{v}_1 = (0, \frac{1}{2}, -\frac{1}{2}, 0) \\
\theta_2 &\rightarrow \vec{v}_2 = (0, 0, \frac{1}{2}, -\frac{1}{2}) \\
\theta_3 &\rightarrow \vec{v}_3 = (0, \frac{1}{2}, 0, -\frac{1}{2})
\end{aligned} \tag{37}$$

we can see from this that θ_3 is a combination of θ_1 and θ_2. From this we can also define the space group as

$$S = \{(kv_1 + lv_2 \ , \ n_a e_a \mid k,l = 0,1 \ , \ n_a \in \mathbb{Z})\} \tag{38}$$

where the twisted sectors are defined as combinations of $k, l = 0, 1$.

3.4 Introduction to Wilson Lines

Finally, I would like to discuss Wilson lines. However similar to the definition of factorisable and non-factorisable lattices, I will not discuss their application in formulating consistent free fermionic models until the next section. Here, I will merely give their definition and any important characteristics that will be needed to discuss their role in the theory.

A Wilson Line is a VEV for A_i; an internal gauge field, where i signifies the direction the VEV acts along the vectors of the root lattice, Λ. The maximum number of Wilson Lines for a given model is the number of independent cycles on the orbifold that are not identified by the lattice vectors. For example, the number of independent cycles on a six-torus is six but since lattices come with point symmetry, some of the Wilson Lines corresponding to these independent cycles are identified, thus lowering their total allowed number.

By including Wilson lines into a model, the effects are such that:

- The modular invariant conditions are much more restricted, resulting in fewer allowed shifts V^I in the root lattice. Note the original modular invariance conditions for orbifold models are as follows:

$$\begin{aligned} N(V^2 - v^2) &\equiv 0 \quad (\text{mod } 2) \\ NV \cdot A_a &\equiv 0 \quad (\text{mod } 1) \\ NA_a \cdot A_b &\equiv 0 \quad (\text{mod } 1) \\ NA_a^2 &\equiv 0 \quad (\text{mod } 2) \end{aligned} \tag{39}$$

- In the untwisted sector, the gauge group is broken, since the Wilson lines introduce new projections.

- In the twisted sectors, the massless equations change with respect to the fixed points on the orbifold, providing different left moving states. I shall quickly define a fixed point as any point on Ω such that it is related to itself via the identification.

This projection is performed under elements of the centralizer, Z_h, for states $h = (\theta^k, n_a e_a)$ in the twisted sectors, introduced in section 3.2.2. In deriving the massless spectrum, left and right moving states are tensored such that the product is invariant under the full space group, S. The difference when Wilson lines are introduced is that now the states have to be invariant under the centralizer $Z_h \in S$. We shall see the usefulness of implementing Wilson lines into our model in section 4.4.

4 Free Fermionic Formulation

For sections 4.1, 4.2 and 4.3, chapter 3 of [9] and chapter 2 of [10] gave key insights.

The most phenomenologically promising string theory models are those constructed under the Free Fermionic Formalism. These are heterotic string theory models with $\mathcal{N} = 1$ supersymmetry in $d = 4$, with three particle generations and canonical $SO(10)$ embedding. In this section, we shall first introduce the formalism, and then discuss the correspondence to our $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and the modifications that make this such a useful theory.

4.1 General Formalism

The first key thing to note regarding the free fermionic formulation, is that is it actually constructed directly in $d = 4$, not $D = 10$ with compactification on the N^6 space. We shall see later how this can still be useful despite not agreeing with the dimensionality of our string theory model.

The difference in this formalism, compared to previous, is that all internal degrees of freedom are fermionised, as opposed to bosonised; producing world-sheet fermions, hence its name. As such, the models under this formalism describe 18 left moving Majorana fermions $\chi^a, a = 1, \dots, 18$, and 44 right moving real fermions $\bar{\Phi}^I, I = 1, \dots, 44$. The spacetime coordinates are described by the left moving coordinates (X^μ, ψ^μ) and right moving bosons $\bar{X^\mu}$.

We also need to introduce left moving local supersymmetry, since under this construction, the supersymmetry is fixed in the left sector to achieve $\mathcal{N} = 1$, and thus make it a phenomenologically useful model. To do this, we impose the supercurrent

$$T_F = \psi^\mu \partial X^\mu + f_{abc}\chi^a\chi^b\chi^c \tag{40}$$

among the left sector, spacetime and internal fields. Here f_{abc} are the structural constants of an 18-dimensional semi-simple Lie group G. χ^a transform under the adjoint representation of G; the representation obtained by linearising the action of G on itself by conjugation.

It was established by Dreiner, Lopez and Nanopoulos in 1989 that $\mathcal{N} = 1$ can be realised in $d = 4$ when $G = SU(2)^6$. Therefore we can split χ^a up into six triplets $(\chi^i, y^i, w^i), i = 1, \dots, 6$ such that each of them transforms as the adjoint representation of $SU(2)$.

4.2 Model Construction

Within the fermionic construction, phenomenologically consistent models can be generated by placing constraints on the boundary conditions. A set of these boundary condition vectors form a group

$$\Xi \sim Z_{N_1} \otimes \cdots \otimes Z_{N_k} \tag{41}$$

generated by a basis $B = \{b_1 \dots b_k\}$, where b_i is a vector form of the spin structure for a non contractible loop on the orbifold. Any fermion appearing in this basis B, is said to have periodic boundary conditions while any fermion not present in the basis is assigned anti-periodic boundary conditions. This basis must be chosen to satisfy the following conditions;

- The identity vector $\vec{\mathbb{I}}$ must be present in the model
- The number of real fermions must be even
- $\sum_i m_i b_i = 0 \;\Leftrightarrow\; m_i \equiv 0 \pmod{N_i}, \;\forall i$
- $N_{ij} b_i \cdot b_j \equiv 0 \pmod 4$

- $N_i b_i \cdot b_i \equiv 0 \pmod 8$

where N_i is the smallest possible integer resulting in $N_i b_i \equiv 0 \pmod 2$ and N_{ij} is the L.C.M of N_i and N_j. Additionally, the Lorentz product above can be explicitly calculated by:

$$b_i \cdot b_j = \left\{ \frac{1}{2} \sum_{L_{\mathbb{R}}} + \sum_{L_{\mathbb{C}}} - \frac{1}{2} \sum_{R_{\mathbb{R}}} - \sum_{R_{\mathbb{C}}} \right\} b_i(f) b_j(f) \tag{42}$$

for f any fermion $(\psi^\mu, \chi^a, \bar{\Phi}^I)$. Also here, $L_{\mathbb{R}}$ corresponds to the real left fermions, $L_{\mathbb{C}}$ corresponds to the complex left fermions, $R_{\mathbb{R}}$ corresponds to the real right fermions and finally $R_{\mathbb{C}}$ corresponds to the complex right fermions.

However these conditions alone are not enough to produce a consistent theory. In order to achieve modular invariance in the model, we must also place restrictions on the phases for the intersection of the basis vectors. These additional requirements are as follows:

- $c \begin{pmatrix} b_i \\ b_j \end{pmatrix} = \delta_{b_i} e^{\frac{2\pi i n_i}{N_j}} = \delta_{b_j} e^{\frac{2\pi i m_i}{N_i}} e^{\frac{i\pi b_i \cdot b_j}{2}}$
- $c \begin{pmatrix} b_i \\ b_i \end{pmatrix} = -e^{\frac{i\pi b_i \cdot b_i}{4}} c \begin{pmatrix} b_i \\ \mathbb{I} \end{pmatrix}$
- $c \begin{pmatrix} b_i \\ b_j \end{pmatrix} = e^{\frac{i\pi b_i \cdot b_j}{2}} c^* \begin{pmatrix} b_i \\ b_j \end{pmatrix}$
- $c \begin{pmatrix} b_i \\ b_j + b_k \end{pmatrix} = \delta_{b_i} c \begin{pmatrix} b_i \\ b_j \end{pmatrix} c \begin{pmatrix} b_i \\ b_k \end{pmatrix}$

where $1 < n_i < N_j$ and $1 < m_i < N_i$.

Furthermore in order to determine the surviving massless states of the spectrum, we can state the GSO projection used:

$$e^{i\pi b_i \cdot F_\alpha} |s\rangle_\alpha = \delta_\alpha c \begin{pmatrix} \alpha \\ b_i \end{pmatrix}^* |s\rangle_\alpha \tag{43}$$

where $|s\rangle_\alpha$ is a arbitrary state in the sector α, produced by the bosonic and fermionic oscillators in the vacuum. δ_α can be defined for the sector α by

$$\delta_\alpha = \begin{cases} 1 & \text{, for } \alpha(\psi^\mu) = 0 \\ -1 & \text{, for } \alpha(\psi^\mu) = 1 \\ 0 & \text{, otherwise} \end{cases}$$

Finally, the operator $(b_i \cdot F_\alpha)$ is defined explicitly by

$$b_i \cdot F_\alpha = \left\{ \sum_L - \sum_R \right\} b_i(f) F_\alpha(f) \tag{44}$$

where F is the fermion number operator, obtained by

$$F(f) = \begin{cases} 1 & \rightarrow \text{for } f \\ -1 & \rightarrow \text{for } f^* \end{cases}$$

If within the sector α, the fermions are periodic, the vacuum is described as degenerate and transforms under a $SO(2n)$ Clifford algebra. This therefore changed the value of the fermion number operator F. Therefore we shall redefine F for when f is a periodic fermion, denoted as $|\pm\rangle$

$$F(f) = \begin{cases} 0 & \rightarrow \text{for } |+\rangle \\ -1 & \rightarrow \text{for } |-\rangle \end{cases}$$

This is all the general analysis we will need to now return to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold.

4.3 $\mathbb{Z}_2 \times \mathbb{Z}_2$ Correspondence

As we have already stated, the construction of semi-realistic free fermionic models is related to the choice of boundary condition vectors $B = \{b_1, \ldots, b_k\}$. We shall consider the NAHE set; the boundary condition vectors corresponding to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. The basis in explicit form is

$$\begin{aligned}
\mathbb{I} &= \{\psi^{1,2}, \chi^{1,\ldots,6}, y^{1,\ldots,6}, w^{1,\ldots,6} | \bar{y}^{3,\ldots,6}, \bar{w}^{1,\ldots,6}, \bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\ldots,8}\} \\
S &= \{\psi^{1,2}, \chi^{1,\ldots,6}\} \\
b_1 &= \{\psi^{1,2}, \chi^{1,2}, y^{3,\ldots,6} | \bar{y}^{3,\ldots,6}, \bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1}\} \\
b_2 &= \{\psi^{1,2}, \chi^{3,4}, y^{1,2}, w^{5,6} | \bar{y}^{1,2}, \bar{w}^{5,6}, \bar{\psi}^{1,\ldots,5}, \bar{\eta}^{2}\} \\
b_3 &= \{\psi^{1,2}, \chi^{5,6}, w^{1,\ldots,4} | \bar{w}^{1,\ldots,4}, \bar{\psi}^{1,\ldots,5}, \bar{\eta}^{3}\}
\end{aligned}$$

where, as stated in section 4.2, only periodic fermions appear in these vectors. The left moving internal coordinates are fermionised by the relation

$$e^{iX^i} = \frac{1}{\sqrt{2}}(y^i + iw^i) \tag{45}$$

with a similar transformation for the right moving internal coordinates. χ^i denotes the superpartners of the left moving bosons, while the complex fermions are given by the 16 extra degrees of freedom $\bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\ldots,8}$. For the NAHE set, the GSO one loop phases are as follows

$$c\begin{pmatrix} b_i \\ b_j \end{pmatrix} = -1, \quad c\begin{pmatrix} 1 \\ S \end{pmatrix} = 1, \quad c\begin{pmatrix} b_i \\ 1, S \end{pmatrix} = -1 \tag{46}$$

The gauge group introduced by the NAHE set is $SO(10) \times SO(6)^3 \times E_8$, thus embedding our SO(10), in $\mathcal{N} = 1$. Explicitly, the $\bar{\psi}^{1,\ldots,5}$ correspond to the $SO(10)$ symmetry, the $\bar{\phi}^{1,\ldots,8}$ result in the E_8 symmetry and the internal fermions $\{\bar{y}^{3,\ldots,6}, \bar{\eta}^1\}, \{\bar{y}^{1,2}, \bar{w}^{5,6}, \bar{\eta}^2\}, \{\bar{w}^{1,\ldots,4}, \bar{\eta}^3\}$ give three copies of the horizontal $SO(6)$.

However as was stated in the beginning of section 4, for the model to be physical, we require the number of generations equal to three. In order to do this and simultaneously break the four dimensional gauge group, we introduce new basis vectors denoted (α, β, γ). This breaking occurs as a result of the boundary conditions for the new vectors corresponding the to generators of the considered subgroup. For example, we can break $SO(10)$ into $SU(5) \times U(1)$, $SO(6) \times SO(4)$ or $SU(3) \times SU(2) \times U(1)^2$, with the latter corresponding to the gauge interactions in the standard model, by choosing specific boundary conditions for $\bar{\psi}^{1,\dots,5}$.

The $SO(6)^3$ symmetries can also be broken into flavour $U(1)$ symmetries. In the visible sector, $U(1)$ charges can be produced by the world-sheet currents $\eta^i\bar{\eta}^i, i = 1, 2, 3$ and further $U(1)^n$ symmetries are realised as a result of pairing real fermions among the right internal sector.

Thus in order to construct a semi-realistic fermionic model corresponding to an orbifold construction, we need to extend the NAHE set by at least one boundary condition vector; namely $\xi_1 = \{\bar{\psi}^{1,\dots,5}, \bar{\eta}^{1,2,3}\}$. By accompanying this with an appropriate GSO projection, the gauge group $SO(4)^3 \times E_6 \times U(1)^2 \times E_8$ is imposed, with $\mathcal{N} = 1$ spacetime supersymmetry.

To construct a model in orbifold formulation we first begin, as before, with toroidal compactification. A given subset of basis vectors

$$\Gamma' = \{\mathbb{I}, S, \xi_1, \xi_2\} \tag{47}$$

where $\xi_2 = \{\bar{\phi}^{1,\dots,8}\}$, provides a toroidally compactified model with $\mathcal{N} = 4$ in $d = 4$ with $SO(12) \times E_8 \times E_8$ gauge group. The metric of the six-dimensional compactified manifold is the Cartan matrix of $SO(12)$, while the anti-symmetric tensor is

$$b_{ij} = \begin{cases} g_{ij} & \text{, for } i > j \\ 0 & \text{, for } i = j \\ -g_{ij} & \text{, for } i < j \end{cases}$$

Adding back the vectors b_1 and b_2 to Γ' we obtain,

$$\Gamma = \{\mathbb{I}, S, \xi_1, \xi_2, b_1, b_2\} \tag{48}$$

where Γ is the set of basis vectors corresponding to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model with standard embedding. It is well established that half the Euler number of a model gives the number of chiral generations. Here, the Euler characteristic is 48, leaving us with a 24 generation model.

The additional basis vector ξ_1 that was added to construct Γ, is used to separate the gauge degrees of freedom spanned by the world-sheet fermions $\{\bar{\psi}^{1,\dots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\dots,8}\}$ from the internal compactified degrees of freedom. In realistic fermionic models, we introduce the vector 2γ to do this task, with its explicit form as follows

$$2\gamma = \{\bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\ldots,4}\} \tag{49}$$

breaking the $E_8 \times E_8$ symmetry into $SO(16) \times SO(16)$. The action of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ twist breaks the symmetry gauge group into $SO(4)^3 \times SO(10) \times U(1)^3 \times SO(16)$. We haven't yet fixed the problem of the model having 24 generations but at least now the representation of $SO(10)$ is the chiral 16 representation. We can actually obtain the same result using Γ rather than 2γ by projecting out the $\mathbf{16} \oplus \bar{\mathbf{16}}$ from the ξ_1 sector by

$$c\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rightarrow -c\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{50}$$

There are two $\mathcal{N} = 4$ models that we can construct using basis Γ' that differ by the sign in (50). We shall define these as Z_+ and Z_-. The first of these, Z_+, produces to the $E_8 \times E_8$ model, while the second produces the $SO(16) \times SO(16)$. The discrete symmetry twist of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts equally in both models, the only difference is in the discrete torsion in (50).

We have now defined how to construct a free fermionic model corresponding to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. However it is far from realistic. This model has be constructed directly in $d = 4$, without the compactification from ten spacetime dimensions, and also, while our model does incorporate the chiral 16 representation of $SO(10)$ from the standard model, it currently has 24 chiral generations, 8 from each of the three twisted sectors - far too many for a physical model. To address this latter issue, we shall reintroduce Wilson lines, to show how they may fix this problem.

4.4 Applications of Wilson Lines

For this final section, a key reference was chapter 5 of the paper on $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic orbifold models of non factorisable six dimensional toroidal manifolds [10] by A. Faraggi.

In theory, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold looks a very promising option for producing a three generation model, since it has three twisted sectors and the number of generations $\sim$ the number of twisted sectors. Unfortunately however, in general, $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models do not lead to three generations since they produce more than one generation per twister sector. We saw this in the free fermionic construction in the previous section; where each twisted sector produced 8 generations.

However, it turns out that we can formulate three generation models with chiral $SO(10)$ embedding if we introduce Wilson lines into our model. By the standard embedding of the orbifold, we saw in section 4.3 that one of the E_8 factors is broken into E_6. We use Wilson lines to then break this E_6 down to $SO(10)$ and remove the 16-dimensional spinors of $SO(10)$ from E_6. This looks promising for fixing our problem since the number of generations in a sector corresponds to the number of 16-dimensional representations of $SO(10)$ within that sector, hence fewer **16** spinors means fewer generations.

For factorisable tori, this Wilson line, breaking E_6 to $SO(10)$, will be invariant under one of the non-trivial elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Within the corresponding twisted sector of this element, the Wilson line will remove all **16** representations of $SO(10)$ making the contribution of that sector to the number of generations equal to zero. In the non-factorisable case, there exists Wilson lines which aren't invariant under any of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ elements, meaning it doesn't necessarily remove the **16** representations from the twisted sector. In this section, I hope to show how the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model with a lattice of $SO(6)^2$ does in fact lead to a three generation model, since the Wilson lines remove all but one of the **16** spinors from each of the three twisted sectors.

In order to do this, we must first define the consistency conditions for the Wilson lines. A general definition of Wilson lines was given in 3.4. For discrete Wilson lines, A_i can only change from its orbifold image by the vectors on the $E_8 \times E_8$ root lattice. To find the consistency conditions, we state the effect on the generating vectors, e_i, of the $SO(6)^2$ lattice by the orbifold action.

$$\theta_1 : \begin{array}{lcl} e_1 & \to & -e_1 \\ e_2 & \to & -e_3 \\ e_3 & \to & -e_2 \\ e_4 & \to & -e_4 \\ e_5 & \to & -e_6 \\ e_6 & \to & -e_5 \end{array} \quad , \quad \theta_2 : \begin{array}{lcl} e_1 & \to & e_1 + e_2 + e_3 \\ e_2 & \to & -e_2 \\ e_3 & \to & -e_3 \\ e_4 & \to & e_4 + e_5 + e_6 \\ e_5 & \to & -e_5 \\ e_6 & \to & -e_6 \end{array} \tag{51}$$

This gives us the following consistency conditions:

$$(A_i, A_2 + A_3, A_5 + A_6) \in \Lambda_{E_8 \times E_8} \quad , \quad i = 1, \ldots, 6 \tag{52}$$

Additionally the modular invariance conditions must also be satisfied. These were given in (39). Now, we characterise the standard embedding by the shift vectors

$$\begin{aligned} V_1 &= (\tfrac{1}{2}, -\tfrac{1}{2}, 0^6)(0^8) \\ V_2 &= (0, \tfrac{1}{2}, -\tfrac{1}{2}, 0^5)(0^8) \end{aligned} \tag{53}$$

where the first three components of the gauge shift are equal to the last three components of the twist vectors θ_1 and θ_2 given in (37). Also in the above notation, I deem it important to explain what I mean by 0^n. This is a shorthand for

$$0^n = \underbrace{(0, \ldots, 0)}_{\text{n times}} \tag{54}$$

Hence we can see that V_i corresponds to two vectors consisting of eight components each, this is due to the gauge symmetry $E_8 \times E_8$.

We can use the same notation on the consistent set of Wilson lines

$$
\begin{aligned}
A_1 &= (0^8)(0^3, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, 0) \\
A_2 = A_3 &= (0^7, 1)(1, 0^7) \\
A_4 &= (0^8)(0, -\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0) \\
A_5 = A_6 &= (0^8)(0, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, 0^3)
\end{aligned} \tag{55}
$$

where the $A_2 = A_3 = (0^7, 1)(1, 0^7)$ wilson lines are responsible for breaking E_6 into $SO(10)$. The others, hopefully, will result in us being left with only one **16** in each twisted sector.

The untwisted spectrum, given by

$$
SO(10) \times U(1)^3 \times SU(2)^8 \tag{56}
$$

results in an $\mathcal{N} = 1$ vector multiplet. In the above product, $SO(10)$ and $U(1)^3$ come from the first E_8 factor in $E_8 \times E_8$ whereas the $SU(2)$ factors come from the second.

We can categorise the Cartan operators H_i of both E_8 groups in the $E_8 \times E_8$ by each of these factors. $H_1, \ldots, H_3$ are generated by the three $U(1)$ factors. The five remaining operators $H_4, \ldots, H_8$ are generated by the $SO(10)$ factor. These make up the first E_8 and obviously the second E_8 consisting of $H_9, \ldots, H_{16}$ is generated by the eight $SU(2)$ factors.

These last eight operators are ordered as follows:

$$
\begin{array}{c}
H_9 + H_{16} \\
H_9 - H_{16} \\
H_{10} + H_{11} \\
H_{10} - H_{11} \\
H_{12} + H_{13} \\
H_{12} - H_{13} \\
H_{14} + H_{15} \\
H_{14} - H_{15}
\end{array} \tag{57}
$$

I will now give representations of (56) by an ordered nonet of the dimensionalities of the $SO(10)$ and $SU(2)^8$ representations. The representation of $U(1)^3$ charges will be given as triple subscripts. Finally I should state that the notation $\mathbf{1}^n$ defines n consecutive entries of $\mathbf{1}$.

Transformations of the chiral multiplets in the **untwisted sector**:

$$
(\mathbf{10}; \mathbf{1}^8)_{1,0,0} + (\mathbf{1}; \mathbf{1}^8)_{0,1,1} + (\mathbf{1}; \mathbf{1}^8)_{0,1,-1} + \text{Charge Conjugate} \tag{58}
$$

$$
(\mathbf{10}; \mathbf{1}^8)_{0,1,0} + (\mathbf{1}; \mathbf{1}^8)_{1,0,1} + (\mathbf{1}; \mathbf{1}^8)_{1,0,-1} + \text{Charge Conjugate} \tag{59}
$$

$$(\mathbf{10};\mathbf{1}^8)_{0,0,1} + (\mathbf{1};\mathbf{1}^8)_{1,1,0} + (\mathbf{1};\mathbf{1}^8)_{1,-1,0} + \text{Charge Conjugate} \tag{60}$$

where (58), (59) and (60) correspond to the first, second and third complex plane. The charge conjugate above refers to the corresponding antiparticles; changing the sign of all the charges.

Before we discuss the twisted sectors, I must first introduce the concept of a fixed torus for a given twist sector. Similar to the idea of a fixed point as mentioned in 3.4, a fixed torus Υ of the sector θ_n is one such that

$$\theta_n \Upsilon = \Upsilon + \sum a_i e_i \tag{61}$$

In order to analyse the twisted sectors, we must give their fixed tori as each chiral multiplet transforms differently under each one. To do this, I shall introduce the notation Υ_i^j, where the lower index i corresponds to the sector in which we are discussing; θ_i. The upper j index is merely a way of cataloging each fixed torus within a given sector.

We shall now discuss the θ_1 **twisted sector** and state its fixed tori as follows:

$$\begin{aligned}
\Upsilon_1^1 &= \left\{(0,0,0,0,x,y) \mid x,y \in \mathbb{R}^2/\Lambda^2\right\} \\
\Upsilon_1^2 &= \left\{(\tfrac{1}{2},0,\tfrac{1}{2},0,x,y) \mid x,y \in \mathbb{R}^2/\Lambda^2\right\} \\
\Upsilon_1^3 &= \left\{(0,\tfrac{1}{2},0,\tfrac{1}{2},x,y) \mid x,y \in \mathbb{R}^2/\Lambda^2\right\} \\
\Upsilon_1^4 &= \left\{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},x,y) \mid x,y \in \mathbb{R}^2/\Lambda^2\right\}
\end{aligned} \tag{62}$$

We can now give the representations in which the chiral multiplets transform on each torus.

For Υ_1^1:

$$\begin{aligned}
&(\mathbf{16};\mathbf{1}^8)_{0,0,\frac{1}{2}} + (\mathbf{10};\mathbf{1}^8)_{-\frac{1}{2},-\frac{1}{2},0} + (\mathbf{1};\mathbf{1}^8)_{\frac{1}{2},\frac{1}{2},1} \\
&+ (\mathbf{1};\mathbf{1}^8)_{\frac{1}{2},\frac{1}{2},-1} + 2(\mathbf{1};\mathbf{1}^8)_{\frac{1}{2},-\frac{1}{2},0} + 2(\mathbf{1};\mathbf{1}^8)_{-\frac{1}{2},\frac{1}{2},0}
\end{aligned} \tag{63}$$

For Υ_1^2:

$$\begin{aligned}
&(\mathbf{1};\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^5)_{-\frac{1}{2},\frac{1}{2},0} + (\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^4)_{-\frac{1}{2},\frac{1}{2},0} \\
&+ (\mathbf{1};\mathbf{1}^4,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})_{\frac{1}{2},-\frac{1}{2},0} + (\mathbf{1};\mathbf{1}^5,\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2},-\frac{1}{2},0}
\end{aligned} \tag{64}$$

For Υ_1^3:

$$\begin{aligned}
&(\mathbf{1};\mathbf{2},\mathbf{1}^3,\mathbf{2},\mathbf{1})_{\frac{1}{2},-\frac{1}{2},0} + (\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{1}^3,\mathbf{2},\mathbf{1})_{\frac{1}{2},-\frac{1}{2},0} \\
&+ (\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1}^3,\mathbf{2})_{-\frac{1}{2},\frac{1}{2},0} + (\mathbf{1};\mathbf{1}^3,\mathbf{2},\mathbf{1}^3,\mathbf{2})_{-\frac{1}{2},\frac{1}{2},0}
\end{aligned} \tag{65}$$

For Υ_1^4:

$$\begin{aligned}&(\mathbf{1};\mathbf{2},\mathbf{1}^5,\mathbf{2},\mathbf{1})_{\frac{1}{2},-\frac{1}{2},0}+(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1}^5,\mathbf{2})_{\frac{1}{2},-\frac{1}{2},0}\\ &+(\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^3)_{-\frac{1}{2},\frac{1}{2},0}+(\mathbf{1};\mathbf{1}^3,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^2)_{-\frac{1}{2},\frac{1}{2},0}\end{aligned} \tag{66}$$

In the θ_2 **twisted sector**, the fixed tori are:

$$\begin{aligned}\Upsilon_2^1 &= \left\{(x,y,0,0,0,0)\mid x,y\in\mathbb{R}^2/\Lambda^2\right\}\\ \Upsilon_2^2 &= \left\{(x,y,\tfrac{1}{2},0,\tfrac{1}{2},0)\mid x,y\in\mathbb{R}^2/\Lambda^2\right\}\\ \Upsilon_2^3 &= \left\{(x,y,0,\tfrac{1}{2},0,\tfrac{1}{2})\mid x,y\in\mathbb{R}^2/\Lambda^2\right\}\\ \Upsilon_2^4 &= \left\{(x,y,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})\mid x,y\in\mathbb{R}^2/\Lambda^2\right\}\end{aligned} \tag{67}$$

The representations are:

For Υ_2^1:

$$\begin{aligned}&(\mathbf{16};\mathbf{1}^8)_{\frac{1}{2},0,0}+(\mathbf{10};\mathbf{1}^8)_{0,-\frac{1}{2},-\frac{1}{2}}+(\mathbf{1};\mathbf{1}^8)_{1,\frac{1}{2},\frac{1}{2}}\\ &+(\mathbf{1};\mathbf{1}^8)_{-1,\frac{1}{2},\frac{1}{2}}+2(\mathbf{1};\mathbf{1}^8)_{0,\frac{1}{2},-\frac{1}{2}}+2(\mathbf{1};\mathbf{1}^8)_{0,-\frac{1}{2},\frac{1}{2}}\end{aligned} \tag{68}$$

For Υ_2^2:

$$\begin{aligned}&(\mathbf{1};\mathbf{2},\mathbf{2},\mathbf{1}^6)_{0,-\frac{1}{2},-\frac{1}{2}}+(\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{2},\mathbf{1}^4)_{0,-\frac{1}{2},-\frac{1}{2}}\\ &+(\mathbf{1};\mathbf{1}^4,\mathbf{2},\mathbf{2},\mathbf{1}^2)_{0,\frac{1}{2},\frac{1}{2}}+(\mathbf{1};\mathbf{1}^6,\mathbf{2},\mathbf{2})_{0,\frac{1}{2},\frac{1}{2}}\end{aligned} \tag{69}$$

For Υ_2^3:

$$\begin{aligned}&(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1}^5,\mathbf{2})_{0,-\frac{1}{2},\frac{1}{2}}+(\mathbf{1};\mathbf{1}^3,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^2)_{0,-\frac{1}{2},\frac{1}{2}}\\ &+(\mathbf{1};\mathbf{2},\mathbf{1}^5,\mathbf{2},\mathbf{1})_{0,\frac{1}{2},-\frac{1}{2}}+(\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^2)_{0,\frac{1}{2},-\frac{1}{2}}\end{aligned} \tag{70}$$

For Υ_2^4:

$$\begin{aligned}&(\mathbf{1};\mathbf{2},\mathbf{1}^6,\mathbf{2})_{0,\frac{1}{2},\frac{1}{2}}+(\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1}^4,\mathbf{2},\mathbf{1})_{0,\frac{1}{2},\frac{1}{2}}\\ &+(\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{1}^2,\mathbf{2},\mathbf{1}^2)_{0,-\frac{1}{2},\frac{1}{2}}+(\mathbf{1};\mathbf{1}^3,\mathbf{2},\mathbf{2},\mathbf{1}^3)_{0,-\frac{1}{2},\frac{1}{2}}\end{aligned} \tag{71}$$

Finally, in the $\theta_3 \sim (\theta_1\theta_2)$ **twisted sector**, the fixed tori are:

$$
\begin{aligned}
\Upsilon_3^1 &= \left\{(0,0,x,y,0,0) \mid x,y \in \mathbb{R}^2/\Lambda^2\right\} \\
\Upsilon_3^2 &= \left\{(\tfrac{1}{2},0,x,y,\tfrac{1}{2},0) \mid x,y \in \mathbb{R}^2/\Lambda^2\right\} \\
\Upsilon_3^3 &= \left\{(0,\tfrac{1}{2},x,y,0,\tfrac{1}{2}) \mid x,y \in \mathbb{R}^2/\Lambda^2\right\} \\
\Upsilon_3^4 &= \left\{(\tfrac{1}{2},\tfrac{1}{2},x,y,\tfrac{1}{2},\tfrac{1}{2}) \mid x,y \in \mathbb{R}^2/\Lambda^2\right\}
\end{aligned}
\tag{72}
$$

The representations are:

For Υ_3^1:

$$
\begin{aligned}
&(\mathbf{16};\mathbf{1}^8)_{0,\frac{1}{2},0} + (\mathbf{10};\mathbf{1}^8)_{-\frac{1}{2},0,-\frac{1}{2}} + (\mathbf{1};\mathbf{1}^8)_{\frac{1}{2},1,\frac{1}{2}} \\
&+ (\mathbf{1};\mathbf{1}^8)_{\frac{1}{2},-1,\frac{1}{2}} + 2(\mathbf{1};\mathbf{1}^8)_{\frac{1}{2},0,-\frac{1}{2}} + 2(\mathbf{1};\mathbf{1}^8)_{-\frac{1}{2},0,\frac{1}{2}}
\end{aligned}
\tag{73}
$$

For Υ_3^2:

$$
\begin{aligned}
&(\mathbf{1};\mathbf{2},\mathbf{1}^2,\mathbf{2},\mathbf{1}^4)_{\frac{1}{2},0,\frac{1}{2}} + (\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}^5)_{\frac{1}{2},0,\frac{1}{2}} \\
&+ (\mathbf{1};\mathbf{1}^4,\mathbf{2},\mathbf{1}^2,\mathbf{2})_{-\frac{1}{2},0,-\frac{1}{2}} + (\mathbf{1};\mathbf{1}^5,\mathbf{2},\mathbf{2},\mathbf{1})_{-\frac{1}{2},0,-\frac{1}{2}}
\end{aligned}
\tag{74}
$$

For Υ_3^3:

$$
\begin{aligned}
&(\mathbf{1};\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^5)_{-\frac{1}{2},0,\frac{1}{2}} + (\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}^4)_{\frac{1}{2},0,-\frac{1}{2}} \\
&+ (\mathbf{1};\mathbf{1}^4,\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})_{-\frac{1}{2},0,\frac{1}{2}} + (\mathbf{1};\mathbf{1}^5,\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2},0,-\frac{1}{2}}
\end{aligned}
\tag{75}
$$

For Υ_3^4:

$$
\begin{aligned}
&(\mathbf{1};\mathbf{2},\mathbf{2},\mathbf{1}^6)_{-\frac{1}{2},0,-\frac{1}{2}} + (\mathbf{1};\mathbf{1}^2,\mathbf{2},\mathbf{2},\mathbf{1}^4)_{\frac{1}{2},0,\frac{1}{2}} \\
&+ (\mathbf{1};\mathbf{1}^4,\mathbf{2},\mathbf{2},\mathbf{1}^2,\mathbf{1}^2)_{\frac{1}{2},0,\frac{1}{2}} + (\mathbf{1};\mathbf{1}^6,\mathbf{2},\mathbf{2})_{-\frac{1}{2},0,-\frac{1}{2}}
\end{aligned}
\tag{76}
$$

Thus we can see that for each twisted sector, only one **16** representation of $SO(10)$ remains, giving us one chiral generation per twisted sector; a total of three as desired. We have successfully formulated a model with three chiral generations, standard $SO(10)$ embedding with $\mathcal{N}=1$ supersymmetry in $d=4$. Hence showing that the $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold along with some other tools can provide us with phenomenologically interesting results.

5 Conclusion

This report began with general overview of string theory, where I gave a motivation and introduced some of the important details. We then explored superstring theories and began analysing some of the problems with these models and how string theorists have overcome them. The dimensionality of the theory was rectified by introducing compactification schemes; beginning with toroidal. Here we showed an extremely important result, that the internal 6-dimensional manifold had to be Ricci-flat; vastly reducing the number of possible schemes. This then lead us on to holonomy groups, where we saw that for a 6-dimensional internal space to have $\mathcal{N} = 1$ supersymmetry, the holonomy group $\mathcal{H}$ must be within $SU(3)$. We also stated that when $\mathcal{H} = SU(3)$, we have defined a Calabi-Yau manifold. Unfortunately we know too little about them to produce any phenomenologically interesting results so we moved on to discuss another, simpler model, but with the same features; the orbifold. We introduced this generally and showed how they are constructed by taking quotients of lattices Λ. These lattices introduce specific symmetry into the model using a distinct set of basis vectors to describe it. The point group P corresponds to the holonomy group $\mathcal{H}$ and so for $\mathcal{N} = 1$ we imposed that $P \subset SU(3)$, leading to the conclusion that $P \equiv \mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_M$. We then introduced the untwisted and twisted sectors of the model and gave some details about the definition of each. We then defined our $\mathbb{Z}_2 \times \mathbb{Z}_2$ as our P from before and stated that, within this orbifold, the $\mathbb{I}$ corresponds to the untwisted sector while θ_i , $i = 1, 2, 3$ relates to the twisted sector. The notion of Wilson lines was then introduced, with a promise of a demonstration of their utility later on. We also gave its modular invariance conditions, and later in section 4 its consistency conditions too. We then moved on to the free fermionic formulation; models with three chiral generations, $SO(10)$ embedding and $\mathcal{N} = 1$ supersymmetry. We discussed the general formalism before showing how models are actually constructed. We gave the conditions for the basis generating the boundary condition vectors and defined some important features of the conditions governing this model. Next, we introduced the link between the main two areas of the report; the correspondence between the free fermionic formulation and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. Here, the NAHE basis was stated and it was showed how extending it leads to a useful breaking of the $E_8 \times E_8$ gauge group. However at this point the theory contained too many generations to be physical, therefore we reintroduced Wilson lines to show how they may fix this issue using a concrete example. We discussed how Wilson lines break the E_6 into an $SO(10)$, removing **16** representations from the E_6. We then explained how this would be useful since the number of **16** spinors in the twisted sectors corresponds to the number of chiral generations, so removing some of them brings us closer to a physical result. We then used the $SO(6)^2$ lattice and calculated the chiral multiplets of all its fixed tori and showed indeed that only one **16** remained in each of the twisted sectors resulting in three in total; matching our experimental observations.

Despite the free fermionic formulation being constructed in just 4 dimensions, with no compactification, the techniques learnt by analysing this model are invaluable, and could lead to significant breakthroughs in the future. The existence of our model with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold correspondence shows that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ deserves to be one of the key players in the theory of string phenomenology and how it may be one of the most promising string vacua we have unearthed.

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