Quantum Mechanics from an Equivalence Principle Project TP12

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Abstract

Recently Alon Faraggi and Marco Matone showed in [1] that the postulate of physical states being equivalent under coordinate transformation leads to the Quantum Stationary Hamilton-Jacobi Equation (QSHJE) in one dimension. This leads to the Schrödinger equation, noting that $\hbar$ plays the part of a covariantizing parameter. The purpose of this paper is to give incite into this development and finally to consider the application of this formalism to the textbook case of the hydrogen atom.

1 Introduction

General relativity and quantum mechanics are central to twentieth century physics, both are unchallenged descriptions of their respective domains. Unfortunately, these two theories have not been successfully meshed together, despite many attempts. General relativity is founded upon two geometric postulates, whilst the text book approach to quantum mechanics relies upon an axiomatic interpretation of the wave function. Thus the difficulties in joining these two very successful theories into a single description is most likely due to their different foundations. The Equivalence Postulate (EP) as formulated by Faraggi and Matone is somewhat reminiscent of Einstein's EP. Whilst the postulated EP cannot be implemented in Classical Mechanics (CM) in a consistent fashion, the EP does lead to a quantum version of the Classical Stationary Hamilton-Jacobi Equation (CSHJE), and then to the familiar Schrödinger equation.

2 Classical Hamilton - Jacobi Theory

The analogue of the CSHJE, the Quantum Stationary Hamilton-Jacobi Equation (QSHJE) is a third-order differential equation, the solution of which defines $\mathcal{S}_0$, the quantum analogue of the classical reduced action $\mathcal{S}_0^{cl}$. It is therefore useful to investigate the properties of canonical transformations in CM, considering that the formulation displays manifest p–q duality. It will be seen that this duality is related to the Möbius symmetry which underlies the EP which fixes the QSHJE.

Canonical transformations (i.e. the equations of motion remain in Hamiltonian form) may be used to give a general method for solving mechanical problems. In the classical theory, generalised coordinates q and momenta p are regarded as being independent quantities. The more general procedure consists of seeking a canonical transformation from coordinates and momenta (p, q) at time t, to a set of $2n$ initial values, for example (p_0, q_0) at time t_0. Thus the equations of transformation are the required solutions of the mechanical problem,

$$q = q(q_0, p_0, t), \tag{1}$$

$$p = p(q_0, p_0, t). \tag{2}$$

But how can such a transformation be found? We may automatically ensure that the transformed variables (P_i, Q_i) are constant in time by making the transformed Hamiltonian K equal to zero. Hence the equations of motion are

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0, \tag{3}$$

$$-\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0. \tag{4}$$

Now, K is related to the original Hamiltonian H and to the generating function of the transformation F [2] by the equation

$$K = H + \frac{\partial F}{\partial t}, \tag{5}$$

which from (3) becomes

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0. \tag{6}$$

We may write this as a function of the same variables as the generating function $F = F(q, P, t)$ using

$$p_i = \frac{\partial F}{\partial q_i}. \tag{7}$$

Note that F results from the condition that the dynamics of the systems in both frames must be equivalent, that is

$$p\dot{q} - H = P\dot{Q} - K + \frac{\partial F}{\partial t}, \tag{8}$$

hence choosing $F = F(q, P, t)$ gives (7).
Now we may write (6) as

$$H\left(q_i, \ldots, q_n, \frac{\partial \mathcal{S}^{cl}}{\partial q_1}, \ldots, \frac{\partial \mathcal{S}^{cl}}{\partial q_n}; t\right) + \frac{\partial \mathcal{S}^{cl}}{\partial t} = 0. \tag{9}$$

This is the classical Hamilton-Jacobi (HJ) equation noting that $\mathcal{S}^{cl} \equiv F$ where $\mathcal{S}^{cl}$ is known as Hamilton's principle function. For a Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(q) \tag{10}$$

the decomposition for a time independent potential

$$\mathcal{S}^{cl}(q, Q, t) = \mathcal{S}^{cl}_0(q, Q) - Et \tag{11}$$

leads to (with $W(q) \equiv \mathcal{S}_0^{cl}(q,Q) - Et$) the classical stationary Hamilton-Jacobi equation (CSHJE)

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}_0^{cl}(q)}{\partial q}\right)^2 + W(q) = 0. \tag{12}$$

This has the surprising property of giving the functional relationship between q and p. Note that unlike with canonical transformations which pass from one pair of independent variables (q,p) to another (Q,P), using Hamilton's principle function as the generating function has the result of collapsing q and p to be dependant variables. If we consider a similar question to that producing the CSHJE, but instead considering

$$p = \frac{\partial \mathcal{S}_0^{cl}(q)}{\partial q}, \tag{13}$$

i.e. looking for the coordinate transformation

$$\mathcal{S}_0^{cl}(q) \rightarrow \tilde{\mathcal{S}}_0^{cl}(\tilde{q}), \tag{14}$$

if we denote the reduced action of the system with zero Hamiltonian by $\tilde{\mathcal{S}}_0^{cl}(\tilde{q})$. Since $\tilde{\mathcal{S}}_0^{cl}(\tilde{q})$ =cnst this is a degenerate transformation- the existence of with is somewhat odd. In CM it is only possible to connect systems by coordinate transformation if neither system is described by a constant reduced action. Thus there is a distinguished frame in the CSHJE description.

3 The v-transformation

Consider the transformation

$$q \rightarrow q^v = v(q) \tag{15}$$

which induces the functional transformation

$$\mathcal{S}_0(q) \rightarrow \mathcal{S}_0^v(q^v) = \mathcal{S}_0(q) \circ v^{-1} \tag{16}$$

assuming v is locally invertible i.e., setting

$$\mathcal{S}_0^v(q^v) = \mathcal{S}_0(q(q^v)) \tag{17}$$

defining a new reduced action associated to these transformations. Note that in general one could define

$$\mathcal{S}_0(q) \rightarrow \mathcal{S}_0^v(q^v) = \mathcal{S}_0(q) \circ v^{-1} + f \tag{18}$$

for a function f, however one can always find a map of the form of (16), which in any case is a more convenient form since with this form as $p_v = \partial_{q^v} \mathcal{S}_0^v(q^v)$ gives

$$p \rightarrow p_v = \left(\frac{\partial q^v}{\partial q}\right)^{-1} p \tag{19}$$

i.e., the resulting formalism is covariant, p transforms like ∂_q.

4 Duality

Considering canonical transformations in CM it is evident that there is a formal p–q duality, which is in general broken in the full solution of the equations of motion. Classical duality is determined by the functional dependance the structure of the relevant equations might have on the canonical variables in the cases that these variables are dependant or independent. The concern over wether the variables are dependant or otherwise is clearly important to us, due to the well known fact that in QM exact position and momentum cannot be simultaneously known. The p–q duality restriction inherent in the classical Hamiltonian formulation hints that a manifest duality demands an alternative description of the physical problem.

There is a structure with such manifest duality, which is a consequence of the involutive character of the Legendre transform. This duality, which may not feature in CM is a direct result of considering positions and momenta as similar coordinates. Thus let us define the dual reduced action, $\mathcal{T}_0(p)$ as

$$\mathcal{T}_0 = q\partial_q \mathcal{S}_0 - \mathcal{S}_0, \quad \mathcal{S}_0 = p\partial_p \mathcal{T}_0 - \mathcal{T}_0, \tag{20}$$

$$p = \partial_q \mathcal{S}_0, \qquad q = \partial_p \mathcal{T}_0 \tag{21}$$

and than consider the same definition but for the Legendre transform of $\mathcal{S}$

$$\mathcal{T} = q\partial_q \mathcal{S} - \mathcal{S}, \quad \mathcal{S} = p\partial_p \mathcal{T} - \mathcal{T}, \tag{22}$$

$$p = \partial_q \mathcal{S}, \qquad q = \partial_p \mathcal{T}. \tag{23}$$

Note that as before the decomposition for the stationary case is

$$\mathcal{S}(q, y) = \mathcal{S}_0 - Et, \quad \mathcal{T}(p, t) = \mathcal{T}_0 + Et. \tag{24}$$

Now let us consider:

$$d\mathcal{S} = \frac{\partial \mathcal{S}}{\partial q}dq + \frac{\partial \mathcal{S}}{\partial t}dt = pdq + \frac{\partial \mathcal{S}}{\partial t}dt \tag{25}$$

and

$$d\mathcal{T} = \frac{\partial \mathcal{T}}{\partial p}dp + \frac{\partial \mathcal{T}}{\partial t}dt = qdp + \frac{\partial \mathcal{T}}{\partial t}dt, \tag{26}$$

from which we see that

$$d\mathcal{S} = d(pq - \mathcal{T}) = pdq - \frac{\partial \mathcal{T}}{\partial t}dt \tag{27}$$

i.e.,

$$\frac{\partial \mathcal{S}}{\partial t} = -\frac{\partial \mathcal{T}}{\partial t} \tag{28}$$

which connects the equivalent $\mathcal{S}$ and $\mathcal{T}$ descriptions through progress of time. Also, $\mathcal{T}_{cl}$ will satisfy the dual of the classical HJ equation.

5 Möbius Symmetry

The construction of Faraggi and Matone is strongly linked to the recently discovered GL $(2, \mathbf{C})$-symmetry of the second order differential equation obtained from the Legendre transformation. It is this symmetry that, in the case of the quantum version of the Hamilton's principle function, picks out distinguished self-dual states which guarantee the existence of this principle function for any given physical system.

Taking the second derivative of the Legendre transform (20) with respect to $s = \mathcal{S}_0(q)$ using the fact that

$$\mathcal{S}_0 = p\frac{\partial \mathcal{T}_0}{\partial q} - \mathcal{T}_0 = \frac{1}{2}\sqrt{p}\frac{\partial \mathcal{T}_0}{\partial \sqrt{q}} - \mathcal{T}_0 \tag{29}$$

gives the canonical equation,

$$({\partial_s}^2 + \mathcal{U}(s))q\sqrt{p} = 0 = ({\partial_s}^2 + \mathcal{U}(s))\sqrt{p}, \tag{30}$$

where $\mathcal{U}(s) = \{q, s\}/2$ and where the Schwarzian derivative is denoted by $\{f(x), x\} = f'''/f' - (3/2)(f''/f')^2$. Note that again the involutivity of (20) and the duality

$$\mathcal{S}_0 \longleftrightarrow \mathcal{T}_0, \quad q \longleftrightarrow p, \tag{31}$$

indicate a dual version of (30),

$$({\partial_t}^2 + \mathcal{V}(t))q\sqrt{p} = 0 = ({\partial_t}^2 + \mathcal{V}(t))\sqrt{p} \tag{32}$$

with $\mathcal{V}(t) = \{p, t\}/2$

It is evident that $\mathcal{U}$ is Möbius invariant, i.e.

$$\{\gamma(f), x)\} = \{f(x), x\} \tag{33}$$

where $\gamma(f)$ is the Möbius transformation of f,

$$\gamma(f) = \frac{Af + B}{Cf + D}. \tag{34}$$

In fact,

$$\{h, x\} = \{f, x\} \tag{35}$$

if and only if $h = \gamma(f)$. Thus, whilst $\mathcal{T}_0$ is invariant under any v-transformation, $\mathcal{V}$ is invariant under Möbius transformations of q.

Another property of (30) is that it may be considered as an equation of motion. Let us consider $\mathcal{V}(s)$ given - if $y_1(s)$ and $y_2(s)$ are linearly independent solutions of (30), the general solution is given by

$$q\sqrt{p} = Ay_1(s) + By_2(s), \quad \sqrt{p} = Cy_1(s) + Dy_2(s). \tag{36}$$

Taking the ratio

$$q = \frac{Ah(s) + B}{Ch(s) + D} \tag{37}$$

with $h(s) = y_1(s)/y_2(s)$. Hence the problem is solved by the inversion

$$\mathcal{S}_0 = h^{-1}(\gamma^{-1}(q)). \tag{38}$$

Note that since $q(s)$ is a ratio of two linearly independent solutions of a second order differential equation, three initial conditions are required to provide a complete solution. It is also evident that the canonical equation is clearly covariant under arbitrary transformations, since $\mathcal{T}_0$ is the Legendre transform of $\mathcal{S}_0$.

6 Self-Dual States

The $\mathcal{S}_0$–$\mathcal{T}_0$ duality gives rise to a complete correspondence between v-transforms and the equivalent transform $p \rightarrow p^u = u(p)$ indicating that any physical system with $W \equiv V(q) - E$ has two descriptions, one being the $\mathcal{S}_0$ picture, the other being the $\mathcal{T}_0$ picture, which are equivalent. Considering the case where both pictures overlap,one would expect the distinguished states, for which $\mathcal{S}_0$ and $\mathcal{T}_0$ have the same functional form, to display slightly odd properties. Considering the interchange of the two pictures by the transformation

$$q \rightarrow \tilde{q} = \alpha p, \quad p \rightarrow \tilde{p} = \beta q, \tag{39}$$

Which implies

$$\frac{\partial \tilde{\mathcal{T}}_0}{\partial \tilde{p}} = \alpha \frac{\partial \mathcal{S}_0}{\partial q}, \qquad \frac{\partial \tilde{\mathcal{S}}_0}{\partial \tilde{q}} = \beta \frac{\partial \mathcal{T}_0}{\partial p}, \tag{40}$$

i.e.,

$$\frac{\partial \tilde{\mathcal{T}}_0}{\partial q} = \alpha\beta \frac{\partial \mathcal{S}_0}{\partial q}, \qquad \frac{\partial \tilde{\mathcal{S}}_0}{\partial p} = \alpha\beta \frac{\partial \mathcal{T}_0}{\partial p}, \tag{41}$$

which when integrated gives

$$\tilde{\mathcal{S}}_0(\tilde{q}) = \alpha\beta \mathcal{T}_0(p) + cnst, \quad \tilde{\mathcal{T}}_0(\tilde{p}) = \alpha\beta \mathcal{S}_0(q) + cnst. \tag{42}$$

These are essentially Legendre transforms. Swapping q and p twice should have no effect upon the relationship between q and p, then we have the following (to plus a constant).

$$\tilde{\tilde{\mathcal{S}}}_0 = \mathcal{S}_0, \quad \tilde{\tilde{\mathcal{T}}}_0 = \mathcal{T}_0, \tag{43}$$

hence

$$(\alpha\beta)^2 = 1. \tag{44}$$

The distinguished states are therefore those that are invariant under the transformation (39). Thus,

$$\tilde{\mathcal{S}}_0(\tilde{q}) = \mathcal{S}_0(q), \quad \tilde{\mathcal{T}}_0(\tilde{p}) = \mathcal{T}_0(p), \tag{45}$$

again, up to plus a constant. Recalling that $\mathcal{S} = \mathcal{S}_0 - Et$ and that $\mathcal{T} = \mathcal{T}_0 + Et$, these states mesh to

$$\mathcal{S} = \pm \mathcal{T} + cnst. \tag{46}$$

The relation (28) fixes the above sign, ensuring stability under time evolution. Thus

$$\alpha\beta = -1 \tag{47}$$

and the distinguished W states mesh to

$$\mathcal{S} = -\mathcal{T} + cnst. \tag{48}$$

Recalling the Legendre transformation $\mathcal{S} = pq - \mathcal{T}$ we have that

$$pq = \gamma, \tag{49}$$

where γ is a constant. In this case the solutions of the canonical and dual canonical equations coincide. Such states are referred to as self-dual, and correspond to

$$\mathcal{S}_0(q) = \gamma \log \gamma_q q, \quad \mathcal{T}_0(p) = \gamma \log \gamma_p p \tag{50}$$

and

$$\mathcal{U}(s) = \mathcal{V}(t) = -\frac{1}{4\gamma^2}. \tag{51}$$

Using symmetry arguments only, we have arrived at the statement that $\alpha = \beta = \pm i$, the imaginary factor being a consequence of the relation connecting time evolution of the $\mathcal{S}$ and $\mathcal{T}$ descriptions. The relation between imaginary factor and time evolution leads to an imaginary factor appearing in the relation between $\mathcal{S}_0$ and the solutions of the Schrödinger Equation (SE) The imaginary factor also hints that the evolution of time in QM is related to the presence of imaginary numbers.

7 The Equivalence Principle

Note that since q and p are not considered as being independent variables transformations to new coordinates will not be canonical. In classical mechanics we search for a canonical transformation which takes one to a system with vanishing Hamiltonian thus leading to the Hamilton-Jacobi equation. Likewise, we may search for the analogous equation found by determining the transformation of q (which induces the transformation of the dependant p), which reduces to the free system with zero energy.

Recalling that under any arbitrary transformation $\tilde{\mathcal{U}}(\tilde{s}) \neq \mathcal{U}(s)$ unless one considers a Möbius transformation. Thus different $\mathcal{U}$'s may be related by a coordinate transformation, as we have seen that two systems may be related by $q \rightarrow \tilde{q}(q)$ as defined by $\tilde{\mathcal{S}}_0(\tilde{q}) = \mathcal{S}_0(q(\tilde{(q)}))$. To find the transformation relating $\tilde{\mathcal{S}}_0$ to $\mathcal{S}_0$ then amounts to the solution of the inversion

$$q \rightarrow \tilde{\mathcal{S}}_0^{-1} \circ \mathcal{S}_0(q). \tag{52}$$

This suggests the equivalence principle of Faraggi and Matone:

> *For each pair W^a, W^b there is a coordinate transformation such that $W^a \rightarrow \tilde{W}^a(\tilde{q}) = W^b(\tilde{q})$*

Thus there will also always be a transformation going to $W = 0$ meshing to the zero energy free system.
Let us consider the (classical) transformation

$$W(q) \rightarrow W^v(q^v) \tag{53}$$

as caused by

$$\mathcal{S}_0^{cl}(q) \rightarrow \mathcal{S}_0^{clv}(q^v) = \mathcal{S}_0^{cl}(q(q^v)). \tag{54}$$

Both reduced actions should satisfy their respective CSHJE

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}_0^{cl}(q)}{\partial q}\right)^2 + W(q) = 0 \tag{55}$$

and

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}_0^{clv}(q^v)}{\partial q^v}\right)^2 + W^v(q^v) = 0. \tag{56}$$

Taking $\mathcal{S}_0^{clv}(q^v) = \mathcal{S}_0^{cl}(q)$ the definition leading to the classical v-transformation $q \rightarrow q^v = v(q)$ along with a comparison of the above two CSHJE's it is evident that

$$W^v(q^v)(dq^v)^2 = W(q)(dq)^2. \tag{57}$$

Hence, in CM consistency insists that W lies within $\mathcal{Q}^{cl}$, the space of functions transforming as quadratic differentials with respect to the classical v-transformations. Let us consider the state of vanishing energy, W^0. We see that

$$W^0(q^0) \rightarrow W^v(q^v) = (\partial_{q^v})^2 W^0(q^0) = 0 \tag{58}$$

ie, W^0 is a fixed point in the space $\mathcal{H}$ of all possible W. In CM the space $\mathcal{H}$ cannot be mapped to a point on factorization by the classical v-transformations - the EP may not be implemented consistently within CM.

A differential equation for $\mathcal{S}_0$ must be covariant under v-transformations, it must reduce to the CSHJE in the applicable limit and all the states $W \in \mathcal{H}$ have equivalency under under the said v-transformations. Covariance is simply a consistency condition and the reduction in a suitable limit is simply due to the existence of CM. However we must address the latter point with more care. The equation we wish will have the form

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}_0(q)}{\partial q}\right)^2 + W(q) + Q(q) = 0. \tag{59}$$

Let $\mathcal{Q}$ be the space of functions transforming as quadratic differentials under the v-transformations. Since we have by consistency

$$\frac{1}{2m}\left(\frac{\partial \tilde{\mathcal{S}}_0^{cl}(\tilde{q})}{\partial \tilde{q}}\right)^2 + \tilde{W}(\tilde{q}) + \tilde{Q}(\tilde{q}) = 0 \tag{60}$$

this implies that $(W + Q) \in \mathcal{Q}$. But (58) and the fact that all states $W \in \mathcal{H}$ show that $W \notin \mathcal{Q}$ and hence it follows that $Q \notin \mathcal{Q}$. The term with the classical limit $Q \rightarrow 0$ may thus be regarded as a term lending the property of covariance. In the case of the vanishing energy free system (59) becomes

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}_0^{cl}(q)}{\partial q}\right)^2 + Q(q) = 0 \tag{61}$$

since $(\partial_q \mathcal{S}_0)^2 \in \mathcal{Q}$ and $Q \notin \mathcal{Q}$ the requirement of covariance indicates that $Q = 0$ such that $\mathcal{S}_0$. In this case, $\tilde{\mathcal{S}}_0(\tilde{q}) = \mathcal{S}_0(q)$ would in all cases give $\tilde{\mathcal{S}}_0 = 0$. This would mean that $\mathcal{S}_0 = cnst$ would comprise a fixed point in the space $\mathcal{K}$ of all possible $\mathcal{S}_0$, in contradiction with the requirement that all states $W \in \mathcal{H}$ be equivalent under v-transformation. This is related to the fact that the $\mathcal{S}_0$–$\mathcal{T}_0$ Legendre duality holds unless $\mathcal{S}_0 = cnst$ or $\mathcal{S}_0 \propto q$. Thus the formalism breaks down in the case of the system corresponding to $W = 0$.

Now, let us consider the identity

$$\left(\frac{\partial \mathcal{S}_0(q)}{\partial q}\right)^2 = \frac{\beta^2}{2}(\{\exp \frac{2i}{\beta}\mathcal{S}_0, q\} - \{\mathcal{S}_0, q\}), \tag{62}$$

where β is an as yet undetermined dimensional constant. Using this identity along with (59) we see

$$W(q) = \frac{\beta^2}{4m}(\{\mathcal{S}_0, q\} - \{\exp \frac{2i}{\beta}\mathcal{S}_0, q\}) - Q(q), \tag{63}$$

This implies that

$$Q(q) = \frac{\beta^2}{4m}\{\mathcal{S}_0, q\} \tag{64}$$

(noting that another term within Q would fail to satisfy the conditions required for the analogue of the CSHJE) and from (59) that

$$W(q) = -\frac{\beta^2}{4m}\{\exp \frac{2i}{\beta}\mathcal{S}_0, q\}, \tag{65}$$

(a potential not having this form is inadmissable) which is equivalent to

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}_0(q)}{\partial q}\right)^2 + V(q) + E + \frac{\beta^2}{4m}\{\mathcal{S}_0, q\} = 0, \tag{66}$$

which becomes the CSHJE in the limit $\beta \rightarrow 0$. (65) with the identity

$$\partial_x h'^{-1/2} = 0 = \partial_x h'^{-1}\partial_x h'^{1/2} h'^{-1/2} h \tag{67}$$

and the identity

$$h'^{1/2}\partial_x h'^{-1}\partial_x h'^{1/2} = \partial_x{}^2 + \{h, x\}/2, \tag{68}$$

with $h = \exp \frac{2i}{\beta}\mathcal{S}_0$ imply [1] that,

$$\exp \frac{2i}{\beta}\mathcal{S}_0 = \frac{A\psi^D + B\psi}{C\psi^D + D\psi}, \tag{69}$$

$AD - BC \neq 0$, where ψ^D and ψ are two linearly independent solutions of the Schrödinger Equation (SE)

$$\left[-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right]\psi = E\psi. \tag{70}$$

Note that $\beta = \hbar$. The solutions to the SE may therefore be used to solve the non-linear QSHJE. Since $\mathcal{S}_0 = cnst, \mathcal{S}_0 \propto q \notin \mathcal{K}$, $\mathcal{K}$ being the space of all possible $\mathcal{S}_0$, the formulation has clear duality. The Möbius invarience of the Schwarzchild derivative allows one to choose constants such that $\mathcal{S}_0 \propto q$ does not occur. From (69) we see that the general solution of the Schrödinger equation is given by

$$\psi = \frac{1}{\sqrt{\mathcal{S}'_0}}\left(A\exp -\frac{i}{\hbar}\mathcal{S}_0 + B\exp \frac{i}{\hbar}\mathcal{S}_0\right). \tag{71}$$

The origin of the $\frac{1}{\sqrt{\mathcal{S}'_0}}$ factor is revealed upon substitution of the general solution

$$\psi = R(\mathcal{S}_0)\left(A\mathrm{exp}-\frac{i}{\hbar}\mathcal{S}_0 + B\mathrm{exp}\,\frac{i}{\hbar}\mathcal{S}_0\right) \tag{72}$$

into the SE. The presence of the self-dual state allows one to find in all physical cases a coordinate $\tilde{q}$ in which $W \in \mathcal{H}$ becomes $\tilde{W}(\tilde{q}) = 0$, in complete analogy with the classical case of the trivializing coordinates defined by a certain canonical transformation implying the CSHJE.

8 Comparison with the Copenhagen Interpretation

Two phenomena intrinsic to QM, namely tunnelling and quantization are direct consequences of the EP. The usual approach to these effects is an axiomatic interpretation of the wavefunction as a probability amplitude, requiring that the wavefunction is a member of $L^2(\mathbf{R})$. It turns out [1] that if the SE has an $L^2(\mathbf{R})$ solution, any other linearly independent solution may not belong to $L^2(\mathbf{R})$. Thus, although each distinguished level in a system has a unique wavefunction, this does not mean that the approach derived from the EP will fail to lend physical meaning to the divergent solution to the SE.

Let us label the solutions of the SE corresponding to the energy level E as Σ_E, and let σ^E be the associated physical discrete spectrum. The statement of the Copenhagen interpretation is that the wavefunction belongs to $L^2(\mathbf{R})$. The physical part of this not involving the axiomatic concept of a wavefunction is as follows:

The SE admits an $L^2(\mathbf{R})$ *solution,*

which automatically leads to the existence of a quantised spectrum $E \in \sigma^E \Leftrightarrow \psi \in L^2(\mathbf{R})$ for a given $\psi \in \Sigma_E$. In the picture of a quantised energy spectrum, the wave function serves to impose the condition that the SE gives a $L^2(\mathbf{R})$ solution, leading to the standard interpretation. It would seem that the Copenhagen interpretation is good enough to over shadow the full nature of QM - it will be observed that the physical result follows from the EP. The fact that both ψ and ψ^D appear in the present formalism is due to the under pinning Legendre duality resulting from the EP.

Quantum Tunnelling arises as a consequence of the EP by the action of the term $Q(q)$ in the QSHJE. This modification of the CSHJE allows the trajectory of a particle to penetrate classically forbidden regions. In fact the only exception to $p \in \mathbf{R}, \forall q \in \mathbf{R}$ where in the case of the QSHJE $p = \pm\sqrt{2m(-V - Q + E)}$ occurs for the infinitely deep potential well, and of course, similarly for $\dot{q}$.

9 Quantisation from the EP

To consider the conditions required for the existence of the QSHJE it is useful to consider the form (65) where $w = \psi^D/\psi$ which is equivalent to

$$\{w, q\} = -\frac{4m}{\hbar}W(q) \tag{73}$$

by Möbius symmetry. This is the consequence of the EP, hence the EP forces conditions upon w: the condition being the existence of the QSHJE, or the condition that $\{w, q\}$ exists such that

$w \neq cnst$, $w \in C^2(\mathbf{R})$, $\partial_q^2 w$ is differentiable on $\mathbf{R}$.

However, these conditions are not complete. Let us consider the chain rule of the Schwarzian derivative

$$\{h(x), x(y)\} = \left(\frac{\partial y}{\partial x}\right)^2 (\{h(x), y\} - \{x, y\}). \tag{74}$$

Therefore there is a duality of sorts relating the entries of the Schwarzian derivative

$$\{w, q\} = \left(\frac{\partial w}{\partial q}\right)^2 (\{w, w\} - \{q, w\}) = -\left(\frac{\partial w}{\partial q}\right)^2 \{q, w\} \tag{75}$$

We see from the above chain rule that $\{w, q^{-1}\} = q^4\{w, q\}$ hence (73) may be written

$$\{w, q^{-1}\} = -\frac{4m}{\hbar} q^4 W(q), \tag{76}$$

hence the EP leads us too either (73) or (76). Since

$$q \to \frac{1}{q} \tag{77}$$

causes $0^{\pm}$ to map to $\pm\infty$, the conditions on w should extend to infinity, i.e., should hold for the extended real line $\hat{\mathbf{R}} = \mathbf{R} \bigcup \{\infty\}$ and not $\mathbf{R}$ alone. Thus we should extend our conditions to

$w \neq cnst$, $w \in C^2(\hat{\mathbf{R}})$, $\partial_q^2 w$ is differentiable on $\hat{\mathbf{R}}$.

(76) also forces us to extend the property of local univalence to $\hat{\mathbf{R}}$, thus giving rise [1] to the joining conditions

$$w(-\infty) = \begin{cases} w(+\infty), & w(-\infty) \neq \pm\infty, \\ -w(+\infty), & w(-\infty) = \pm\infty. \end{cases} \tag{78}$$

Thus the conditions for use of the EP reduce to conditions S_0 must satisfy for to satisfy the QSHJE. The derivation of such conditions leads to quantization of energy without the need for any axiomatic concept of a wave function.
The formalism easily extend to systems in which

$$V(q) = \sum_{k=1}^{n} V_k(q_k), \tag{79}$$

since the Legendre transform

$$\mathcal{S}_0 = \sum_{k=1}^{n} p_k \frac{\partial \mathcal{T}_0}{\partial p_k} - \mathcal{T}_0 \tag{80}$$

is defined for any such physical system, with its involutivity giving $\mathcal{S}_0$–$\mathcal{T}_0$ duality. Thus in n dimensions for such potentials one is required to construct an equation equivalent to the decoupled one-dimensional SEs.

10 The Central Potential

Our method of solving this physical problem will therefore consist of deriving the equivalent of three Schrödinger equations and then applying the aforementioned conditions to determine the physical $L^2(\mathbf{R})$ solution which admits the correct energy spectrum.

The corresponding ratio w may then be placed into the form of the reduced action proposed by Bouda [3]:

$$\mathcal{S}_0 = \hbar \arctan\left(\frac{\sigma\psi + \nu\psi^D}{\mu\psi + \gamma\psi^D}\right) + \hbar\varsigma \tag{81}$$

this may be shown to be a solution of the QSHJE and as shown in [3] leads easily to the form of the wavefunction (71). $(\mu, \nu, \sigma, \gamma, \varsigma)$ are arbitrary real constants such that $\mu\nu \neq \sigma\gamma$. Since the SE is linear, by a rescaling of its solutions either the pair (μ, ν) or the pair (σ, γ) may be absorbed. Thus we may set for example $\sigma = \gamma = 1$ then consider μ, ν and ς $(\mu\nu \neq 1)$ as constants of integration. Note than ς is a hidden variable attributed to so-called microstates.

The hydrogenic potential is clearly spherically symmetric. Let us therefore consider the stationary SE written in the guise of spherical coordinates

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{2m}{\hbar^2}[E - V(r)]\Psi = 0, \tag{82}$$

in the case that the potential is central. Let us assume a separable wavefunction of the form

$$\Psi = R(r)S(\theta)T(\phi), \tag{83}$$

R being the radial wavefunction , S and T being the angular wavefunctions. Substituting this form of the wavefunction into (82) leads to

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} + \frac{2mr^2}{\hbar^2}(E - V(r)) = \lambda \tag{84}$$

with $\lambda = l(1+l)$ the eigenvalue of L^2, L denoting the angular momentum operator, such that $l \geq 0$,

$$\frac{d^2S}{d\theta^2} + \cot\theta\frac{dS}{d\theta} + \left(\lambda - \frac{m_l^2}{\sin^2\theta}\right)S = 0 \tag{85}$$

with m_l the eigenvalue of L_z the component operator of L in the z axis, such that $-l \leq m_l \leq l$ and

$$\frac{d^2T}{d\phi^2} + m_l^2 T = 0. \tag{86}$$

Now, (86) clearly has the correct form, however, for the other two we must find a suitable substitution such that they take the form equivalent to the one dimensional SE. Considering (84) we find that the textbook substitution

$$R(r) = \frac{X(r)}{r}, \tag{87}$$

leads to the correct form,

$$\frac{-\hbar^2}{2m}\frac{d^2X}{dr^2} + \left[V(r) + \frac{\lambda\hbar^2}{2mr^2}\right]X = EX. \tag{88}$$

Similarly, we see that considering (85) the substitution

$$Y(\theta) = \sin^{\frac{1}{2}} S(\theta), \tag{89}$$

leads to the correct form

$$\frac{d^2Y}{d\theta^2} + (\lambda + \frac{1}{4})Y + \frac{(\frac{1}{4} - m_l^2)}{\sin^2\theta} Y = 0. \tag{90}$$

Thus we may write the functions $R(r)$, $S(\theta)$ and $T(\phi)$ may be written in the form

$$X(r) = A(r)(\alpha \exp -\frac{i}{\hbar} L(r) + \beta \exp \frac{i}{\hbar} L(r)), \tag{91}$$

along with

$$Y(\theta) = B(\theta)(\gamma \exp -\frac{i}{\hbar} M(\theta) + \epsilon \exp \frac{i}{\hbar} M(\theta)) \tag{92}$$

and

$$T(\phi) = C(\phi)(\sigma \exp -\frac{i}{\hbar} N(\phi) + \omega \exp \frac{i}{\hbar} N(\phi)). \tag{93}$$

By replacing (91), (92) and (93) respectively in (88), (90) and (86) we may obtain

$$\frac{1}{2m}\left(\frac{dL}{dr}\right)^2 - \frac{\hbar^2}{4m}\{L, r\} + V(r) + \frac{\lambda\hbar^2}{2mr^2} = E, \tag{94}$$

$$\left(\frac{dM}{d\theta}\right)^2 - \frac{\hbar^2}{2}\{M, \theta\} + \frac{(\frac{1}{4} - m_l^2)}{\sin^2\theta}\hbar^2 = \left(\lambda + \frac{1}{4}\right)\hbar^2, \tag{95}$$

and

$$\left(\frac{dN}{d\phi}\right)^2 - \frac{\hbar^2}{2}\{N, \phi\} = m_l^2\hbar^2. \tag{96}$$

Substituting the value of λ and m_l deduced from (95) and (96) into (94) leads to

$$\begin{aligned} &\frac{1}{2m}\left[\left(\frac{dL}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dM}{d\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{dN}{d\phi}\right)^2\right] \\ &-\frac{\hbar^2}{4m}\left[\{L, r\} + \frac{1}{r^2}\{M, \theta\} + \frac{1}{r^2\sin^2\theta}\{N, \phi\}\right] \\ &+V(r) - \frac{\hbar^2}{8mr^2} - \frac{\hbar^2}{8mr^2\sin^2\theta} = E. \end{aligned} \tag{97}$$

It is therefore convenient to define the reduced action as

$$\mathcal{S}_0(r, \theta, \phi) = L(r) + M(\theta) + N(\phi), \tag{98}$$

the substitution of which into (97) leads to the representation of the QSHJE in three dimensions for a central potential

$$\frac{1}{2m}\nabla^2_{r,\theta,\phi}\mathcal{S}_0 - \frac{\hbar^2}{4m}\left[\{\mathcal{S}_0, r\} + \frac{1}{r^2}\{\mathcal{S}_0, \theta\} + \frac{1}{r^2\sin^2\theta}\{\mathcal{S}_0, \phi\}\right] + V(r) - \frac{\hbar^2}{8mr^2} - \frac{\hbar^2}{8mr^2\sin^2\theta} = E. \tag{99}$$

Note that only by taking the classical limit ($\hbar \to 0$) in the above equation (and not in (94), (95) and (96)) one may reduce to the CSHJE

$$\frac{1}{2m}\nabla^2_{r,\theta,\phi}\mathcal{S}_0 + V(r) = E. \tag{100}$$

(94), (95) and (96) are trivial when (84), (85) and (86) are solvable. In this case we may use the notation (81)giving

$$L(r) = \hbar \arctan\left(\frac{R(r) + \nu_r R(r)^D}{\mu_r R(r) + R(r)^D}\right), \tag{101}$$

along with

$$M(\theta) = \hbar \arctan\left(\frac{S(\theta) + \nu_\theta S(\theta)^D}{\mu_\theta S(\theta) + S(\theta)^D}\right) \tag{102}$$

and

$$N(r) = \hbar \arctan\left(\frac{\sin(m_l\phi) + \nu_\phi \cos(m_l\phi)}{\mu_\phi \sin(m_l\phi) + \cos(m_l\phi)}\right). \tag{103}$$

Here the solutions to (94), (95) and (96) are written in terms of the two independent solutions to (84), (85) and (86) respectively.

11 The Hydrogenic Potential

For the hydrogenic case we insert the well known potential

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}. \tag{104}$$

Thus we may write (84) as

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \left[\frac{\tilde{\lambda}}{\rho} - \frac{1}{4} - \frac{l(l+1)}{\rho^2}\right]R = 0 \tag{105}$$

with the convenient change of variables

$$\rho = \alpha r, \tag{106}$$

$$\alpha^2 = \frac{8m|E|}{\hbar^2}, \tag{107}$$

$$\tilde{\lambda} = \frac{2mZe^2}{4\pi\varepsilon_0\alpha\hbar^2} = \frac{Ze^2}{4\pi\varepsilon_0\hbar}\left(\frac{m}{2|E|}\right)^{\frac{1}{2}}. \tag{108}$$

Considering the dominant behavior $R(\rho)$ in the asymptotic region $\rho \to \infty$, we find that for large enough ρ, it is evident that as far as leading order terms are concerned $R(\rho) = \rho^n \exp \pm\frac{1}{2}\rho$ is a solution of (105) for finite n. This implies we look for solutions of the form

$$R(\rho) = F(\rho)\exp\left(\pm\frac{\rho}{2}\right), \tag{109}$$

$F(\rho)$ being a polynomial of finite order in ρ. Substitution of this into (105) leads to

$$\frac{d^2F}{d\rho^2} + \left(\frac{2}{\rho} \pm 1\right) \frac{dF}{d\rho} + \left(\frac{\tilde{\lambda} \pm 1}{\rho} - \frac{(1 \pm 1)}{4} - \frac{l(l+1)}{\rho^2}\right) F = 0. \tag{110}$$

This may be solved by application of the method of Froebinius [4].

Substitution of the series solution

$$F = \sum_{n=0}^{\infty} a_n \rho^{s+n} \tag{111}$$

into (110) leads (by assumption) to the relation

$$a_0 \neq 0, \tag{112}$$

$$a_1\{(s+1)s + 2(s+1) - l(l+1)\} + a_0\{(\tilde{\lambda} \pm 1) \pm s\} = 0 \tag{113}$$

and the recursion relation

$$a_{n+2}\{(s+n+2)(s+n+1)+2(s+n+2)-l(l+1)\}+a_{n+1}\{(\tilde{\lambda}\pm1)\pm(s+n+1)\}-\frac{1}{4}(1\pm1)a_n = 0. \tag{114}$$

This method leads to the conclusion that four independant solutions exist:

$$F = F_-, \tag{115}$$

such that the initial exponent of the series is l,

$$F = F_-^D, \tag{116}$$

such that the initial exponent of the series is $-l-1$,

$$F = F_+, \tag{117}$$

such that the initial exponent of the series is l,

$$F = F_+^D, \tag{118}$$

such that the initial exponent of the series is $-l-1$,
where subscripts denote the sign of the exponent in the substitution (109).
Now since the variable ρ runs from 0 to ∞ we must alter the conditions (78) to

$$w(0) = \begin{cases} w(+\infty), & w(0) \neq \pm\infty, \\ -w(+\infty), & w(0) = \pm\infty. \end{cases} \tag{119}$$

It is evident that the only form of ratio

$$w = \frac{X^D}{X} = \frac{R^D}{R} \tag{120}$$

that may satisfy the conditions (119) contains the above solution $F = F_-$ as denominator with the condition that this series solution terminates. In textbook fashion [5] this termination implies the quantisation of energy levels. The functions $R(\rho) = F_- \exp(-\frac{\rho}{2})$ are therefore the Laguerre polynomials, representing the familiar radial wavefunctions.

12 Conclusions

It is evident that the familiar spectra of energy levels of the hydrogenic potential does indeed follow directly from the application of the EP of Faraggi and Matone thus illustrating a test of the formalism. We have seen that the EP leads to the QSHJE which implies the SE. The SE then has a clear interpretation in terms of trajectories (see [6]) depending upon initial conditions, which disappear in the SE. This shows that the Copenhagen interpretation of QM hides some of its true nature in that the hidden variables we have seen depend upon fundamental constants (e.g. $\hbar$); which suggests that fundamental interactions could possibly arise within the EP . It has been shown that this new formulation of QM does reproduce the experimental phenomena of tunneling and quantisation of energy as one would require, this fact may be considered a basic test of this formalism.

The fact that QM can be derived from a principle similar to Einstein's EP implies the difficulties in quantizing gravity may be inherent in the differences between the fundamental principles of general relativity and conventional textbook QM. In fact in the formalism of Faraggi and Matone we see that the quantum potential is never zero - akin to the relativistic mass energy. Note that in [1] it is shown that one may easily generalise the QSHJE to include both the time-dependant and relativistic cases, the former leading to the time-dependant SE, the latter leading to the Klein-Gordon equation.

Thus [1] represents what may be a starting point to the end of a unification of fundamental interactions. To obtain such unification it will first be required to reconsider our present ideas regarding the nature of time, the application of trajectories and their use in regarding the concept of force at a distance. It could be the case that the nature of interaction is related to quantum potential, which may be considered to be a particles self-energy depending upon the external potential - in other words, the particle responding to external changes.

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