Third Year Project

Quantum Mechanics from an Equivalence Principle

Name: Yuhang Tia

ID: 201369855

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Abstract

Quantum Mechanics is incompatible with the postulates of General Relativity. It is known that the fundamental features of Quantum Mechanics are the quantisation of energy and their probabilistic nature. Several methods have been developed to solve the contradictions of both theories, with one being the method of approaching Quantum Mechanics from structures of General Relativity. This report will focus on reconstructing Quantum Mechanics from Classical Mechanics formulation, with the Equivalence Postulate being formulated and applied and is somewhat parallel to the formulation of Equivalence Principle for General Relativity.

1. Introduction

Quantum Mechanics and General Relativity have played a significant role in the development of various disciplines of physics. They have been tested through time and remain dominant in explaining laws of nature; of course, in each domain they have always been in excellent agreement with experiments. Unfortunately, both theories have contradicted each other as each foundations differ significantly; General Relativity relies on differential geometry as language, while Quantum Mechanics relies on the probabilistic interpretation of wave function. Uniting both theories remain an important topic among the physics community.

As shown in works [1], [2], [3] from Alon Faraggi and Marco Matone, Quantum Mechanics can be derived from Equivalence Postulate. While the Equivalence Postulate cannot be implemented in Classical Mechanics, there exist a formulation of Classical Mechanics which can lead to the formulation of the Schrödinger equation. We modify that formulation such that it is compatible to Equivalence Postulates and we use the modified equation to derive the quantum variant and eventually the Schrödinger equation.

2. Hamilton-Jacobi Equation

In Classical Mechanics, there exists various formulation of which we focus on Hamiltonian mechanics. Although Hamilton's approach is often not as convenient as other methods for solving practical problems, nevertheless it is useful for theoretical studies relevant to this report. In this section, we will derive Hamilton-Jacobi Equation with a slight modification instead that leads to Classical Stationary Hamilton-Jacobi Equation (CSHJE) which can be formulated to the quantum analogue of CSHJE known as Quantum Stationary Hamilton-Jacobi Equation (QSHJE).

Canonical transformation is a type of coordinate transformation that preserves the form of Hamilton's equation for which the integration of the equations of motion is trivial. Consequently, the problem of obtaining the integrated equations of motion is reduced to allow us to obtain the general solution of equation via generating function. In Classical Mechanics, generalised coordinates q and momenta p are regarded as being independent quantities. Consider the transformation equation with initial condition q_0 coordinates and p_0 momenta,

$$q = q(q_0, p_0, t), \tag{1}$$

and

$$p = p(q_0, p_0, t), \tag{2}$$

where p, q are canonical variables with p_0, q_0 as a new set. Reminder that p_0, q_0 are constant.

Now let $K = K(P, Q, t)$ be transformed Hamiltonian from $H = H(p, q, t)$ via canonical transformation, where

$$\dot{Q}_\iota = \frac{\partial K}{\partial P_i}, \tag{3}$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{4}$$

And

$$K = H + \frac{\partial F}{\partial t}, \tag{5}$$

where F is the generating function. Since the new variables are constant, $\dot{P}_i = \dot{Q}_i = 0$ which is only true if we set $K = 0$, giving

$$0 = H + \frac{\partial F}{\partial t}. \tag{6}$$

We now have a differential equation that can be solved for F. This shows that canonical transformation reduces the problem of integrating equation of motion to general solution with generating function F.

Let $F = F(q, P, t)$. According to canonical transformation, F can be written as

$$p_i = \frac{\partial F}{\partial q_i}, \tag{7}$$

$$Q_i = \frac{\partial F}{\partial P_i}, \tag{8}$$

$$K = H + \frac{\partial F}{\partial t}. \tag{9}$$

Note that the condition of both frames $H = H(p, q, t)$ and $K = K(P, Q, t)$ must be equivalent, thus

$$pq_i - H = PQ_i - K + \frac{dF}{dt}. \tag{10}$$

Now we can write (6) as

$$H(q_1, \dots, q_n;\ p_1, \dots, p_n; t) + \frac{\partial F}{\partial t} = 0. \tag{11}$$

Let p_i be

$$p_i = \frac{\partial S^{cl}}{\partial q_i}, \tag{12}$$

where $S^{cl}(q, Q, t)$ is called Hamilton's principal function. Substitute (12) to (11) to give

$$H\left(q_1, \dots, q_n;\ \frac{\partial S^{cl}}{\partial q_1}, \dots, \frac{\partial S^{cl}}{\partial q_n}; t\right) + \frac{\partial S^{cl}}{\partial t} = 0. \tag{13}$$

We now use the basic interpretation of Hamiltonian mechanics

$$H = \frac{p^2}{2m} + V(q,t) \tag{14}$$

To convert equation (13) to

$$\frac{1}{2m}\left(\frac{\partial S^{cl}}{\partial q}\right)^2 + V(q,t) + \frac{\partial S^{cl}}{\partial t} = 0. \tag{15}$$

For a time–independent potential there is the decomposition

$$S^{cl}(q,Q,t) = \ S_0^{cl}\,(q,Q) - \ Et, \tag{16}$$

with E the energy of the stationary state. The function S_0^{cl} is called Hamilton's characteristic function or reduced action that by (13) satisfies the Classical Stationary Hamilton-Jacobi Equation (CSHJE)

$$H\left(q, \frac{\partial S_0^{cl}}{\partial q}\right) - E = 0. \tag{17}$$

Consider $\mathcal{W}(q) \equiv V(q) - E$, simplify (17) along with (14) to become

$$\frac{1}{2m}\left(\frac{\partial S^{cl}}{\partial q}\right)^2 + \mathcal{W}(q) = 0. \tag{18}$$

It is important to notice that while for an arbitrary canonical transformation one passes from a couple of independent variables (q, p) to another one (Q, P), in the case the Hamilton's principal function is used as generating function in (5), one has $K = K(Q,P,t)$ from (5) vanishes, that is $K = 0$, to produce another relation,

$$p = \partial_q S^{cl}(q,Q,t) \tag{19}$$

which shows that p and q become dependent.

Now let us apply a similar equation (12)

$$p = \frac{\partial S_0^{cl}}{\partial q}, \tag{20}$$

rather than with p and q being independent. This allows us to look for another coordinate transformation in the form

$$S_0^{cl}(q) \ \ \rightarrow \ \ \tilde{S}_0^{cl}(\tilde{q}) \tag{21}$$

with $\tilde{S}_0^{cl}(\tilde{q})$ denoting a reduced action of system as $K(Q,P,t) \rightarrow 0$. Since one requirement of canonical transformation is the independence of set coordinates, the transformation (21) is not a canonical transformation as p and q are dependent. Even if we specify the structure of transformation, $\tilde{S}_0^{cl}(\tilde{q}) = cnst$ shows that (21) is a degenerate transformation. The existence of such degenerate transformation is an odd one – applying reduced actions from two systems shows that there is no coordinate transformation making the two systems equivalent. However, it is possible to connect different systems by a coordinate transformation without applying constant reduced action. This means that in Classical Mechanics equivalence under coordinate transformations is frame dependent, and from page 15-16 of [1] it has been shown to leads to the conclusion that there is a distinguished frame in the CSHJE description.

3. v – transformation

We now introduce a locally invertible coordinate transformation onto the case of the reduced action S_0

$$q \to q^v = v(q). \tag{22}$$

Set

$$S_0^v(q^v) = S_0\big(q(q^v)\big). \tag{23}$$

Equation (23) defines a new reduced action S_0^v derived by the v-transformation.

Now, considering

$$S_0 \to S_0^v = S_0 \circ v^{-1}, \tag{24}$$

which is equivalent to (23) and retain its generality. Note that equation (24) is equivalent to saying that for every $v = S_0^{v-1} \circ S_0$ there exist the induced map $v^{-1^*} : \; S_0 \; \mapsto v^{-1^*}(S_0)$. In other words, S_0^v is the pullback of S_0 by v^{-1^*}.

Let us simplify the formalism previously obtained in this section with the induced transformation in the form (24). For the following form

$$p_v = \partial_{q^v} S_0^v(q_v), \tag{25}$$

we have

$$p \to p_v = \left(\partial_q q^v\right)^{-1} p. \tag{26}$$

The result shows that the formalism will be covariant.

4. Legendre Transformation and Duality

In Classical Mechanics, there exist a feature where considering canonical transformations and phase space produces $p - q$ duality which is broken under explicit solutions of equation of motions. However, this is not always true for Hamiltonians where the solutions still show explicit $p - q$ duality. Rather, we want to know whether the descriptions of p and q share the same structure which we currently lack a formulation in Classical Mechanics.

To understand the duality of p and q and their structure, we need to investigate the formalism of Classical Mechanics. Doing so allows us to connect the issue of dependence – independence of the canonical variables with dynamical features. As shown in one of the reports from Alon Faraggi and Marco Matone, a connection is given between conservation of energy and $p - q$ duality by considering that Hamiltonian equations reduce independent variables to dependent ones. Part of the investigation is analysing relation between the structures of equations of independent canonical variables and dependent canonical variables. The result shows that the duality $p \leftrightarrow q$ is broken into

$$q \to -p, \qquad p \to q. \tag{27}$$

This proofs that $p - q$ duality in Classical Mechanics is related to structures of equations of independent canonical variables in one side and dependent canonical variables in other side. According to Heisenberg's Uncertainty Principle, position and momentum cannot be simultaneously measured. This tells us that the issue of dependence and independence of the canonical variables will be a basic feature in formulating quantum analogue of CSHJE and the $p - q$ duality of Hamilton mechanics is a restrictive one. This lack of Hamiltonian formalism indicates that an alternative transformation is needed for descripting manifest $p - q$ duality. In this next section, we now consider canonical variables on equal footings which is more helpful in formulating the Equivalence Postulate. Such duality can only arise from the involutive nature of Legendre transformation, and we will use such principle to derive QSHJE.

Consider the dual reduced action $T_0(p)$:

$$T_0 = q\frac{\partial S_0}{\partial q} - S_0, \qquad S_0 = p\frac{\partial T_0}{\partial p} - T_0 \tag{28}$$

where $T_0(p)$ are the Legendre transformation of $S_0(q)$.

Also note that

$$p = \frac{\partial S_0}{\partial q}, \qquad q = \frac{\partial T_0}{\partial p}. \tag{29}$$

Now consider the Legendre transform of the Hamilton principal function S

$$T = q\frac{\partial S}{\partial q} - S, \qquad S = p\frac{\partial T}{\partial p} - T \tag{30}$$

$$p = \frac{\partial S}{\partial q}, \qquad q = \frac{\partial T}{\partial p}. \tag{31}$$

For the stationary case

$$S(q,t) = S_0(q) - Et, \qquad T(p,t) = T_0(p) + Et. \tag{32}$$

Now consider the differential

$$dS = \frac{\partial S}{\partial q}dq + \frac{\partial S}{\partial t}dt = pdq + \frac{\partial S}{\partial t}dt, \tag{33}$$

$$dT = \frac{\partial T}{\partial p}dp + \frac{\partial T}{\partial t}dt = qdp + \frac{\partial T}{\partial t}dt. \tag{34}$$

This implies

$$dS = d(pq - T) = pdq + qdp - qdp - \frac{\partial T}{\partial t}dt, \tag{35}$$

Which can be simplified as

$$\frac{\partial S}{\partial t} = -\frac{\partial T}{\partial t}. \tag{36}$$

The above equation shows that S and T have time variables and can be used to determine the self-dual state on the later section. Furthermore, it shows that T^{cl} satisfies the dual version of the classical HJ equation.

5. Möbius symmetry and Legendre duality

We now use the second derivative of the Legendre transformation to discover a $GL(2,\mathbb{C})$-transformation. More specifically, we analyse the Legendre transformation of T_0 induced by v via S_0 since S_0 and T_0 and Legendre pair. Note that the transformation $S_0(q) \to \tilde{S}_0(q)$ does not have to be v-type and in this case v-transformation is determined to be a symmetry of the Legendre transform of T_0.

Let us consider the Möbius transformation

$$q^v = \frac{Aq+B}{Cq+D}, \tag{37}$$

Equation (23) with (37) implies that

$$p^v = \rho^{-1}(Cq+D)^2 p, \tag{38}$$

where $\rho \equiv AD - BC$. Note that

$$\delta_v T_0 = \rho^{-1}(ACq^2 + 2BCq + BD)p. \tag{39}$$

Equation (37) and (38) are equivalent to

$$q^v\sqrt{p_v} = \epsilon\left(Aq\sqrt{p} + B\sqrt{p}\right), \quad \sqrt{p_v} = \epsilon\left(Cq\sqrt{p} + D\sqrt{p}\right), \tag{40}$$

where $\epsilon = \pm\sqrt{1/\rho}$.

Now take the second derivative of (28) with respect to $s = S_0(q)$ to obtain

$$\frac{1}{q\sqrt{p}}\frac{\partial^2\left(q\sqrt{p}\right)}{\partial s^2} = \frac{1}{\sqrt{p}}\frac{\partial^2 p}{\partial s^2}, \tag{41}$$

which is also equivalent to the ''canonical equation''

$$\left(\frac{\partial^2}{\partial s^2} + \mathcal{U}(s)\right)q\sqrt{p} = 0 = \left(\frac{\partial^2}{\partial s^2} + \mathcal{U}(s)\right)\sqrt{p}. \tag{42}$$

Here $\mathcal{U}(s)$ is the ''canonical potential''

$$\mathcal{U}(s) = \frac{1}{2}\left\{\frac{q\sqrt{p}}{\sqrt{p}}, s\right\} = \frac{1}{2}\,\{q, s\}, \tag{43}$$

with

$$\{h(x), x\} = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 = (\ln h')'' - \frac{1}{2}(\ln h')'^2 \tag{44}$$

denoting the Schwarzian derivative.

Reminder of the involutivity of the Legendre transformation and its duality

$$S_0 \leftrightarrow T_0 \qquad q \leftrightarrow p. \tag{45}$$

This implies another $GL(2,\mathbb{C})$- symmetry with the u-transformation corresponding to Möbius transformation instead

$$p^u \rightarrow \frac{Ap+B}{Cp+D}. \tag{46}$$

Repeat the process like equation (37) onward to obtain the dual version of (42) with respect to $t = T_0(p)$

$$\left(\frac{\partial^2}{\partial t^2} + \mathcal{V}(t)\right) p\sqrt{q} = 0 = \left(\frac{\partial^2}{\partial t^2} + \mathcal{V}(t)\right)\sqrt{q}, \tag{47}$$

where

$$\mathcal{V}(t) = \frac{1}{2}\left\{\frac{p\sqrt{q}}{q}, t\right\} = \frac{1}{2}\,\{p, t\}. \tag{48}$$

Continuing with the v-transformation variant, we notice that equations (42) and (43), which is from (41), shows that the Schwarzian derivative is twice the potential function $V(x)$. In the form of the equation,

$$\{\gamma(h), x\} = \{h, x\}, \tag{49}$$

where

$$\gamma(h) = \frac{Ah+B}{Ch+D}. \tag{50}$$

By considering the chain rule of the Schwarzian derivative along with (49) and (50), we got

$$\{f, x\} = \{h, x\}, \tag{51}$$

where $f = \gamma(h)$.

This implies the following relation under v-transformation

$$\mathcal{U}^v(s^v) = \frac{1}{2}\,\{q^v, s^v\} = \frac{1}{2}\{\gamma(q), s\} = \mathcal{U}(s), \tag{52}$$

where equation (23) implies that $s^v = s$. This shows that $\mathcal{U}$ is invariant under $GL(2,\mathbb{C})$ Möbius transformation of q.

One property of canonical equation (42) is that it can be used to solve dynamical problems by treating it as equation of motion. Let $y_1(s)$ and $y_2(s)$ be linearly independent solutions of the canonical equation

$$q\sqrt{p} = Ay_1(s) + By_2(s), \qquad \sqrt{p} = Cy_1(s) + Dy_2(s). \tag{53}$$

Take the ratio of (53) to become

$$q = \frac{Ah(s) + B}{Ch(s) + D}, \tag{54}$$

where $h(s) = y_1(s)/y_2(s)$. Note the similarity with (50). Consider the inverse of q and $\gamma(h)$ and we got the following equation

$$S_0(q) = h^{-1}\big(\gamma^{-1}(q)\big). \tag{55}$$

This shows that reduced action S_0^v are inverse of ratio of two linearly independent solutions given by the canonical equation. Note that while the canonical is a second-order linear differential equation, i.e., it needs two initial conditions to obtain solution, (54) requires three initial conditions to be solved.

6. Self-dual state

The establishment of $S_0 - T_0$ duality indicates correspondence of u- and v- transformation. This duality and their transformations indicate that for a given system $\mathcal{W}$ there are two equivalent descriptions of S_0 and T_0. This raises a question of whether dual structures select the distinguished states.

Consider the special case of $\mathcal{W}$ where pictures of S_0 and T_0 overlaps as shown below

$$q \to \tilde{q} \; = \; \alpha p, \qquad p \; \to \tilde{p} \; = \; \beta q. \tag{56}$$

This implies that

$$\frac{\partial \tilde{T}_0}{\partial \tilde{p}} = \alpha \frac{\partial S_0}{\partial q}, \qquad \frac{\partial \tilde{S}_0}{\partial \tilde{q}} = \beta \frac{\partial T_0}{\partial p}, \tag{57}$$

or

$$\frac{\partial \tilde{T}_0}{\partial q} = \alpha\beta \frac{\partial S_0}{\partial q}, \qquad \frac{\partial \tilde{S}_0}{\partial p} = \alpha\beta \frac{\partial T_0}{\partial p}. \tag{58}$$

Integrate (58) to get

$$\tilde{S}_0(\tilde{q}) = \; \alpha\beta T_0(p) + \; cnst, \qquad \tilde{T}_0(\tilde{p}) = \; \alpha\beta S_0(q) + \; cnst, \tag{59}$$

showing that $\tilde{S}_0(\tilde{q})$ and $\tilde{T}_0(\tilde{p})$ are Legendre transformation of $S_0(q)$ and $T_0(p)$ respectively.

Notice that roles of p and q have no effect on their functionals when interchanged twice, so we can write the following equation

$$\tilde{\tilde{S}}_0 = S_0, \qquad \tilde{\tilde{T}}_0 = T_0, \tag{60}$$

so that

$$(\alpha\beta)^2 = 1 \; \to \; \alpha\beta = \pm 1 \tag{61}$$

The distinguished states are shown to be invariant by (56) and (59) thus

$$\tilde{S}_0(\tilde{q}) = S_0(q), \qquad \tilde{T}_0(\tilde{p}) = T_0(p). \tag{62}$$

Substitute (61) and (62) to (59) and by considering the equation $S = S_0 - Et$ and $T = T_0 + Et$, we will get

$$S = \pm T + cnst. \tag{63}$$

Notice the sign of (36), for this implies (61) is actually

$$\alpha\beta = -1. \tag{64}$$

thus

$$S = -T + cnst. \tag{65}$$

Since $S = pq - T$, (65) becomes

$$pq = \gamma, \tag{66}$$

where $\gamma = cnst$.

Substitute (66) to (56) to give

$$q \to \tilde{q} = \frac{\alpha\gamma}{q}, \qquad p \to \tilde{p} = \frac{\beta\gamma}{p}. \tag{67}$$

Note that equation (56) remains invariant with the highest symmetry for $p \leftrightarrow q$, which is true if $\alpha = \beta$. Compare with (64) to obtain $\alpha = \beta = \pm i$ which gives the highest symmetric state associated with (56). This shows that the stability of Legendre transformation is related to time evolution, and by imposing conditions related to highest symmetry like $q^v = q_u$ and $p^u = p_v$, we have found the relation between time evolution and S and T pictures. This implies that the imaginary factor does appear in the relation between S_0 and solutions of the Schrödinger Equation. On the other hand, this factor also implies that time evolution in Quantum Mechanics is related to imaginary numbers and thus related to S and T pictures.

Because of the highest symmetry imposed by the conditions, the self-dual states are now distinguished. Thus, the states with equation (66) corresponds to

$$S_0(q) = \gamma \ln \gamma_q q, \qquad T_0(p) = \gamma \ln \gamma_p p\,. \tag{68}$$

Reminder that, for

$$S_0 + T_0 = pq = \gamma, \tag{69}$$

we obtain the following relationships of constants

$$\gamma_p \gamma_q \gamma = e. \tag{70}$$

By coinciding the solutions of canonical equation (42) and its dual version (47) with $p = \gamma/q$, we got

$$\mathcal{U}(s) = -\frac{1}{4\gamma^2} = \mathcal{V}(t). \tag{71}$$

Combine (71) with (42) and (47) to obtain the canonical equations with self-dual states.

$$\left(\frac{\partial^2}{\partial s^2}-\frac{1}{4\gamma^2}\right)q\sqrt{p}=0=\left(\frac{\partial^2}{\partial s^2}-\frac{1}{4\gamma^2}\right)\sqrt{p} \tag{72}$$

$$\left(\frac{\partial^2}{\partial t^2}-\frac{1}{4\gamma^2}\right)p\sqrt{q}=0=\left(\frac{\partial^2}{\partial t^2}-\frac{1}{4\gamma^2}\right)\sqrt{q} \tag{73}$$

7. Equivalence Principle and its implementation

We now deal with the core of the Equivalence Principle (EP). Like the method of obtaining Hamilton-Jacobi equation as seen in section 2 and 3, we look for transformations of dependent variables q and p which reduce to the free system with zero energy.

Recall from section 3 we obtain the following statement regarding the transformation $S_0 \to S_0^v$:

Given a locally invertible coordinate transformation $q \to q^v = v(q)$ onto the case of the reduced action S_0, set

$$S_0^v(q^v)=S_0\big(q(q^v)\big) \tag{74}$$

to find the coordinate transformation with the system containing $V-E=0$.

Now set $\mathcal{W}^0(q^0)$ as the state corresponding to $\mathcal{W}=0$ along with the new notations $q^0 \equiv q^v$ and $S_0^0 \equiv S_0^v$ to express the following transformation

$$q \to q^0 = S_0^{0^{-1}} \circ S_0(q). \tag{75}$$

This suggests the following EP:

For each pair $\mathcal{W}^a$, $\mathcal{W}^b$, there is a v–transformation $q^a \to q^b = v(q^a)$ such that

$$\mathcal{W}^a(q^a) \to \mathcal{W}^b\big(q^b\big). \tag{76}$$

With the Equivalence Principle defined, we now look at implementing EP into Classical Mechanics (CM). Set the following transformation based on (76)

$$\mathcal{W}(q) \to \mathcal{W}^v(q^v), \tag{77}$$

which is induced by

$$S_0^{cl}(q) \to S_0^{cl\,v}(q^v) = S_0^{cl}\big(q(q^v)\big). \tag{78}$$

Now substitute (77) and (78) into CSHJE expressed in (18) to obtain

$$\frac{1}{2m}\left(\frac{\partial S_0^{cl\,v}(q^v)}{\partial q^v}\right)^2+\mathcal{W}^v(q^v)=0. \tag{79}$$

Compare (18) with (79) and with the knowledge from (78), we got

$$\mathcal{W}(q) \to \mathcal{W}^v(q^v) = \big(\partial_{q^v} q\big)^2 \mathcal{W}(q), \tag{80}$$

that is,

$$\mathcal{W}^v(q^v)(dq^v)^2 = \mathcal{W}(q)(dq)^2. \tag{81}$$

Therefore, for consistency's sake $\mathcal{W}(q)$ must belong to Q^{cl}, the space of functions transforming as quadratic differentials under v^{cl}–transformations.

Now consider (80) but with the state W^0

$$\mathcal{W}^0(q^0) \to \mathcal{W}^v(q^v) = \left(\partial_{q^v} q^0\right)^2 \mathcal{W}^0(q^0) = 0. \tag{82}$$

This shows that $\mathcal{W}^0$ is a fixed point in space $\mathcal{H}$, however in CM $\mathcal{H}$ cannot be reduced to a point upon factorization by v^{cl}–transformations. Thus, the EP (76) cannot be implemented in CM without consistency issue.

For EP to be implemented consistently we need to modify CSHJE. Let the general equation of S_0 be

$$F\left(\frac{\partial S_0}{\partial q}, \frac{\partial^2 S_0}{\partial q^2}, \frac{\partial^3 S_0}{\partial q^3}, \dots\right) = 0, \tag{83}$$

noting that from (28) adding constants to S_0 or T_0 does not change the system. One key principle between Classical and Quantum Mechanics is the correspondence principle which states that, as the system approaches classical limit, results derived from quantum theory is being reduced to those of classical system. Consider (83) in the classical limit

$$F\left(\frac{\partial S_0}{\partial q}, \frac{\partial^2 S_0}{\partial q^2}, \dots\right) = 0 \to \frac{1}{2m}\left(\frac{\partial S_0^{cl}}{\partial q}\right)^2 + \mathcal{W}(q) = 0, \tag{84}$$

equation (83) can be written in the form

$$\frac{1}{2m}\left(\frac{\partial S_0(q)}{\partial q}\right)^2 + \mathcal{W}(q) + Q(q) = 0 \tag{85}$$

and by comparing with (84) we notice that, as we approach classical limit

$$Q \to 0 \tag{86}$$

which reduces the quantum equation we intend to find to CSHJE.

What about $\mathcal{W}$ and Q during transformation? Based on (85), $\mathcal{W} + Q$ must be consistent upon transformation to maintain consistency on (85). Let us consider the transformed version of (85)

$$\frac{1}{2m}\left(\frac{\partial S_0^v(q^v)}{\partial q^v}\right)^2 + \mathcal{W}^v(q^v) + Q^v(q^v) = 0 \tag{87}$$

Implement square of (74) onto (85) and (87) to give

$$\mathcal{W}^v(q^v) + Q^v(q^v) = (\partial_{q^v} q)^2\big(\mathcal{W}(q) + Q(q)\big). \tag{88}$$

This shows that $\mathcal{W} + Q$ are set memberships of space $\mathcal{Q}$ under v-transformation i.e. (85) is covariant, or consistent under the v-transformation. Note that the statement (86) already indicates covariance at the breaking limit.

According to EP, all $\mathcal{W}$ are connected to each other by coordinate transformation. However, (82) states that $\mathcal{W}^0$ is a fixed point in space $\mathcal{H}$and $\mathcal{H}$ cannot be reduced to a point. Therefore,

$$\mathcal{W} \notin \mathcal{Q} \qquad Q \notin \mathcal{Q}. \tag{89}$$

This is also true for Q since $\mathcal{W} + Q \in \mathcal{Q}$.

Recall from the beginning of this section we consider transformation which reduces system with vanishing energy, or $\mathcal{W} = 0$. In this case equation (85) becomes $(\partial_q S_0)^2 = -2mQ$. The fact that $(\partial_q S_0)^2 \in \mathcal{Q}$, and the relation on (89) imply that $Q = 0$ and therefore $S_0 = cnst$. Based on general form of equation (74), $S_0 = \tilde{S}_0 = cnst$ for any coordinates, which implies that S_0 is a fixed point in space $\mathcal{K}$. This causes consistency issue in CM. On the other hand, for the case of $S_0 \propto q$, one has the free particle with non-zero energy and has the same consistency. This shows that the problem only applies with free particles and has to do with defining Legendre transformation. In fact, the problem is related to $S_0 - T_0$ duality and it can be solved with the knowledge of self-duality as shown in section 6. Staring with (68) and their characteristics associated with highest symmetry $q^v = q_u$, $p^u = p_v$ and $\partial_v T_o = \partial_u S_0$, we denote $\mathcal{W}^{sd}$ as distinguished self-dual state $\mathcal{W}$ corresponding to $S_0^{sd}(q^{sd})$ and by deriving their reduced action, we obtain the following transformation S_0 to S_0^{sd}

$$q \rightarrow q^{sd} = v_{sd}(q) = \gamma_{sd}^{-1} e^{\frac{1}{\gamma_{sd}} S_0(q)}. \tag{90}$$

Note that transformations associated with EP should be locally invertible. Thus (90) can also be written as the transformation of S_0^{sd} to S_0 inversing $v_{sd}(q)$ and switching q and q^{sd}. This shows interesting properties that is only possible under EP.

8. Deriving Quantum Stationary Hamilton-Jacobi Equation

With consistency issue resolved, we can determine the explicit expression of (85) by first considering the Schwarzian derivative $\{e^h, x\}$ whose chain rule is

$$\{e^h, x\} = \left(\frac{\partial e^h}{\partial x}\right)^2 \left(\{e^h, e^h\} - \{x, e^h\}\right) = -\left(\frac{\partial e^h}{\partial x}\right)^2 \{x, e^h\}. \tag{91}$$

We also consider for $\{x, e^h\}$,

$$\{x, e^h\} = e^{-2h}\{x, h\} - \{e^h, h\} = e^{-2h}\{x, h\} + \frac{1}{2}. \tag{92}$$

Combine (91) and (92) to give

$$\{e^h, x\} = \{x, h\} - \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2. \tag{93}$$

One interesting property of $\{h, x\}$ is that due to the equation (49) it may take real value despite h and x taking complex value. In other words, if h and x are either real or purely imaginary values then $\{e^{ih}, x\} \in \mathbb{R}$. For α be independent of x,

$$\{e^{i\alpha h}, x\} = \{x, h\} - \frac{\alpha^2}{2}\left(\frac{\partial h}{\partial x}\right)^2. \tag{94}$$

Note that for h being independent of α,

$$\left(\frac{\partial h}{\partial x}\right)^2 = \alpha^{-1}\frac{\partial\{e^{i\alpha h}, x\}}{\partial \alpha}. \tag{95}$$

(94) and (95) imply the following equation with two different Schwarzian derivatives

$$\left(\frac{\partial S_0}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\{e^{\frac{2i}{\beta}S_0}, q\} - \{S_0, q\}\right), \tag{96}$$

with β as dimensional constant.

We now introduce the cocycle condition that arises from the fact that the transformation $\mathcal{W}^0 \to \mathcal{W}^v$ must have an inhomogeneous term within. For the basic cocycle condition corresponding to EP,

$$(q^a; q^c) = \left(\partial_{q^c} q^b\right)^2 \left[\left(q^a; q^b\right) - \left(q^c; q^b\right)\right]. \tag{97}$$

We also obtain the following theorem:

The cocycle condition (97) uniquely defines the Schwarzian derivative up to a multiplicative constant and a coboundary term.

With the theorem in mind, consider the equation that satisfies (97)

$$[q^a; q^b] = \left(q^a; q^b\right) - c_1\{q^a; q^b\}, \tag{98}$$

we can derive the following result

$$\left(q^a; q^b\right) = -\frac{\beta^2}{4m}\{q^a; q^b\}, \tag{99}$$

with $c_1 = -\beta^2/4m$.

Before implementing the above equations, we need to know whether they satisfy the classical limit. For the case of $\mathcal{W} \to \mathcal{W}^v$ we have $\left(q^a; q^b\right) \to 0$ and thus

$$\frac{\beta^2}{4m}\{q; q^v\} \to 0. \tag{100}$$

Since in classical limit $Q \to 0$ and with constant β we get

$$\lim_{\beta \to 0} Q = 0, \tag{101}$$

thus, we obtain the solution related to CSHJE

$$\lim_{\beta \to 0} S_0 = S_0^{cl}. \tag{102}$$

By substituting (96), (85) we get

$$\mathcal{W}(q) = \frac{\beta^2}{4m}\left(\{S_0, q\} - \{e^{\frac{2i}{\beta}S_0}, q\}\right) - Q(q). \tag{103}$$

Since $\mathcal{W}(q) = (q^0; q)$, by (99) we got

$$\mathcal{W}(q) = -\frac{\beta^2}{4m}\{q^0; q\}, \tag{104}$$

Before continuing, we need to determine q^0. Since $S_0^0(q^0) = S_0(q)$, we obtain $q^0 = v_0(q)$ and by Möbius symmetry we have obtain the unique solutions of q^0.

Under Möbius symmetry $\{h, x\}$ equation (104) becomes

$$\mathcal{W}(q) = -\frac{\beta^2}{4m}\left\{e^{\frac{2i}{\beta}S_0}; q\right\}, \tag{105}$$

and (103) becomes

$$Q(q) = \frac{\beta^2}{4m}\{S_0, q\}. \tag{106}$$

Finally, equation (85) with (104) becomes

$$\frac{1}{2m}\left(\frac{\partial S_0(q)}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0, q\} = 0. \tag{107}$$

Note that as $\beta \to 0$, (107) becomes CSHJE.

Recall from section 5 the method involving Möbius transformation. Equation (104) has the following identities

$$\partial_x h'^{1/2} h'^{1/2} = 0 = \partial_x h'^{-1} \partial_x h'^{1/2} h'^{1/2} h \tag{108}$$

and

$$h'^{1/2} \partial_x h'^{-1} \partial_x h'^{1/2} = \partial_x^2 + \{h, x\}/2, \tag{109}$$

with $h = e^{\frac{2i}{\beta}S_0}$. This implies the following

$$e^{\frac{2i}{\beta}S_0} = \frac{A\psi^D + B\psi}{C\psi^D + D\psi}, \tag{110}$$

where $AD - BC \neq 0$ and ψ^D and ψ are linearly independent solutions of the stationary Schrödinger equation (SE)

$$\left(-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right)\psi = E\psi. \tag{111}$$

Thus, for the "covariantizing parameter" β, $\beta = \hbar$, where $\hbar = h/2\pi$ is Planck's reduced constant. We can now formally call equation (107) the Quantum Stationary Hamilton-Jacobi Equation (QSHJE) of which SE linearize.

The solution of (111) related to (110) will be

$$\psi = \frac{1}{\sqrt{S_0'}}\left(Ae^{-\frac{i}{\hbar}S_0} + Be^{\frac{i}{\hbar}S_0}\right). \tag{112}$$

9. QSHJE and the Legendre transformation

As shown on the ending part of section 8, we notice that not all S_0 descriptions fit well in our formalism without consistency issue. We will now show that QSHJE can solve the issue of defining the Legendre transformation for *any* states of reduced actions S_0 and T_0. For the case of $\mathcal{W}^0$, QSHJE becomes

$$\frac{1}{2m}\left(\frac{\partial S_0^0}{\partial q^0}\right)^2 + \frac{\hbar^2}{4m}\{S_0^0, q^0\} = 0, \tag{113}$$

which is equivalent to the following equation

$$\{e^{\frac{2i}{\hbar}S_0^0}, q^0\} = 0. \tag{114}$$

One possible solution with a well-defined Legendre transformation of S_0^0 is

$$e^{\frac{2i}{\hbar}S_0^0} = \gamma_q q^0. \tag{115}$$

Compare equation (115) with the self-dual states (68) and we realize that we can identify the reduced action S_0^0 associated with $\mathcal{W}^0$ with the following distinguished self–dual state

$$S_0(q) = S_0^{sd}(q^{sd}) = \gamma_{sd} \ln \gamma_q q^{sd}(q). \tag{116}$$

For the case of $\mathcal{W}^0 = \mathcal{W}^{sd}$, we set

$$\gamma_{sd} = \pm\frac{\hbar}{2i}, \tag{117}$$

then substitute to (116) to get

$$S_0^0(q^0) = S_0^{sd}(q^{sd}) = \pm\frac{\hbar}{2i}\ln \gamma_q q^0\,, \qquad q^{sd} = q^0. \tag{118}$$

We can now write with (114) and (115) in mind

$$S_0^0 = \frac{\hbar}{2i}\ln\left(\frac{Aq^0 + B}{Cq^0 + D}\right). \tag{119}$$

We now consider cases where the Legendre transformation is not defined. For the case of $S_0 \propto q$ we take $S_0 = Aq + B$ as an example. This corresponds to $\mathcal{W} = -A^2/2m$ where for $V = 0$ this represents a free system with energy $E = A^2/2m$. Now consider equation (104) which produces $\mathcal{W}$ previously

$$\mathcal{W}(q) = -\frac{\hbar^2}{4m}\left\{e^{\frac{2i}{\hbar}S_0}; q\right\} = -\frac{\hbar^2}{4m}\left\{\frac{Ae^{\frac{2i}{\hbar}S_0} + B\psi}{Ce^{\frac{2i}{\hbar}S_0} + D\psi}; q\right\}, \tag{120}$$

where $AD - BC \neq 0$.

Should S_0 be the solution of QSHJE, we have the following solution

$$\tilde{S}_0 = \frac{\hbar}{2i}\ln\left(\frac{Ae^{\frac{2i}{\hbar}S_0} + B}{Ce^{\frac{2i}{\hbar}S_0} + D}\right). \tag{121}$$

The solution above coincides with (110). This explains the relationship between kinetic term in Quantum Mechanics and $Q(q)$; they mix under the symmetry (120) of $\mathcal{W}$. We can now solve the issue of defining the Legendre transformation in the case $\mathcal{W} = cnst$. For $S_0 = \pm\sqrt{2mE}q$ we got the equation

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 - E + \frac{\hbar^2}{4m}\{S_0, q\} = 0, \tag{122}$$

with solution

$$S_0 = \frac{\hbar}{2i}\ln\left(\frac{Ae^{\frac{2i}{\hbar}\sqrt{2mE}q} + B}{Ce^{\frac{2i}{\hbar}\sqrt{2mE}q} + D}\right), \tag{123}$$

where S_0 is not proportional to q. Since $S_0^{cl} = \pm\sqrt{2mE}q$ we have

$$\lim_{\hbar\to 0}\ln\left(\frac{Ae^{\frac{2i}{\hbar}\sqrt{2mE}q} + B}{Ce^{\frac{2i}{\hbar}\sqrt{2mE}q} + D}\right)^{\frac{\hbar}{2i}} = \pm\sqrt{2mE}q. \tag{124}$$

It is shown that for solution $S_0 = \pm\sqrt{2mE}q$, $S_0 = 0$ as $E \to 0$. This shows that the Schwarzian derivative and the Legendre transformation corresponding to $\mathcal{W} = cnst$ are undefined.

10. Quantised Energy

One important concept of Quantum Mechanics is the discrete energy level, or quantised energy, for bound states. This arises from its probabilistic nature that the wave function and its derivative are continuous and is square integrable. We now derive the same concept from our Equivalence Principle approach. We start with trivializing coordinates as it plays a crucial role in canonical transformation and is related to self-dual states. Consider the Schwarzian equation equivalent to QSHJE (107)

$$\left\{e^{\frac{2i}{\beta}S_0}, q\right\} = -\frac{4m\mathcal{W}}{\hbar^2}. \tag{125}$$

Now given the trivializing transformation

$$w = q^0 = \frac{\psi^D}{\psi}, \tag{126}$$

and (125) becomes

$$\{q^0, q\} = -\frac{4m}{\hbar^2}(V(q) - E). \tag{127}$$

Based on (126), $w \neq cnst$; $w \in C^2(R)$ and $\partial_q^2 w$ differentiable, and by considering the inverse of (127) we got $\{q^0, q^{-1}\} = q^4\{q^0, q\}$. Thus, conditions for trivializing transformation to be consistent must be applied beyond the real line i.e., on the real line plus the point at infinity, or

$$w \neq cnst; w \in C^2(\hat{R}) and\ \partial_q^2 w\ differentiable\ on\ \hat{R}\ where\ \hat{R} = R \cup \{\infty\}.$$

This implies the two solutions of Schrödinger equation and their derivatives are continuous within the equivalent postulate, or to be more specific,

$$w(-\infty) = \begin{cases} w(+\infty), & for\ w(-\infty) \neq \pm\infty, \\ w(-\infty), & for\ w(-\infty) = \pm\infty. \end{cases} \tag{128}$$

We now state the following theorem:

If the potential function $W(q)$ is bounding in some interval, then the ratio $q^0 = \psi^D/\psi$ is continuous on the extended real line if and only if the Schrödinger equation admits a square integrable solution.

To test the theorem above, consider the case of a particle in a potential well

$$V(q) = \begin{cases} 0, & |q| \leq L, \\ V_0, & |q| > L. \end{cases} \tag{129}$$

Note that $V(q)$ is an even function.

Let us set two linearly independent solutions of SE

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad K = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \tag{130}$$

This allows us the find the expression of conjugate momentum as needed in QM. For the solutions inside the potential well $|q| \leq L$, choose either $\psi_1^1 = \cos{(kq)}$ or $\psi_2^1 = \sin{(kq)}$. Outside the potential well $q > L$, choose either $\psi_1^2 = e^{-Kq}$ or $\psi_2^2 = e^{Kq}$. Parity fixes the solutions for $q \leq -L$ and for $q > L$ we can choose the solution to be linear combination of ψ_1^2 and ψ_2^2. At $q = L$ continuity across the boundary implies that $\psi_i^1(L) = \psi_j^2(L)$ and $\partial_q \psi_i^1(L) = \partial_q \psi_i^2(L)$. Let us denote such solutions as (i, j) and consider the case $(i, j) = (1,1)$. Here we have

$$\psi = \begin{cases} \cos(kL) \exp{[K(q + L)]}, & q < -L, \\ \cos(kq), & |q| \leq L, \\ \cos{(kL)} \exp{[-K(q - L)]}, & q < L, \end{cases} \tag{131}$$

with the quantisation condition

$$k \tan(kL) = K. \tag{132}$$

A linearly independent solution is given by

$$\psi^D = [2k \sin{(kL)}]^{-1} \cdot \begin{cases} \cos(2kL) \exp{[K(q + L)]} - \exp{[-K(q + L)]}, & q < -L, \\ 2\sin{(kL)} \sin(kq), & |q| \leq L, \\ \exp{[K(q - L)]} - \cos(2kL) \exp{[-K(q - L)]}, & q < L. \end{cases} \tag{133}$$

Now consider the trivializing map q^0. The solution associated with (1,1) is

$$\frac{\psi^D}{\psi} = [k \sin{(2kL)}]^{-1} \cdot \begin{cases} \cos(2kL) - \exp{[-2K(q + L)]}, & q < -L, \\ \sin{(2kL)} \tan{(kq)}, & |q| \leq L, \\ \exp{[2K(q - L)]} - \cos{(2kL)}, & q < L. \end{cases} \tag{134}$$

At the limit of (131),

$$\lim_{q \to \pm\infty} \frac{\psi^D}{\psi} = \pm\infty. \tag{135}$$

Hence, we have shown that for the case (1,1) the trivializing map is continuous on $\hat{R}$ which is needed for consistency of equivalence postulate and solutions imposed by (130) become physical energy levels.

Now consider the case for (1,2). Repeat the same process as done for (1,1) case.

$$\psi = \begin{cases} \cos(kL)\exp[-K(q+L)], & q < -L, \\ \cos(kq), & |q| \leq L, \\ \cos(kL)\exp[K(q-L)], & q < L, \end{cases} \tag{136}$$

with the quantisation condition

$$k\tan(kL) = -K. \tag{137}$$

The solution will be

$$\psi^D = [2k\sin(kL)]^{-1} \cdot \begin{cases} \cos(2kL)\exp[-K(q+L)] - \exp[K(q+L)], & q < -L, \\ 2\sin(kL)\sin(kq), & |q| \leq L, \\ \exp[-K(q-L)] - \cos(2kL)\exp[K(q-L)], & q < L, \end{cases} \tag{138}$$

and the trivializing transformation q^0 will be

$$\frac{\psi^D}{\psi} = [k\sin(2kL)]^{-1} \cdot \begin{cases} \cos(2kL) - \exp[2K(q+L)], & q < -L, \\ \sin(2kL)\tan(kq), & |q| \leq L, \\ \exp[-2K(q-L)] - \cos(2kL), & q < L. \end{cases} \tag{139}$$

whose asymptotic behaviour is

$$\lim_{q\to\pm\infty} \frac{\psi^D}{\psi} = \mp k^{-1}\cot(2kL). \tag{140}$$

Recall (128) that at points at infinity, $w(-\infty) = w(+\infty)$, and from (138) it is only possible if $k^{-1}\cot(2kL) = 0$. However, it is incompatible with the quantisation condition (135) and therefore $w(-\infty) \neq w(+\infty)$. We have now shown that, for case of (1,2), energy eigenvalues associated with solutions of case of (1,2) are inconsistent against the equivalence postulate and therefore does not make solutions of (135) physical.

We can now conclude that the same physical eigenstates that are selected in conventional Quantum Mechanics by the probability interpretation of the wave function, are selected in the equivalence postulate approach by mathematical consistency. Essentially, we have shown that our formulation above remains consistent with the Möbius symmetry which is the foundation of Quantum Mechanics and much of our report. We further note that the requirement for the trivialising transformation to be continuous on the extended real line creates another requirement that the real line is compact. Therefore, energy quantisation and square integrability arises from the consistency of the equivalence postulate and the compactness of space, which is mandated by the Möbius symmetry underlying QM. This however implies that parameterisation of particle propagation is deterministic, which goes against the probabilistic nature of QM. This raises questions whether such parameterisation is consistent with quantum Hamilton-Jacobi formalism and the Möbius symmetry that underlies it.

As shown on page 94 of [1] and section 6 of [3] by Alon E. Faraggi and Marco Matone, there are two approaches to define time parameterisation related to our problem. The first is the Bohmian mechanics where time parameterisation is defined with $p = \partial_q S^{NS} = m\dot{q}$ where conjugate momentum p is related to mechanical momentum $m\dot{q}$ and solutions of QHJE S^{NS}. The second is the Floyd's definition of time parameterisation by using Jacobi's theorem,$t = \partial_E S_0$ where S_0 is the solution of the QSHJE. Whilst it is shown that both approaches work in Classical Mechanics, they fail to meet any conditions needed to answer our questions. With Bohmian mechanics, its definition of time did not coincide with Floyd's definition of time and is incompatible with the Möbius symmetry and compactness of space. And while definition of time $t = \partial_E S_0$ does provide time parameterisation by inverting $t(q) \to q(t)$, energy levels corresponding to our solutions are always quantised when considering space compactness. This shows that differentiation with respect to E is undefined and time parameterisation corresponding to Jacobi's theorem is incompatible with our formalism related to QSHJE.

11. Geometrical Quantum Hamilton-Jacobi Theory

To solve the issue of time parameterisation presented in the previous section, we need to derive quantum Hamilton-Jacobi theory with geometrical formulation. The idea is that such formulation reproduces energy quantisation as a result without using any probabilistic interpretation of the wave function, which creates consistency issue regarding parameterisation. Another idea of Geometrical Quantum Hamilton-Jacobi (GQHJ) theory is that if space is compact, then there is no notion of particle trajectory, which again eliminates compatibility issue with QHJ formalism. This allows us to reproduce results of QM without the axiomatic interpretation of the wave function as probability amplitude. We begin with deriving the Wheeler–DeWitt (WDW) equation, a quantum gravity equation with no time variable. Then we derive the Hamilton-Jacobi equation based on WDW equation.

We start with the Arnowitt, Deser and Misner (ADM) formulation, a Hamiltonian formulation with space-time being foliated into a family of closed space-like hypersurfaces $\sum_t$ with respect to time t and coordinates x on each slice. Now take dynamic variables of the formulation as the metric tensor of three-dimensional spatial slices g_{ij} and modify some metric to give the Einstein-Hilbert Lagrangian density

$$\mathcal{L} = \frac{1}{2\kappa^2} N\sqrt{g}\left({}^3R - 2\Lambda + K^{jk}K_{jk} - K^2\right), \tag{141}$$

where ${}^3R$ is the intrinsic spatial scalar curvature, Λ the cosmological constant, and K the trace of the extrinsic curvature. Note that K is

$$K_{jk} = \frac{1}{N}\left(\frac{1}{2} g_{jk,0} - D_j N_k\right), \tag{142}$$

where D_j represent the j component of the covariant derivative.

Let π^0 and π^k be the conjugate momenta of N and N_k respectively. Since $\mathcal{L}$ is independent of both $\partial_{x_0} N_k$ and $\partial_{x_0} N$, we obtain the following primary constraints $\pi^0 \approx 0$ and $\pi^k \approx 0$. By considering the time conservation in the primary constrains, we got secondary constrains in the form of super momentum,

$$\mathcal{H}_k = -2D_j \pi_k^j \approx 0, \tag{143}$$

and super-Hamiltonian,

$$\mathcal{H} = 2\kappa^2 G_{ijkl}\pi^{ij}\pi^{kl} - \frac{1}{2\kappa^2}\sqrt{g}\left({}^3R - 2\Lambda\right) \approx 0, \tag{144}$$

where π^{jk} is the conjugate momentum of g_{jk}

$$\pi^{jk} = -\frac{1}{2\kappa^2}\sqrt{\bar{g}}\left(K^{jk} - g^{jk}K\right), \tag{145}$$

and the DeWitt supermetric

$$G_{ijkl} = \frac{1}{2\sqrt{\bar{g}}}\left(g_{ik}g_{jl} + g_{il}g_{jk} + g_{ij}g_{kl}\right). \tag{146}$$

With Legendre transformation we obtain the Hamiltonian

$$H = \int d^3\mathrm{x}\left(N\mathcal{H} + N^k\mathcal{H}_k\right). \tag{147}$$

Notice that N and N^k are the Lagrange multipliers of $\mathcal{H}$ and $\mathcal{H}_k$ respectively. Let Ψ be the wave function of the Schrödinger equation. Set momenta as

$$\hat{\pi}^0 = -i\hbar\frac{\delta}{\delta N}, \qquad \hat{\pi}^k = -i\hbar\frac{\delta}{\delta N_k}, \tag{148}$$

so that,

$$-i\hbar\frac{\delta\Psi}{\delta N} = 0, \qquad -i\hbar\frac{\delta\Psi}{\delta N_k} = 0 \tag{149}$$

showing that Ψ only depends on g_{jk}.

The conjugate of the field ϕ corresponds to $-i\hbar S_\phi$, thus within the configuration space of ϕ we have $\left[\delta^{(3)}\right] = L^{-3}$. From (145) we also have $\left[\pi_{ij}\right] = MT^{-2}$, which is different from $\left[-i\hbar\delta_{g_{jk}}\right] = ML^{-1}T^{-1}$, the dimension of the canonical choice of $\hat{\pi}^{jk}$. Based on the information above, we have

$$\hat{\pi}^{jk} = -i\hbar c\frac{\delta}{\delta g_{jk}}, \tag{150}$$

with the normalization of classical variant being fixed

$$\pi^{jk} = c\frac{\delta S}{\delta g_{jk}}, \tag{151}$$

where S in this case is the functional analogue of Hamilton's characteristic function. From (150), we obtain the constraint of super-momentum

$$\widehat{\mathcal{H}}_k\Psi = 2\mathrm{i}\hbar c g_{jk}D_l\frac{\delta\Psi}{\delta g_{lj}}. \tag{152}$$

For the secondary constraint in the form $\widehat{\mathcal{H}}\Psi = 0$, we obtain the WDW equation

$$\hbar c\left[-2{\ell_P}^2 G_{ijkl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}} - \frac{1}{2{\ell_P}^2}\sqrt{\bar{g}}\left({}^3R - 2\Lambda\right)\right]\Psi\left[g_{ij}\right] = 0, \tag{153}$$

where $\ell_P = \sqrt{8\pi\hbar G/c^3} = \kappa\sqrt{\hbar c}$ is the rationalized Planck length. With the secondary constraint we can write $\widehat{H}\Psi = 0$ so that $\partial_t\Psi = 0$, which shows the origin of the time problem.

Consider the key identity

$$\frac{1}{Ae^{\beta S}}\frac{\delta^2\left(Ae^{\beta S}\right)}{\delta g_{ij}\delta g_{kl}} = \beta^2\frac{\delta S}{\delta g_{ij}}\frac{\delta S}{\delta g_{kl}} + \frac{1}{A}\frac{\delta^2 A}{\delta g_{ij}\delta g_{kl}} + + \frac{\beta}{2A^2}\left[\frac{\delta}{\delta g_{ij}}\left(A^2\frac{\delta S}{\delta g_{kl}}\right) + \frac{\delta}{\delta g_{kl}}\left(A^2\frac{\delta S}{\delta g_{ij}}\right)\right], \quad (154)$$

with constant $\beta \in \mathbb{C}$. Set $\beta = i/\hbar$ and

$$\Psi = \mathrm{A}e^{\frac{i}{\hbar}S}, \quad (155)$$

where $A, S \in \mathbb{R}$. Note that if (155) is a solution, then by the nature of WDW operator $Ae^{-\frac{i}{\hbar}S}$ is also a solution.

Substitute in (155) to WDW equation (153) to give the WDW Hamilton-Jacobi equation

$$\hbar c\left[-2{\ell_P}^2 G_{ijkl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}} - \frac{1}{2{\ell_P}^2}\sqrt{\bar{g}}({}^3R - 2\Lambda)\right]\mathrm{A}e^{\frac{i}{\hbar}S} = 0, \quad (156)$$

where its quantum deformation is

$$2(c\kappa)^2 G_{ijkl}\frac{\delta S}{\delta g_{ij}}\frac{\delta S}{\delta g_{kl}} - \frac{1}{2\kappa^2}\sqrt{\bar{g}}({}^3R - 2\Lambda) - 2(c\kappa\hbar)^2\frac{1}{A}G_{ijkl}\frac{\delta^2 A}{\delta g_{ij}\delta g_{kl}} = 0, \quad (157)$$

along with the continuity equation

$$G_{ijkl}\frac{\delta}{\delta g_{ij}}\left(A^2\frac{\delta S}{\delta g_{kl}}\right) = 0. \quad (158)$$

Finally, the last term in (157) is called the quantum potential

$$Q = -2(c\kappa\hbar)^2\frac{1}{A}G_{ijkl}\frac{\delta^2 A}{\delta g_{ij}\delta g_{kl}}. \quad (159)$$

Note that (157) reduces to (144) as it approaches classical limit.

From section 3 of [4] it has been shown that GQHJ formulation has basic Möbius symmetry associated with the Schwarzian equation unlike the Bohmian formulation. GQHJ formulation also avoided the issue related to compact space by stating that there is no notion of particle trajectory within the space. Finally, no interpretation of the wave function is needed as energy is quantised with GQHJ formulation.

12. Conclusion

From the start we focus on developing Equivalence Postulate in Quantum Mechanics with one-dimensional and time-independent case only. We start with Hamilton-Jacobi equation, a formulation of Classical Mechanics that allows us to solve the entire system of equations of motion by reducing the dynamical problem to a single partial differential equation. The issue of duality in Classical Mechanics is solved when we base the formulation with the involute nature of Legendre transformation which allows us the obtain the $p - q$ and $S_0 - T_0$ dualities. With the duality we can generate the canonical potential in the form of Schwarzian derivative by considering Möbius symmetry. Like the case of reduced action S_0 the canonical potential is invariant under $GL(2, \mathbb{C})$ Möbius transformation and thus the Schwarzian derivative remains unchanged.

We then apply the Hamilton-Jacobi equation with the Equivalence Postulate we have formualted but this creates inconsistency in Classical Mechanics regarding $\mathcal{W} = 0$ state, so we need to modify the Hamilton-Jacobi Equation. Such modification is achieved by applying the self-dual states corresponding to $\mathcal{W}$ states, adding inhomogeneous term into $\mathcal{W}$ and by defining the Schwarzian derivative with the basic cocycle condition. As a result, we obtain the Quantum Stationary Hamilton-Jacobi Equation

$$\frac{1}{2m}\left(\frac{\partial S_0(q)}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0, q\} = 0. \tag{160}$$

As the quantum Hamilton-Jacobi formalism is being tested for the quantisation of energy, parameterisation of particle propagation is found to be deterministic, which goes against the nature of Quantum Mechanics. We consider two approaches to time parameterisation, Bohmian mechanics and Floyd's definition of time parameterisation, but neither can fully satisfy the consistency condition of both Möbius symmetry and space compactness.

We resolve the inconsistency issue by taking the geometrical approach to quantum Hamilton-Jacobi theory. We start with the quantum gravity equation known as Wheeler–DeWitt (WDW) equation and derive WDW Hamilton-Jacobi Equation. In WDW HJ equation, no probabilistic interpretation of the wave function is needed and there is no particle trajectory in compact space. WDW equation being time-independent also avoids issue relating to time parameterisation. In the quantum deformation of WDW HJ equation there exist a term (159) known as the quantum potential. From section 4 of [4] it is shown to be never trivial, so that it plays the role of intrinsic energy. Such property selects the quantum potential corresponding to WDW equation as the candidate of dark energy which will have further implication in relating General Relativity and Quantum Mechanics. Researching the quantum potential may give us a starting point on understanding the nature and relationship of General Relativity and Quantum Mechanics.

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