Ecole Polytechnique
Promotion 2010
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RAPPORT DE STAGE DE RECHERCHE

Classification of Free Fermionic Models

Non confidentiel

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Abstract

Our subject is the classification of some free-fermionic models. After a brief introduction to the string and the heterotic string, we describe the free-fermionic formalism and the model construction. We next expose one way to obtain models with $SO(10)$ GTU symmetry, or models where $SO(10)$ is broken into $SU(5) \times U(1)$ or $SU(3) \times SU(2) \times U(1)^2$ at the string level. At last, we see an algorithm and a specific computer program which can analyse the entire spectrum of a large class of models in a sensible time.

Résumé

Notre sujet est la classification de certains *free-fermionic models* (modèles de corde hétérotique de type fermions libres). Après une brève introduction à la théorie des cordes et cordes hétérotiques, nous détaillons le formalisme *free-fermionic* et la construction de modèles. Nous décrivons ensuite le moyen d'obtenir des modèles ayant pour groupe de jauge le groupe de grande unification $SO(10)$, ou bien des modèles où celui-ci est brisé en $SU(5) \times U(1)$ ou $SU(3) \times SU(2) \times U(1)^2$ à l'échelle d'énergie des cordes. Enfin, nous nous intéressons à un algorithme et un programme java particulier permettant d'analyser le spectre d'une large classe de modèles en temps raisonnable.

Remerciements

Je souhaite tout d'abord remercier Pierre Fayet pour m'avoir permis de faire ce stage à Liverpool.

I would like to thank Alon Faraggi for giving me the opportunity to have this first contact with string theory, and for his answers when I had questions.

At last, I would like to thank warmly all the PhD students of the TP department of the university of Liverpool.

Merci beaucoup à Hasan, Panos, Viraf, Paul, David, Owen, Stephen, Josh, Tomas, Jaclyn et Siraj; pour votre aide lorsque j'avais des questions, pour votre accueil chaleureux et votre hospitalité. Sans vous, ce stage n'aurait pas été le même.

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Introduction

Many popular science works take an interest in what is called the quest for a "theory of everything". The main question is often: "is string theory the good candidate?". This formulation is probably grandiloquent... This "quest" is more about the next step we should take, beyond the standard model, in our unification process.

The Standard Model was finalized in its current formulation in the mid-1970s. The discovery of the bottom quark (1977), the top quark (1995), the tau neutrino (2000), the possible detection of the Higgs boson (2012), and its success in explaining a wide variety of experimental results have given to the standard model the status of a theory of "almost" everything. "Almost" because the standard model doesn't take into account the general theory of relativity, and leave some unexplained physical phenomena, such as the universe expansion, the dark matter or the neutrino oscillation. Moreover, it seems arbitrary, and some possible embedding of the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ in a bigger group such as $SU(5)$ or $SO(10)$, reducing the number of free parameter lets physicists think that this theory may be the tip of the iceberg of some more general theory... This theory should deal with big energy scale, where the gravity is no more negligible compared to other forces: this theory will have to encompass gravity.

Superstring may be this theory. It has one undisputable success: it provides the only known consistent theory of quantum gravity. The basis of the theory is quite seducing by it simplicity: we used to describe punctual particles by a one-dimension trajectory (most often, with the "time"). What happens if we suppose that physical object have now a two dimensional trajectory? We have now one space dimension objects, strings, which have some new degrees of freedom: they can oscillate. These different oscillation modes can be interpreted as different particles.

There exist five different string theories, which have all been known since the middle of the 1980s. Some relationships between them were found, and the M-theory, limit of one of those five theories, has made it clear that we indeed have here just one theory: these five string theories, plus the M-theory, are just limits of a more general theory.

We will here focus on one of the five different theory, the $E_8 \times E_8$ heterotic string. This theory makes easier the embedding of the Standard Model gauge group in $SO(10)$. We will use there the free-fermionic formulation, which has been studied since the 80s.

With this formulation, we can break $SO(10)$ into the Standard Model Gauge group at the string level, or choose to keep it intact, or even break it partially and explain the observation of the Standard Model gauge group by an other breaking at a lower level.

In the first part of this report, we will introduce a quick introduction to string theory. We will then expose the free-fermionic formalism, how we can justify it physically and how it allows us to create models. The third part will be dedicated to a program made to compute and test different free fermionic models.

Chapter 1

String theory: introduction to the bosonic and fermionic string

1.1 The classical relativistic string

We will describe here a basic introduction to string theory. We will not try to justify what we assert. We will just introduce the first concepts and ideas of string theory.

In quantum mechanics, we obtain the physical laws, the Hamiltonian, the following way: We first study the classical laws of motion. With this, we find an action $\mathcal{S}$ and a Lagrangian $\mathcal{L}$, and conjugate variables q_i, p_i. We then use the correspondence principle to define some operators $\hat{q}_i, \hat{p}_i$.

In string theory, the way to proceed is more or less the same: the physical system we start with is the classical string. We first go from the nonrelativistic string to the relativistic one, and (often after a change of coordinates to express the theory in "light cone coordinates", a new set of coordinates) use the correspondence principle. The theory we obtain, the relativistic bosonic quantum string, is not enough to describe all particles: We add to its action an other contribution, the standard Dirac action for fermions. We obtain then what is called the Superstring.

The motion of any point particle can be described by only one time-like parameter. In classical nonrelativistic physics, we usually use what we call the time, a parameter t with which all physical trajectories can be parameterized. In relativistic physics, time has lost its mysterious universal quality, but proper time can be used for almost all physical trajectories (not for the light), and every physical trajectory of particles can always be parameterised by only one parameter. With strings, it is now different: If, in space-time, the trajectory of a point particle is of dimension one, it is obvious that it is of dimension two for a string. Thus, to describe a string, one need two parameters τ and σ: The string propagates in a D-dimensional Lorentzian space-time (we will not suppose $D = 4$) which sweeps out a two-dimensional surface, the world-sheet. The position of the string is described by a mapping function $X^\mu(\tau, \sigma)$. The world sheet is bounded in the σ direction

(we will take $\sigma \in [0, \pi]$ for bosons and $\sigma \in [-\pi, \pi]$ for fermions), and infinite in the τ one (we will take $\tau \in (-\infty, +\infty)$).

1.1.1 The Nambu-Goto action for the relativistic string

The relativistic point particle action is $S = -mc \int ds$: the trajectory of the particle is the one which minimizes the proper time $\frac{1}{c} \int ds$: we can show that this action is Lorentz invariant. The action of the string is obtained in the same way: the trajectory of the string will be the one which minimizes the "proper area":

$$A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}\right)^2}. \tag{1.1}$$

The action of the bosonic string is then the Nambu-Goto action:

$$S = -\frac{T_0}{c} \int d\tau d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}\right)^2}, \tag{1.2}$$

where T_0 can be interpreted as a tension, the "string tension", and "$\cdot$" is the Lorentz product.

This action has to be invariant under reparameterization. This is easy to prove ([1]) with a new way to write this action: The induced (by the Lorentz space in which the string lives) metric on the world-sheet is $\gamma_{\alpha\beta} = \frac{\partial X}{\partial \xi^\alpha} . \frac{\partial X}{\partial \xi^\beta}$, where $\xi^1 = \tau$ and $\xi^2 = \sigma$. Actually, lengths in the target Minkowski space (which metric is $\eta_{\mu\nu}$) can be written as:

$$-ds^2 = dX^\mu dX_\mu = \eta_{\mu\nu} dX^\mu dX^\nu = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} d\xi^\alpha d\xi^\beta \tag{1.3}$$

Then the Nambu-Goto action can be written as:

$$S = -\frac{T_0}{c} \int d\tau d\sigma \sqrt{-\gamma}, \quad \gamma = det(\gamma_{\alpha\beta}) \tag{1.4}$$

Which is invariant, thanks to the change-of-variable theorem.

This action is associated to the following density of Lagrangian $\mathcal{L}$:

$$\mathcal{L}(\dot{X}^\mu, X^{\mu\prime}) = -\frac{T_0}{c} \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}, \tag{1.5}$$

where he have used the notation $\dot{X}^\mu \equiv \frac{\partial X}{\partial \tau}$ and $X^{\mu\prime} \equiv \frac{\partial X}{\partial \sigma}$.

1.1.2 The equation of motion

With the classical manipulation of variation calculus, from $\mathcal{L}$ we can find the equation of motion:

$$0 = \frac{\partial \mathcal{P}^{\tau}_{\mu}}{\partial \tau} + \frac{\partial \mathcal{P}^{\sigma}_{\mu}}{\partial \sigma} = \partial_{\alpha} \mathcal{P}^{\alpha}_{\mu}, \tag{1.6}$$

Where

$$\mathcal{P}^{\tau}_{\mu} \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}} \qquad \mathcal{P}^{\sigma}_{\mu} \equiv \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}}. \tag{1.7}$$

This can also be seen as the conservation of a current.

In the general case, the expressions of the equation of motion, in terms of X, are quite complicate. Nevertheless, it is possible to find ([1]) some parameterization where this equation becomes:

$$\ddot{X}^{\mu} - X^{\mu\prime\prime} = 0 : \tag{1.8}$$

this is just a wave equation.

There exist two types of strings:

- **The closed strings**, where X is periodic in σ: $X(\tau, \sigma) = X(\tau, \sigma + \pi)$. In this case, the string can be seen as a loop, with no end.

- **The open strings**. In this case, the string can have two boundary conditions:

  1. The Dirichlet boundary condition: the ends of the string are fixed, the string is attached to some structure, a D-Brane:

  $$X^{\mu}(\sigma = 0) = X^{\mu}_{0}, \qquad X^{\mu}(\sigma = \pi) = X^{\mu}_{\pi}. \tag{1.9}$$

  2. The Neumann boundary condition: the ends of the string are free:

  $$\frac{\partial X^{\mu}}{\partial \sigma}(\sigma = 0) = \frac{\partial X^{\mu}}{\partial \sigma}(\sigma = \pi) = 0. \tag{1.10}$$

In the following, *we will only consider closed strings*. We will use the new variables:

$$u = \tau + \sigma, \qquad v = \tau - \sigma. \tag{1.11}$$

The solution of the wave equation becomes:

$$X^{\mu} = X^{\mu}_{L}(u) + X^{\mu}_{R}(v). \tag{1.12}$$

Using the mode expansion, we find:

$$\begin{aligned} X_L^\mu(u) &= \frac{1}{2}x_{L,0}^\mu + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^\mu u + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar{\alpha}_n^\mu}{n}e^{-inu}, \\ X_R^\mu(v) &= \frac{1}{2}x_{R,0}^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu v + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-inv}. \end{aligned} \tag{1.13}$$

The periodicity condition gives: $\bar{\alpha}_0^\mu = \alpha_0^\mu$.
Finally, coming back to τ and σ:

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}). \tag{1.14}$$

1.2 The quantized string

1.2.1 Quantization of the bosonic string

The method to quantize the string is similar to what is done in quantum field theory, with the electromagnetic field for example. We define the canonical commutation relation first:

$$[X^I(\tau,\sigma),(P)^{\tau,J}(\tau,\sigma')] = i\delta(\sigma-\sigma')\eta^{IJ}, \tag{1.15a}$$
$$[x_0^-(\tau),p^+(\tau)] = -i, \tag{1.15b}$$
$$[X^I(\tau,\sigma),X^J(\tau,\sigma')] = [\mathcal{P}^{\tau I}(\tau,\sigma),\mathcal{P}^{\tau J}(\tau,\sigma')] = 0, \tag{1.15c}$$
$$[x_0^-(\tau),X^I(\tau,\sigma)] =, [x_0^-(\tau),\mathcal{P}^{\tau I}(\tau,\sigma)] = 0, \tag{1.15d}$$
$$[p^+(\tau),X^I(\tau,\sigma)] =, [x^+(\tau),\mathcal{P}^{\tau I}(\tau,\sigma)] = 0. \tag{1.15e}$$

Here $x_0^-(\tau)$ and $p^+(\tau)$ are just (the operators associated to the) coordinates and momentum of the string in the light-cone coordinates: the light cone coordinates are just a smart new way to replace the coordinates x^μ and p^μ for $\mu = 0,1$ with new ones $x^\pm$ and $p^\pm$.

This leads to some commutation relations for the α_n^μ which can be interpreted as creation and annihilation operators (up to a factor $\sqrt{n}$):

$$[\bar{a}_m^I,\bar{a}_n^{J\dagger}] = \delta_{m,n}\eta^{IJ}, \quad [a_m^I,a_n^{J\dagger}] = \delta_{m,n}\eta^{IJ}, \tag{1.16a}$$
$$[x_0^I,p^J] = i\eta^{IJ}, \tag{1.16b}$$

with:

$$\alpha_n^I = a_n^I\sqrt{n} \quad and \quad \alpha_{-n}^I = a_n^{I\dagger}\sqrt{n}, \tag{1.17a}$$

$$\bar{\alpha}_n^I = \bar{a}_n^I\sqrt{n} \quad and \quad \bar{\alpha}_{-n}^I = \bar{a}_n^{I\dagger}\sqrt{n}. \tag{1.17b}$$

We can also define $N^\perp$ and $\bar{N}^\perp$, the number operators:

$$N^\perp \equiv \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I, \quad and \quad \bar{N}^\perp \equiv \sum_{n=1}^{\infty} n \bar{a}_n^{I\dagger} \bar{a}_n^I. \tag{1.18}$$

1.2.2 The states space

We are now able to build the whole state space for the quantum closed string. The ground states (with fixed energy and momentum) are $|p^+, \vec{p}_T\rangle$, they are annihilated by the left and right moving annihilation operators. The general basis vector is any:

$$|\lambda, \bar{\lambda}\rangle = \left[\prod_{n=1}^{\infty}\prod_{I=2}^{25}(a_n^{I\dagger})^{\lambda_{n,I}}\right]\left[\prod_{n=1}^{\infty}\prod_{I=2}^{25}(\bar{a}_n^{I\dagger})^{\bar{\lambda}_{n,I}}\right]|p^+, \vec{p}_T\rangle, \tag{1.19}$$

with the condition that this element is such that $\bar{N}^\perp = N^\perp$. Here the $\lambda_{n,I}$ and $\bar{\lambda}_{n,I}$ are non-negative integers. Our space coordinates go from 2 to 25: This comes from the fact that the coordinate $I = 1$ was used to create the light cones coordinates. As we can see, *we have 25 space dimensions* (plus one time dimension). This comes from the requirement that the quantum theory must be Lorentz invariant: This gives a critical dimension, which is $D = 26$ ([1], [2]).

A general state is obtained by a linear combination of these elements, and an integration over all the $|p^+, \vec{p}_T\rangle$.

1.2.3 Superstring theories

Bosonic string theory can't give a realistic theory: it doesn't contain any fermionic particle. Moreover, it contains tachyons (particles that moves faster than light) which make the theory unstable. We will describe here quickly how we go from string theory (with just bosons) to superstring theory (with bosons and fermions). To add some fermionic fields to our theory, we introduce some more world-sheet fields $\psi^\mu(\tau, \sigma)$. They are anticommuting variables:

$$\{\psi_A^\mu(\tau, \sigma), \psi_B^\nu(\tau, \sigma')\} = \pi\delta_{AB}\delta(\sigma - \sigma')\eta^{\mu\nu}. \tag{1.20}$$

The superstring action is just the string one we defined in (1.2), plus the standard Dirac action for fermions, ie:

$$S = -\frac{T}{2}\int d\tau d\sigma(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\mu), \tag{1.21}$$

with ρ^0 and ρ^1 the Dirac matrices:

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.22}$$

For superstrings, the critical dimension is different: here $D = 10$ ([1], [2]).

The equations of motion for fermions, which can be derived from the Dirac action, impose some boundary conditions on the fermions. We have two possibilities for the boundary conditions: periodic ("Ramond boundary conditions") or antiperiodic ("Neveu-Schwartz boundary conditions"):

- $\psi^I(\tau, \pi) = +\psi^I(\tau, -\pi)$: R boundary condition.
- $\psi^I(\tau, \pi) = -\psi^I(\tau, -\pi)$: NS boundary condition.

Here we choose to define the world-sheet for $\sigma \in [-\pi, \pi]$.

The mode expansion gives:

- For R states:
$$\psi^I(\tau, \pi) = \sum_{n\in\mathbb{Z}} d_r^I e^{-ir(\tau-\sigma)}.$$
- For NS states:
$$\psi^I(\tau, \pi) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_r^I e^{-ir(\tau-\sigma)}.$$

There is one ground state $|NS\rangle$ for NS states, whereas Ramond ground states $|R^A\rangle$ are degenerate. Thus, the states in NS and R sectors are of the form:

- R states:
$$\prod_{n=1}^{\infty}\prod_{I=2}^{9}(\alpha_{-n}^I)^{\lambda_{n,I}} \prod_{m\in\mathbb{N}}\prod_{J=2}^{9}(d_{-m}^I)^{\rho_{m,J}}|R^A\rangle \oplus |p^+, \vec{p}_T\rangle.$$
- NS states:
$$\prod_{n=1}^{\infty}\prod_{I=2}^{9}(\alpha_{-n}^I)^{\lambda_{n,I}} \prod_{r\in\mathbb{N}+\frac{1}{2}}\prod_{J=2}^{9}(b_{-r}^I)^{\rho_{r,J}}|NS\rangle \oplus |p^+, \vec{p}_T\rangle.$$

Here the ρ are either 0 or 1, and the b and d operators anticommute.

To obtain the full closed string theory, we have to combine a left moving sector (NS or R) with a right moving one (NS or R). There is, as for the bosonic string, a constraint on $(N, N^{\perp})$.

Chapter 2

Free Fermionic models

The construction of a free-fermionic model in 4D-heterotic string is a computing problem, which can be done without any knowledge of string theory. First, we will give here the way we can physically justify it. Then, we will describe how the algorithm works.

2.1 From the physics to the mathematical structure

2.1.1 What is a Free Fermionic model?

As we have seen, the classical solution to the string equation of motion can be expanded in term of "left-moving" and "right-moving" modes. These two sectors are almost decoupled: they just have to contribute equally to the mass of the string (this is equivalent to the condition $N = N^{\perp}$).

We are free to take different theories for the left and the right sector. For the heterotic string (the one we use to build Free Fermionic models), we choose the left sector to be supersymetric and the right sector to be bosonic (non-supersymetric).

The critical dimension is 26 for the non-supersymetric string, and 10 for the supersymetric: as our world seems to be 4 dimensional, we have to explain how our theory can have these extra dimensions we can't see.

In the Free Fermionic formulation, the idea is to formulate the string theory directly in 4 dimension: the extra degrees of freedom have different interpretations.

We first go from 26 to 10 dimensions in the right sector, by interpreting the 16 extra dimensions as linked to the matter gauge group, and some other extra gauge group. Then, we have 10 dimensions on the left and on the right sector. We interpret 6 of them (on each sector) as compatified space-like dimensions.

In order to respect the critical dimension, and to compensate for the conformal anomaly which appears after quantization, we must have $n_L = 44$ right-moving fermions ϕ^a and $n_L = 18$ left-moving fermions χ^I, in addition to the four superpartners ψ^μ of the

space-time coordinates ([3]).

Those conditions are dictated by "invariance under local super-reparameterizations of the world-sheet". This also leads to the supercharge:

$$T_F = \psi^\mu \partial X_\mu + f_{IJK}\chi^I\chi^J\chi^K, \tag{2.1}$$

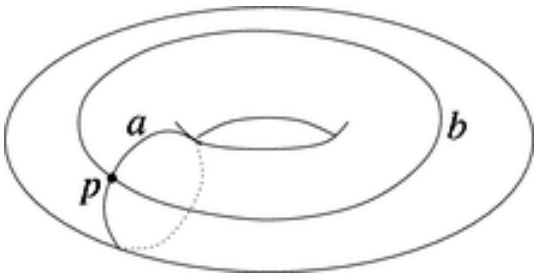
where f_{IJK} are some normalized structure constant of a semi simple Lie group with 18 generators ([6]).

String perturbation theory is an expansion in the topology of the two-dimensional world-sheet. Topologically distinct surfaces have a different genus (the "number of hole"). String scattering amplitudes are defined as path integrals over the two-dimensional quantum field theory on the world-sheet. This takes into account the infinite number of massive modes which could be exchanged in the scattering process.

2.1.2 The parallel transport around the world-sheet

Let us consider a genus g compact Riemann surface M which describe the contribution to the g-loop vacuum to vacuum string amplitude. Then the fundamental group $\pi_1(M)$ of the surface is generated by a canonical basis $(\alpha_i, \beta_i)_{i=1..g}$

For the torus (g=1), we are just considering the basis (a,b):



The transport of right-moving fermions ϕ^a around some loop $\alpha \in \pi_1(M)$ is linear: there exist some matrices $R(\alpha)$ such as:

$$\phi^a \xrightarrow{after\ transport\ around\ \alpha} R(\alpha)^a{}_b\phi^b. \tag{2.2}$$

Here the free energy-momentum current $\phi^a\phi_a$ should be unchanged, so, by definition, $R(\alpha)$ is orthogonal.

The transformation of the left movers must in addition leave the supercurrent T_F invariant up to a sign. Then there exist some matrices $L(\alpha)$ such that:

$$\begin{aligned} \psi^I &\xrightarrow{after\ transport\ around\ \alpha} \psi^\mu, \\ \chi^I &\xrightarrow{after\ transport\ around\ \alpha} L(\alpha)^I{}_J\chi^J, \end{aligned} \tag{2.3}$$

with $L(\alpha)$ an orthogonal matrix leaving $\psi^\mu \partial X_\mu$ and $f_{IJK}\chi^I\chi^J\chi^K$ unchanged up to a sign.

Then we have $f_{I'J'K'}\chi^{I'}\chi^{J'}\chi^{K'} = \pm f_{IJK} L(\alpha)^{I'}_{\ I} L(\alpha)^{J'}_{\ J} L(\alpha)^{K'}_{\ K} \chi^I\chi^J\chi^K$, so:

$$f_{IJK} L(\alpha)^{I'}_{\ I} L(\alpha)^{J'}_{\ J} L(\alpha)^{K'}_{\ K} = \pm f_{I'J'K'}. \tag{2.4}$$

This sign will from now be written as $\delta_a = \pm 1$.

We see here that $L(\alpha)$ and $R(\alpha)$ define two different representations of $\pi_1(M)$, we will call them $\mathcal{L}$ and $\mathcal{R}$: the $L(\alpha)$ acts on the vector space $\mathcal{L}$ generated by the (ψ^μ, χ^I) and the $R(\alpha)$ acts on $\mathcal{R}$ generated by the (ϕ^a). Then we can extend the $L(\alpha), R(\alpha)$ to $L(\alpha)\oplus R(\alpha)$ acting on $\mathcal{L}\oplus\mathcal{R}$: we still have orthogonal operators, which are diagonalizable in $\mathbb{C}$.

We would like them to commute with each other, in order to be able to co-diagonalize all of them. We can't expect a priori that any $L(\alpha_1)$ will commute with any $L(\alpha_2)$: this would be always true if $\pi_1(M)$ was commutative. Indeed, $\pi_1(M)$ is freely generated by the $(\alpha_i, \beta_i)_{i=1..g}$ up to the only constraint:

$$\prod_{i=1}^{g} \alpha_i\beta_1\alpha_i^{-1}\beta_i^{-1} = 1. \tag{2.5}$$

This is the presentation of the group $\pi_1(M)$ (this group is the "biggest" group satisfying this constraint): we see here that $\pi_1(M)$ is non-abelian for $g \geq 2$. Therefore, nothing indicates that $L(\alpha_1)L(\alpha_2)f = L(\alpha_2)L(\alpha_1)f$ in general...

But nobody has found a way to deal with the general non commutative case ([4]).

We will choose here to study a simpler case, when those matrices commute: we then obtain commuting boundary conditions for fermions. We can co-diagonalize the $L(\alpha)$ and $R(\alpha)$ in a same (complex) basis of $\mathcal{L}\oplus\mathcal{R}$: $(f_1^r, ..., f_k^r, f_1^c, ..., f_l^c, f_1^{c*}, ..., f_l^{c*})$, where the f_i^r are real fermions and the f_j^c are complex. We must have $k + 2l = 66$, and as $L(\alpha)\oplus R(\alpha)$ is a real orthogonal matrix, it is only determined by its image on $f_1^r, ..., f_k^r, f_1^c, ..., f_l^c$.

On f_i^r, it can only be ± 1 (the only eigenvalues possible for an orthogonal operator in $\mathbb{R}$). On f_i^c, it can be anything in $\mathcal{U} = \{z \in \mathbb{C} \mid |z| = 1\}$, which will give the angle of the rotation $L(\alpha)\oplus R(\alpha)$ on the plane f_i^c, f_i^{c*}.

Thus for any $\alpha \in \pi_1(M)$, we can resume the boundary conditions by a $(k+l)$ vector:

$$\alpha = [\alpha(f_1^r), ..., \alpha(f_k^r), \alpha(f_1^c), ..., \alpha(f_l^c)]. \tag{2.6}$$

Under a parallel transport around this loop, for a fermion of the basis:

$$f \xrightarrow{\alpha} -e^{i\pi\alpha(f)} f. \tag{2.7}$$

Thus, only α modulo 2 has a meaning, and a "complete spin-structure assignment to all fermions in a genus-g surface" can be specified by $2g$ vectors $\alpha_1 ..., \beta_g$.

We have here almost all the ingredients to describe our model. Indeed, our "physical world" can be describe by all the partition functions Z_g associated to surfaces M_g of different genus g. Here, Z_g is, in the case were we only have complex fermions:

$$Z_g = \int_{M_g} [D\Omega] \sum_{\substack{spin \\ structure}} C \begin{bmatrix} \alpha_1 & ... & \alpha_g \\ \beta_1 & ... & \beta_g \end{bmatrix} X \begin{bmatrix} \alpha_1(\psi) & ... & \alpha_g(\psi) \\ \beta_1(\psi) & ... & \beta_g(\psi) \end{bmatrix} (\Omega) \times \prod_f \Theta_0 \begin{bmatrix} \alpha_1(f) & ... & \alpha_g(f) \\ \beta_1(f) & ... & \beta_g(f) \end{bmatrix} (\Omega), \tag{2.8}$$

where Ω is the period matrix of the surface, and:

$$\Theta_0 \begin{bmatrix} \alpha_1 & ... & \alpha_g \\ \beta_1 & ... & \beta_g \end{bmatrix} (\Omega) = \sum_{\mathbf{n} \in \mathbb{Z}^g} exp(i\pi(\mathbf{n} + \frac{1}{2}\alpha)^\top \Omega(\mathbf{n} + \frac{1}{2}\alpha) + 2i\pi(\mathbf{n} + \frac{1}{2}\alpha).(\frac{1}{2}\beta) - \frac{1}{2}i\pi\alpha.\beta). \tag{2.9}$$

The $C \begin{bmatrix} \alpha_1 & ... & \alpha_g \\ \beta_1 & ... & \beta_g \end{bmatrix}$ are the coefficients of spin structure, they have to factorize in a product of one-loop coefficients $C \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \times ... \times C \begin{bmatrix} \alpha_g \\ \beta_g \end{bmatrix}$. Other coefficients appear in Z_g, but we will not talk about them here.

In string theory, we have some symmetries of the physical world: the "modular transformations". For one torus ($g = 1$), those transformations are $\Omega \to \Omega + 1$ (for a torus of length 1 along one of the axes), which leave the torus invariant, and $\Omega \to -\frac{1}{\Omega}$, which swaps the two coordinates and reorients the torus. In general, for any g-genus surface M_g, thanks to factorisation, we just have to impose invariance under some non-trivial transformation of the double torus, such as $\Omega \to \Omega - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Those symmetries are symmetries of our description of the world, and to each set of partition functions Z_g is associated one possible world: then the partition functions must be invariant under the modular transformations.

The three modular conditions give three conditions on the spin-structure coefficients:

$$C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = e^{i\pi(\alpha.\alpha+\mathbb{1}.\mathbb{1})/4} C\begin{bmatrix}\alpha\\ \beta-\alpha+\mathbb{1}\end{bmatrix}, \tag{2.10a}$$

$$C\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix} = e^{i\pi(\alpha.\beta)/2} C\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}, \tag{2.10b}$$

$$C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} C\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix} = \delta_\alpha\delta_{\alpha'} e^{-i\pi(\alpha.\alpha')/2} C\begin{bmatrix}\alpha\\ \beta+\alpha'\end{bmatrix} C\begin{bmatrix}\alpha'\\ \beta'+\alpha\end{bmatrix}. \tag{2.10c}$$

$$\tag{2.10d}$$

Here the scalar product between vectors is defined as (if some fermions are not complex):

$$\alpha.\beta = \frac{1}{2}\sum_{\substack{real\\ transv\ left}} \alpha(f)\beta(f) + \sum_{\substack{complex\\ transv\ left}} \alpha(f)\beta(f) - \frac{1}{2}\sum_{\substack{real\\ transv\ right}} \alpha(f)\beta(f) - \sum_{\substack{complex\\ transv\ right}} \alpha(f)\beta(f). \tag{2.11}$$

(The fermions ψ are the only not-transverse fermions.)

2.1.3 The underlying group structure

The equation (2.10) hide an interesting mathematical structure: an abelian group. In fact, (2.10b) and (2.10c) give that:

$$C\begin{bmatrix}\alpha\\ \beta\end{bmatrix} C\begin{bmatrix}\alpha\\ \gamma\end{bmatrix} = \delta_\alpha\delta_\gamma C\begin{bmatrix}\alpha\\ \beta+\gamma\end{bmatrix} C\begin{bmatrix}\gamma\\ 0\end{bmatrix}. \tag{2.12}$$

Then, with first $\beta=\gamma=0$, we see that either $C\begin{bmatrix}\alpha\\ 0\end{bmatrix}=0$ or $C\begin{bmatrix}\alpha\\ 0\end{bmatrix} = \delta_\alpha C\begin{bmatrix}0\\ 0\end{bmatrix}$. Here we can normalize $C\begin{bmatrix}0\\ 0\end{bmatrix}$ to 1. Then:

$$\Xi = \{\alpha | C\begin{bmatrix}\alpha\\ 0\end{bmatrix} = \delta_\alpha\} \tag{2.13}$$

is an abelian group: 2.12 with $\alpha=0$ says that it is stable by $+$. And with (2.10a), $\mathbb{1}\in\Xi$. Moreover, for all nonzero $C\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$, we have $\alpha,\beta\in\Xi$.

Ξ could be infinite. For simplicity, we will just focus on the case where Ξ is finite. In

this case, with in particular the fundamental theorem of finitely generated abelian groups, two theorems have been shown ([4]), which give a way to build free fermionic models. The first one is about the group Ξ. Given a group Ξ, the second is about the models we can build. I recall here the main result, which will be useful in our future model building:

Theorem 1. *To any consistent fermionic string theory there correspond a finite additive group of vectors of boundary conditions:*

$$\Xi \simeq \mathbb{Z}_{N_1} \oplus ... \oplus \mathbb{Z}_{N_k},$$

generated by a canonical basis $b_1, ..., b_k$ such as:

1. $\forall i,\ \sum m_i b_i = 0 \iff m_i \equiv 0[N_i]$,
2. $\frac{1}{2} N_1 b_1 = \mathbb{1}$,
3. *With* $N_{ij} = N_i \vee N_j$, $N_{ij} b_i . b_j \equiv 0[4]$,
4. $N_i b_i^2 \equiv 0[8]$ *if* N *even.*
5. *some other conditions...*

(here $\vee$: least common multiple and $\wedge$: greater common divisor).

Theorem 2. *For any such group Ξ, there exist $2\prod_{i>j} N_i \wedge N_j$ consistent string theories. These correspond to the different choices for every one-loop coefficients $C\begin{bmatrix} b_i \\ b_j \end{bmatrix}, i > j$, such that:*

1. $C\begin{bmatrix} b_i \\ b_j \end{bmatrix} = \delta_{b_i} e^{2\pi i n_i/N_j} = \delta_{b_j} e^{i\pi b_i . b_j/2} e^{2\pi i m_j/N_i}$, *$m_j$ and n_i integers,*
2. $C\begin{bmatrix} b_1 \\ b_1 \end{bmatrix} = \pm e^{i\pi b_1^2/4}$*: two choices for this c.*

To find the coefficients on all pairs of element of Ξ, we have the following rules:

1. $C\begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = e^{i\pi(\alpha.\alpha + \mathbb{1}.\mathbb{1})/4} C\begin{bmatrix} \alpha \\ \beta_1 \end{bmatrix}^{N_1/2}$,
2. $C\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = e^{i\pi(\alpha.\beta)/2} C\begin{bmatrix} \beta \\ \alpha \end{bmatrix}^*$,
3. $C\begin{bmatrix} \alpha \\ \beta + \gamma \end{bmatrix} = \delta_\alpha C\begin{bmatrix} \alpha \\ \beta \end{bmatrix} C\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}$.

2.1.4 The Hilbert space

To each sector α in Ξ will correspond an Hilbert space $\mathcal{H}_\alpha$. The states in this space have to satisfy the Virasoro conditions, ie must have the same mass in the left and right sectors:

$$M_L^2 = -\frac{1}{2} + \frac{\alpha_L.\alpha_L}{8} + N_L = -1 + \frac{\alpha_R.\alpha_R}{8} + N_R = M_R^2, \tag{2.14}$$

where α_L and α_R correspond to the left and right part of alpha and N_L and N_R are the total left and right oscillator number acting on the vacuum $|0\rangle_\alpha$. They are determined by the frequencies of the oscillators, which are given respectively by:

$$\nu(f) = \frac{1+\alpha(f)}{2} \quad \text{and} \quad \nu(f^*) = \frac{1-\alpha(f)}{2}, \tag{2.15}$$

for a fermion f and its conjugate f^*. Then the expression of N_L and N_R are:

$$N_L = \sum_{\substack{f \\ L-osc}} \nu_f + \sum_{\substack{f^* \\ L-osc}} \nu_f, \qquad N_R = \sum_{\substack{f \\ R-osc}} \nu_f + \sum_{\substack{f^* \\ R-osc}} \nu_f. \tag{2.16}$$

In our model construction, we are interested in model with no negative-mass states, and in the massless states: Since the states with a mass would have a mass of the order of the Plank mass (which is enormous, unattainable), they have no phenomenological meaning. Each complex fermion generates a U(1) current, with a charge with respect to the gauge group of the model given by:

$$Q_\nu(f) = \frac{\alpha(f)}{2} + F. \tag{2.17}$$

Here $F_\alpha(f)$ is $+1$ on an f oscillator and -1 on a conjugate oscillator f^*. On a degenerated vacuum with Ramond states, it is 0 on $|+\rangle$ and -1 on $|-\rangle$.

Not all the states we can build in the sector alpha are in $\mathcal{H}_\alpha$: some are projected out by the GSO projection.

If we come back to the partition function, which can be written as:

$$Z = \int \frac{d\tau d\bar{\tau}}{Im(\tau)^2} \sum_{\alpha,\beta\in\Xi} C\begin{bmatrix}\alpha \\ \beta\end{bmatrix} Tr_{\mathcal{H}_\alpha}(e^{i\pi\beta.F_\alpha} e^{i\pi\tau.H_\alpha}) \tag{2.18}$$

Where H_α is the halmitonian in $\mathcal{H}_\alpha$, and

$$\beta.F_\alpha \equiv \{\sum_{left} - \sum_{right}\}\beta(f)F_\alpha(f). \tag{2.19}$$

If we decompose all β in the basis: $\beta = \sum_{n_1,...,n_k} n_1 b_1 + ... + n_k b_k$ with $0 \leq n_i < N_i$, and use the formula $C \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = e^{i\pi(\alpha.\beta)/2} C \begin{bmatrix} \beta \\ \alpha \end{bmatrix}^*$, we obtain:

$$Z = \int \frac{d\tau d\bar{\tau}}{Im(\tau)^2} \sum_{\alpha \in \Xi} \delta_\alpha Tr_{\mathcal{H}_\alpha} \Big\{ \prod_{b_i} (1 + \delta_\alpha C \begin{bmatrix} \alpha \\ b_i \end{bmatrix} e^{i\pi b_i . F_\alpha} + \Big[\delta_\alpha C \begin{bmatrix} \alpha \\ b_i \end{bmatrix} e^{i\pi b_i . F_\alpha} \Big]^2 + \\ ... + \Big[\delta_\alpha C \begin{bmatrix} \alpha \\ b_i \end{bmatrix} e^{i\pi b_i . F_\alpha} \Big]^{N_i - 1}) \; e^{i\pi\tau . H_\alpha} \Big\}. \quad (2.20)$$

The operator $(1 + \delta_\alpha C \begin{bmatrix} \alpha \\ b_i \end{bmatrix} e^{i\pi b_i . F_\alpha} + \Big[\delta_\alpha C \begin{bmatrix} \alpha \\ b_i \end{bmatrix} e^{i\pi b_i . F_\alpha} \Big]^2 + ... + \Big[\delta_\alpha C \begin{bmatrix} \alpha \\ b_i \end{bmatrix} e^{i\pi b_i . F_\alpha} \Big]^{N_i - 1})$ is (proportional to) a projector, because it's square is (proportional to) himself. Then the trace is obtained on the image of this operator (subspace of $\mathcal{H}_\alpha$), ie the states $|s\rangle$ such as

$$\delta_\alpha C \begin{bmatrix} \alpha \\ b_i \end{bmatrix}^* |s\rangle = e^{i\pi b_i . F_\alpha} |s\rangle : \quad (2.21)$$

We have here the GSO projection. All states $|s\rangle$ which do not satisfy (2.21) are "projected out", because they don't participate to the trace in the partition function.

2.2 A more particular case: toward the classification of heterotic string models

2.2.1 A special set of real and complex fermions

In the following, we will consider a fixed set of real and complex fermions. On the left sector, all fermions $(\psi_i, \chi_i, y_i, \omega_i)$ will be real. On the right the first 12 internal fermions $(\bar{y}_i, \bar{\omega}_i)$ will be real and the remaining 16 $(\bar{\psi}_i, \bar{\eta}_i, \bar{\phi}_i)$ complex.

This choice is not made randomly. As we have seen, the number of complex and real fermions correspond to the diagonalisation of some orthogonal real operators $L(\alpha) \oplus R(\alpha)$. Such an operator is not always diagonalizable in $\mathbb{R}$, but is always diagonalizable in $\mathbb{C}$. In $\mathbb{R}$, it can be reduced into:

- a diagonal matrix of $+1$ or -1 on some space F,
- a quasi-diagonal matrix with 2 by 2 diagonal factors of rotation on different planes $\mathcal{P}_i$ on $F^\perp = \overset{\perp}{\oplus} \mathcal{P}_i$.

When we diagonlize in $\mathbb{C}$ we just diagonalize each 2 by 2 rotation matrix. Thus, from two

real fermions f_1, f_2 corresponding to the same eigenvalue ± 1, we can make two complex ones: we just have to interpret the action of this operator on f_1, f_2 as a rotation of angle 0 or π. But from two complex fermions we can't always create two real ones. The most general case would be to consider that all fermions are complex. Nevertheless, our case of 10 dimensions on the left sector and 26 on the right leads to a special structure.

We first want to go from 26 to 10 dimensions on the right sector: here 16 dimensions will be considered as internal, and corresponds to the complex fermions $(\bar{\psi}_i, \bar{\eta}_i, \bar{\phi}_i)$. We don't suppose anything about the restriction of $L(\alpha) \oplus R(\alpha)$ on this internal vector space. Then, we have two sectors of dimension 10: we have to compactify 6 dimensions on each sectors: this corresponds to the fermions y_i, ω_i and $\bar{y}_i, \bar{\omega}_i$. On this vector space, we make a big assumption on $L(\alpha) \oplus R(\alpha)$: it must be diagonalizable in $\mathbb{R}$. We impose this to be able to take pairs of fermions with the same transformation property around the loops and to interpret those pairs as bosons: these six dimensions are interpreted as compatified space dimensions. The χ_i are the superpartner of the corresponding bosons.

On the right sector, we don't use the same notation for all the fermions. First, we want to be able to interpret some of these fermions as the gauge group of matter. We know that the standard model gauge group $SU(3) \times SU(2) \times U(1)$ can easily be embedded in $SO(10)$: this correspond to $\psi^{1\ldots5}$. For the heterotic string in 10 dimensions, only two gauge group are allowed: $E_8 \times E_8$ and $SO(32)$. Here we want to find an $E_8 \times E_8$ theory. One of the E_8 will be broken into the matter gauge group and something else (this correspond to $\psi^{1\ldots5}$ and $\eta^{1\ldots3}$), and the remaining E_8 correspond to $\phi^{1\ldots8}$.

With this set of fermions, we will consider some vectors of $\Xi, S, e_1, ..., e_6, b_1, b_2, z_1, z_2, \alpha_1, \alpha_2$. Here are the coordinates (the empty boxes contain a 0):

- On the left sector:

	ψ_1	ψ_2	χ_1	y_1	ω_1	χ_2	y_2	ω_2	χ_3	y_3	ω_3	χ_4	y_4	ω_4	χ_5	y_5	ω_5	χ_6	y_6	ω_6
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
S	1	1	1			1			1			1			1			1		
e_1				1	1															
e_2							1	1												
e_3										1	1									
e_4													1	1						
e_5																1	1			
e_6																			1	1
b_1									1	1		1	1		1	1		1	1	
b_2			1	1		1	1								1	1		1	1	
z_1																				
z_2																				
α_1																				
α_2																				

- On the right sector:

	$\bar{y}_1$	$\bar{\omega}_1$	$\bar{y}_2$	$\bar{\omega}_2$	$\bar{y}_3$	$\bar{\omega}_3$	$\bar{y}_4$	$\bar{\omega}_4$	$\bar{y}_5$	$\bar{\omega}_5$	$\bar{y}_6$	$\bar{\omega}_6$	$\bar{\psi}_1$	$\bar{\psi}_2$	$\bar{\psi}_3$	$\bar{\psi}_4$	$\bar{\psi}_5$	$\bar{\eta}_1$	$\bar{\eta}_2$	$\bar{\eta}_3$	$\bar{\phi}_1$	$\bar{\phi}_2$	$\bar{\phi}_3$	$\bar{\phi}_4$	$\bar{\phi}_5$	$\bar{\phi}_6$	$\bar{\phi}_7$	$\bar{\phi}_8$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
S																												
e_1	1	1																										
e_2			1	1																								
e_3					1	1																						
e_4							1	1																				
e_5									1	1																		
e_6											1	1																
b_1					1		1		1		1		1	1	1	1	1	1										
b_2	1		1						1		1		1	1	1	1	1		1									
z_1																					1	1	1	1				
z_2																									1	1	1	1
α_1																1	1				1	1						
α_2													1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1			

2.2.2 A first example

We will describe here one example of free fermionic model to see how the construction works.

We will first consider the simplest basis: $\mathcal{B} = \{\mathbb{1}\}$. Then the additive group Ξ is given by $\Xi = \{\mathbb{1}, 0\}$. We will also call the 0 sector the Neveu-Schwartz sector (NS).

We are interested in the negative mass states, and the massless states: phenomenologically, we want no negative mass states, and we want all the particles of the standard model in the massless states.

For any state in the sector $\mathbb{1}$:

$$M_L^2 = -\frac{1}{2} + \frac{10}{8} + \sum_{left\ movers} \nu. \tag{2.22}$$

M_L^2 is always > 0: this sector contains no massless state.

For the NS (ie 0) sector we have:

$$M_L^2 = -\frac{1}{2} + \frac{0}{8} + \sum_{left\ movers} \nu, \tag{2.23}$$

$$M_R^2 = -1 + \frac{0}{8} + \sum_{right\ movers} \nu. \tag{2.24}$$

The frequencies of the oscillators here are always $\nu_{f,f^*} = \frac{1}{2}$.

We must also satisfy the Viasoro condition: $M_L = M_R$. Thus we have a tachyonic state (of mass $M^2 = -\frac{1}{2}$) with only one right-moving fermionic oscillator. We also get a massless state by acting with one left-moving fermionic oscillator and two right-moving oscillators, or one right-moving bosonic oscillator.

The tachyonic states are $\bar{\phi}^a_{1/2}|0\rangle_0$.

The massless states are:

- $\psi^\mu_{1/2}\partial\bar{X}^\nu_1|0\rangle_0$: the graviton, the dilaton and the antisymmetric tensor.
- $\psi^\mu_{1/2}\bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}|0\rangle_0$: the gauge bosons of $SO(44)$ (here, on the right, all fermions $\bar{\phi}^a_{1/2}$ are real).
- $\{\chi^i_{1/2}, y^i_{1/2}, \omega^i_{1/2}\}\bar{X}^\mu_1|0\rangle_0$: the gauge bosons of $SU(2)^6$.
- $\{\chi^i_{1/2}, y^i_{1/2}, \omega^i_{1/2}\}\bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}|0\rangle_0$: scalars in adjoint of $SU(2)^6 \times SO(44)$.

Here $\{\chi^i_{1/2}, y^i_{1/2}, \omega^i_{1/2}\}$ means that we take one of the $\chi^i_{1/2}$, $y^i_{1/2}$ or $\omega^i_{1/2}$, and the 1/2 indicates "half oscillator".

Now we have to do the GSO projections on each state, to see if they are in or not. On the first massless state, for example, we have:

$$e^{i\pi\mathbb{1}.F_\alpha}\psi^\mu_{1/2}\partial\bar{X}^\nu_1|0\rangle_0 = -\psi^\mu_{1/2}\partial\bar{X}^\nu_1|0\rangle_0 \tag{2.25}$$

As $C\binom{0}{\mathbb{1}} = \delta_{\mathbb{1}} = -1$, with the rule 3 of the second theorem, we have:

$$\delta_0 C\binom{0}{\mathbb{1}}^* \psi^\mu_{1/2}\partial\bar{X}^\nu_1|0\rangle_0 = -\psi^\mu_{1/2}\partial\bar{X}^\nu_1|0\rangle_0 : \tag{2.26}$$

this state survives the GSO projection. Indeed, all the other states of this model survive the GSO projection, even the tachyons.

We are now interested in the models associated to a special gauge group, for example the standard model gauge group $SU(3) \times SU(2) \times U(1)$. We will here use a specific basis of vectors made to obtain this gauge group (up to a $U(1)$ factor), and try to classify all the different models we can find depending on the one-loop coefficients $C\binom{\alpha}{\beta}$. First, we will briefly explain the choice of the basis vectors. There are different way to choose basis vectors to obtain some specific gauge group, see [5].

2.2.3 A general basis for $SO(10)$ gauge group models

We will first use this specific basis:

- On the left sector:

	ψ_1	ψ_2	χ_1	y_1	ω_1	χ_2	y_2	ω_2	χ_3	y_3	ω_3	χ_4	y_4	ω_4	χ_5	y_5	ω_5	χ_6	y_6	ω_6
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
S	1	1	1			1			1			1			1			1		
e_1				1	1															
e_2							1	1												
e_3										1	1									
e_4													1	1						
e_5																1	1			
e_6																			1	1
b_1									1	1		1	1		1	1		1	1	
b_2			1	1		1	1								1	1		1	1	
z_1																				
z_2																				

- On the right sector:

	$\bar{y}_1$	$\bar{\omega}_1$	$\bar{y}_2$	$\bar{\omega}_2$	$\bar{y}_3$	$\bar{\omega}_3$	$\bar{y}_4$	$\bar{\omega}_4$	$\bar{y}_5$	$\bar{\omega}_5$	$\bar{y}_6$	$\bar{\omega}_6$	$\bar{\psi}_1$	$\bar{\psi}_2$	$\bar{\psi}_3$	$\bar{\psi}_4$	$\bar{\psi}_5$	$\bar{\eta}_1$	$\bar{\eta}_2$	$\bar{\eta}_3$	$\bar{\varphi}_1$	$\bar{\varphi}_2$	$\bar{\varphi}_3$	$\bar{\varphi}_4$	$\bar{\varphi}_5$	$\bar{\varphi}_6$	$\bar{\varphi}_7$	$\bar{\varphi}_8$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
S																												
e_1	1	1																										
e_2			1	1																								
e_3					1	1																						
e_4							1	1																				
e_5									1	1																		
e_6											1	1																
b_1					1		1		1		1		1	1	1	1	1	1										
b_2	1		1						1		1		1	1	1	1	1		1									
z_1																					1	1	1	1				
z_2																									1	1	1	1

i with this basis, the vectors $\mathbb{1}, S$ generate a model with SUSY $\mathcal{N} = 4$, with $SO(44)$ gauge symmetry.

ii when we add the $e_1, ..., e_6$, we allow all possible symmetric shifts of the six internal coordinates (y_i, ω_i): we break $SO(44)$, but keep $\mathcal{N} = 4$ SUSY. The coordinates (y_i, ω_i) correspond to the six compatify dimension: we just impose there that the 6 compatified dimensions are equivalent.

iii b_1 and b_2 allow us to have an $SO(10)$ gauge symmetry, and break $\mathcal{N} = 4$ to $\mathcal{N} = 1$ supersymmetry.

iv $\phi_1, ..., \phi_8$, which are not affected by the action of the previous vectors, are part of the hidden sector (the hidden sector are the complex fermions which doesn't form the matter gauge group, η_i and ϕ_i). z_1 and z_2 breaks the corresponding group from $SO(16)$ to $SO(8) \times SO(8)$.

This basis is the most general one with symmetric shift of the fermions (y_i, ω_i), $\mathcal{N} = 1$ and $SO(10)$ gauge symmetry.

After GSO projection, the massless states of the all models with this basis (but different one-loop coefficients) forms a vector space on which $SO(10) \times U(1)^3 \times SO(8)^2$ acts.

Here we can see $SO(10)$ as the gauge group of matter, because this group can be broken into $SU(3) \times SU(2) \times U(1)$. The $U(1)^3 \times SO(8)^2$ is then the gauge group for the states in the hidden sector: the "hidden matter".

2.2.4 From $SO(10)$ to $SU(3) \times SU(2) \times U(1)^2$

We now want to break $SO(10)$ into $SU(3) \times SU(2) \times U(1)^2$, using some new basis vectors. Here the breaking will be done *at the string level*. The two new vectors we add to our basis are:

	ψ_1	ψ_2	χ_1	y_1	ω_1	χ_2	y_2	ω_2	χ_3	y_3	ω_3	χ_4	y_4	ω_4	χ_5	y_5	ω_5	χ_6	y_6	ω_6
α_1																				
α_2																				

	$\bar{y}_1$	$\bar{\omega}_1$	$\bar{y}_2$	$\bar{\omega}_2$	$\bar{y}_3$	$\bar{\omega}_3$	$\bar{y}_4$	$\bar{\omega}_4$	$\bar{y}_5$	$\bar{\omega}_5$	$\bar{y}_6$	$\bar{\omega}_6$	$\bar{\psi}_1$	$\bar{\psi}_2$	$\bar{\psi}_3$	$\bar{\psi}_4$	$\bar{\psi}_5$	$\bar{\eta}_1$	$\bar{\eta}_2$	$\bar{\eta}_3$	$\bar{\varphi}_1$	$\bar{\varphi}_2$	$\bar{\varphi}_3$	$\bar{\varphi}_4$	$\bar{\varphi}_5$	$\bar{\varphi}_6$	$\bar{\varphi}_7$	$\bar{\varphi}_8$
α_1																1	1				1	1						
α_2													1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1			

Now $\mathbb{1} = S + e_1 + ... + e_6 + 2\alpha_2 + z_2$: we must take off one vector from our basis, we choose $\mathbb{1}$.

This is not the only model we can study. We could also keep the basis $(\mathbb{1}, S, e_1, ..., e_6, b_1, b_2, z_1, z_2)$ corresponding to a matter group $SO(10)$. Then the observation of the Standard Model gauge group would be explained by an other group breaking, not at the string level (at a lower energy). Or take the basis $(S, e_1, ..., e_6, b_1, b_2, z_1, z_2, \alpha_2)$, which correspond to the flipped $SU(5) \times U(1)$ group: this group would be then broken into $SU(3) \times SU(2) \times U(1)$ at a lower energy level than the Planck energy.

For all these bases, we can use Theorem 2 and try to classify the different models we can obtain with the differents one-loop coefficients $C\binom{b_i}{b_j}$. One interesting question is: do we obtain 3 full generations of particles? With the basis $(\mathbb{1}, S, e_1, ..., e_6, b_1, b_2, z_1, z_2)$, we know from Theorem 2 that we have a priori 2^{67} different consistent string theories. If we impose the supersymmetry, the 11 coefficients $C\binom{S}{b}$ (for $b \neq S$ basis vector) are fixed. We still have $2^{56} \approx 7.10^{16}$ different theories to study: this is to much, even for a computer. It is worse with the two other bases. All we can expect to do is to examine a sample of these different theories. The main objective of my internship was to create a computer program to do this. I tried to do it quite generally: it works for those three examples of bases, and many others. It could be generalized to any model described by theorems 1 and 2 without any conceptual issue.

Chapter 3

The program

We will describe here a program made to compute and test different free fermionic models. Firstly, we will outline the whole functioning of the algorithm: this is not so important for a user who just wants to use the program. Next, we will see a "user guide" where we will just give the way to adapt it to a particular model: this should be the most useful for users.

This program is sometimes not well coded. This is due to the fact that I coded this program during a (short) three months internship, and I first wanted it to be just a way to help me in the tedious calculations I had to do. I also didn't know java very well and maybe didn't have good enough computer skills.

3.1 How does it work? The basic concept

As we have seen in the presentation of the algorithm, in the last chapter, a free fermionic model is given by:

- A basis $\mathcal{B} = (\beta_i)$ of Ξ.
- The matrix $\mathcal{C}$ of the one-loop phases on the basis vectors, $(C\binom{\beta_i}{\beta_j})_{i,j}$.

Now, we will fix a special basis, made to give a specific gauge group. We want to take many different matrices $(C\binom{\beta_i}{\beta_j})_{i,j}$ and analyse the spectrum we obtain: after the GSO projections, which states remain? Do we still have states in different sectors we are interested in (for physical reason)? Sometimes, we also want to know what these states are. For example, do we have states corresponding to the 3 generations of particles in the Standard Model?

The first objective of the project was to work mainly on the 13-vector basis: $\mathcal{B} = (S, e_1, ..., e_6, b_1, b_2, z_1, z_2, \alpha_1\alpha_2)$. Thus, according to Theorem 2, this gives $2 \times \frac{13\times12}{2} = 79$ degrees of freedom. 12 of them are fixed in order to give a model consistent with $\mathcal{N} = 1$

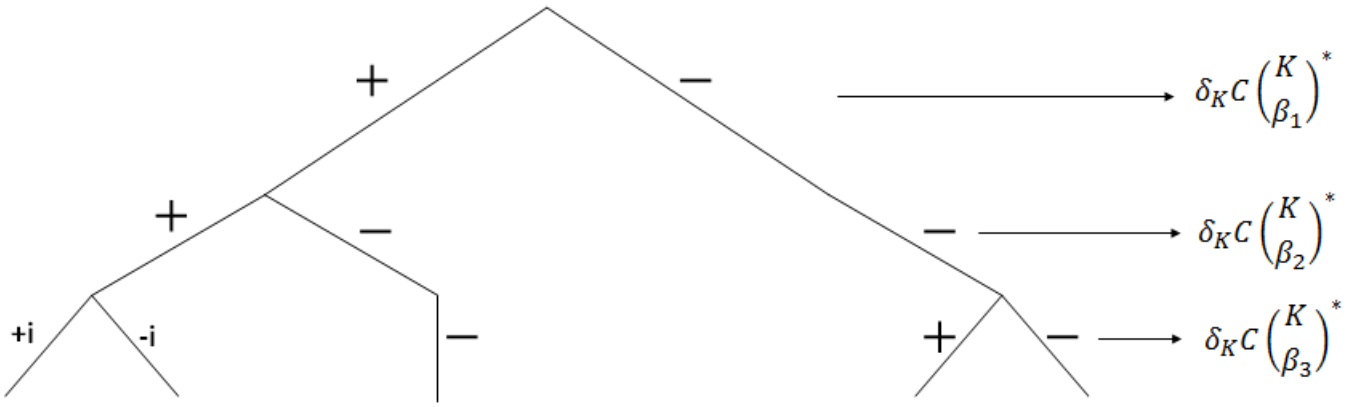
SUSY. Each degree of freedom takes values ± 1: this gives $2^{67} \approx 10^{20}$ different possibilities for $\mathcal{C}$. This is many, even for a computer... All we can manage is to analyse a sample of them.

Roughly speaking, our program will be a loop. At the beginning, we take a random matrix, do some GSO projections on the sectors we are interested in, and analyse what we obtain. The GSO projections must be done in a fast way, to be able to look over a big sample. The main idea of the algorithm, to accelerate the computation of the GSOs, is to do some preparatory work before the loop. It is based on the "GSO trees".

3.1.1 The GSO trees

The "GSO tree" of $K \in \Xi$ is the object which, depending on the coefficients $C\binom{K}{\beta}$ (for $\beta \in \mathcal{B}$), describes if some states are still in this sector after the GSO projections (in the case of my program, I use $\delta_K C\binom{K}{\beta}^*$).

Here is an example of a GSO tree, with a basis $\mathcal{B} = (\beta_1, \beta_2, \beta_3)$:



The $\pm$ or $\pm i$ on the branch of the tree represent the value of $\delta_K C\binom{K}{\beta}^*$. If the tree has no branch for some vector $\left(\delta_K C\binom{K}{\beta_1}^*, \delta_K C\binom{K}{\beta_2}^*, \delta_K C\binom{K}{\beta_3}^*\right)$, then a matrix $\mathcal{C}$ which give this vector is associated to a physical model when no states are in this sector.

This vector $\left(\delta_K C\binom{K}{\beta_1}^*, \delta_K C\binom{K}{\beta_2}^*, \delta_K C\binom{K}{\beta_3}^*\right)$ can easily be computed from $\mathcal{C}$, the decomposition of K in $\mathcal{B}$ and the rule $C\binom{\alpha}{\beta+\gamma} = \delta_\alpha C\binom{\alpha}{\beta} C\binom{\alpha}{\gamma}$.

In this example, we see that if the $\delta_K C\binom{K}{\beta}^*$ are all $+$, we have no states in, and if they are all $-$, we have some states in. For most of the sectors K, we are just interested in the question "is there something in?" (if yes, we just discard the matrix and take an other one). For some of them, what is "in" is interesting: in this case, the tree have at their end the description of the states which are in.

The GSO trees can be computed independently of the coefficents of the matrix $\mathcal{C}$, because they take into account all the different possibilities for these coefficients. They can be computed outside of the loop. Inside the loop, we just have to compute the $\delta_K C\binom{K}{\beta}^*$

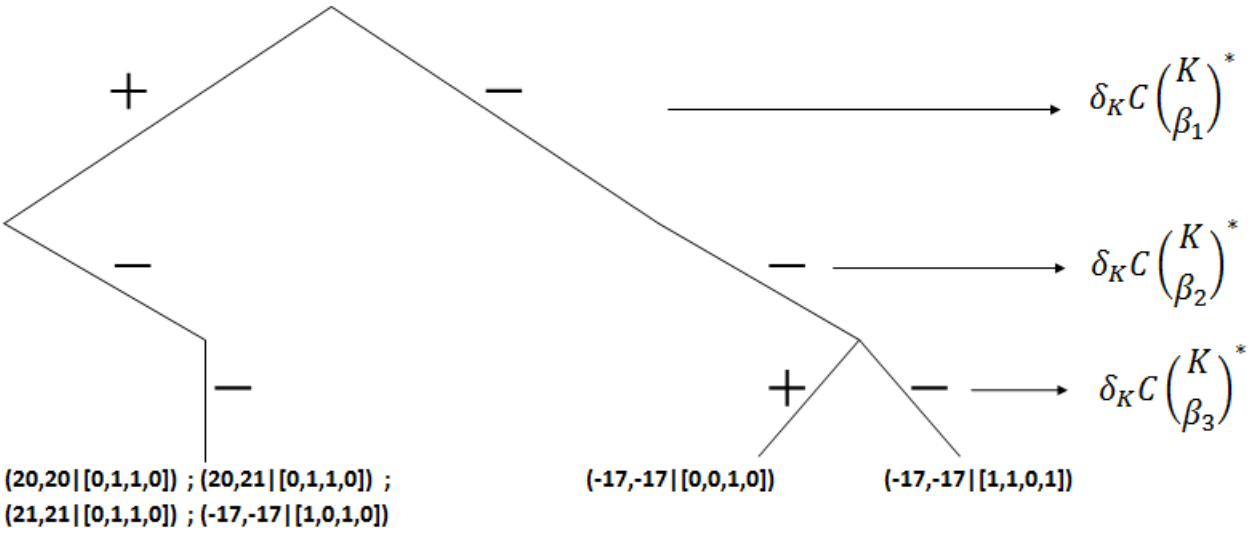
from $\mathcal{C}$, thanks to the rule on the one-loop phases $C\binom{\alpha_1}{\alpha_2}$, and look to the corresponding tree to see if the coefficients $\left(\delta_K C\binom{K}{\beta_1}^*, \delta_K C\binom{K}{\beta_2}^*, \delta_K C\binom{K}{\beta_3}^*\right)$ give some states or not.

3.1.2 Implementation of the trees

In the Java program, the trees are implemented in some classes. The program doesn't use the diamond operator (it should, it would be simpler!): It use many classes of tree (`Tree1Vectorial`, `Tree2Vectorial`, `TreeSimple`, `TreeSpinorial`, `TreeVS_1osc`, `TreeVS_2osc`), when it should use just one. All those classes have the same structure: a tree has 4 sons "Son1, Son_1, Soni, Son_i". The difference is in the data at the end of the tree:

- `TreeSimple` has no data.
- `Tree1Vectorial` has a list of integers, standing for the vectorial part of the state (if just one vectorial oscillator).
- `Tree2Vectorial` has a list of pair of integers, standing for the vectorial part of the state (if two vectorial oscillators).
- `TreeSpinorial` has a list of vectors of integers ± 1, standing for the spinorial part of the state.
- `TreeVS_1osc` has a list of pair (integer, vector), standing for the whole state (if just one vectorial oscillator).
- `TreeVS_2osc` has a list of triplet (integer, integer, vector), standing for the whole state (if two vectorial oscillator).

Thus, the last picture of tree represented a `TreeSimple`. A `TreeVS_2osc` could be represented as:



Here, $(20, 21|[0, 1, 1, 0])$ represent the state $\phi_1\phi_2|+,-,-,+\rangle$ and $(-17|-17|[1, 0, 1, 0])$ the state $\eta_1^*\eta_2^*|+,-,-,+\rangle$: The oscillators are encoded by an integer: $\bar{y}_1$ is 0, $\bar{\omega}_1$ is 1,

..., $\bar{\psi}_1$ is 12, ..., $\bar{\phi}_8$ is 27. (We are not interested in the left sector). A minus sign in front of it indicates the conjugate oscillator. The spinorials are encoded by a vector: 0 is + and 1 is -. The coordinate here is not indicated, but it is always in the same order: $\psi_{12}, \chi_{12}, \chi_{34}, \chi_{56}, y\bar{y}_1, \omega\bar{\omega}_1, ..., y\bar{y}_6, \omega\bar{\omega}_6, \bar{\psi}_1, ...\bar{\psi}_5, \bar{\eta}_1, ..., \bar{\eta}_3, \bar{\phi}_1, ..., \bar{\phi}_8$.

3.1.3 Construction of the trees

The GSO tree of a sector $K \in \Xi$ is always computed in the same order. The GSO on the basis vectors which have nonzero coefficients on the coordinates of K equal to 1 are done at the end: those vectors have an action on the spinorial part of states (We will now call it "*the vacuum*"). The remaining ones have no effect on the vacuum, the GSO is done on them first: we take the "nonvaccum basis vectors first".

It is easier to understand the way it works with an example: if we have those basis vectors (I have reduced the number of coordinates, and we only consider the right sector for simplification), $S, e_1, e_2, e_3, b, z, \alpha$:

	$\bar{y}_1$	$\bar{\omega}_1$	$\bar{y}_2$	$\bar{\omega}_2$	$\bar{y}_3$	$\bar{\omega}_3$	$\bar{\psi}_1$	$\bar{\psi}_2$	$\bar{\psi}_3$	$\bar{\eta}_1$	$\bar{\eta}_2$	$\bar{\varphi}_1$	$\bar{\varphi}_2$
S													
e_1	1	1											
e_2			1	1									
e_3					1	1							
b							1	1	1	1			
z												1	1
α							1/2	1/2	1/2	1/2	1/2	1/2	1/2

If we consider the sector $K = e_2 + b$:

	$\bar{y}_1$	$\bar{\omega}_1$	$\bar{y}_2$	$\bar{\omega}_2$	$\bar{y}_3$	$\bar{\omega}_3$	$\bar{\psi}_1$	$\bar{\psi}_2$	$\bar{\psi}_3$	$\bar{\eta}_1$	$\bar{\eta}_2$	$\bar{\varphi}_1$	$\bar{\varphi}_2$
K			1	1			1	1	1	1			

The "vacuum coordinates" are $\bar{y}_2, \bar{\omega}_2, \bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3, \bar{\eta}_1$. So the GSO tree will be built in this order: $S, e_1, e_3, z, b, e_2, \alpha$.

The construction of a GSO tree is done in three steps:

1. **The tree of vectorial**: here we suppose that our state has no spinorials. It can be one or two vectorial, half or quarter oscillators (the case two half oscillators is not taken into account by the program: this case is not present with the sector that we are interested in).

2. **The tree of spinorial**: here we suppose that our states have no vectorials.

3. **The whole tree of GSO**: from those two trees.

This method is based on the fact that we can decompose the GSO formula (on a state S, in sector K, with the basis vector β):

$$\delta_K C\binom{K}{\beta}^* = e^{i\pi\beta.F_S} = e^{i\pi\beta.F_S^{vac}} \times e^{i\pi\beta.F_S^{nvac}}, \tag{3.1}$$

where F_S^{vac} is the vacuum part of F_S, and F_S^{nvac} is the nonvacuum part of it.

For example, with $K = e_2 + b$, $\beta = \alpha$ and the state $\bar{\eta}_2\bar{\phi}_1^*|-,-,+,-,-,-\rangle$, we have

$$\beta.F_S^{vac} = 0 \times -1 + 0 \times -1 + \frac{1}{2} \times 0 + \frac{1}{2} \times -1 + \frac{1}{2} \times -1 + \frac{1}{2} \times -1,$$

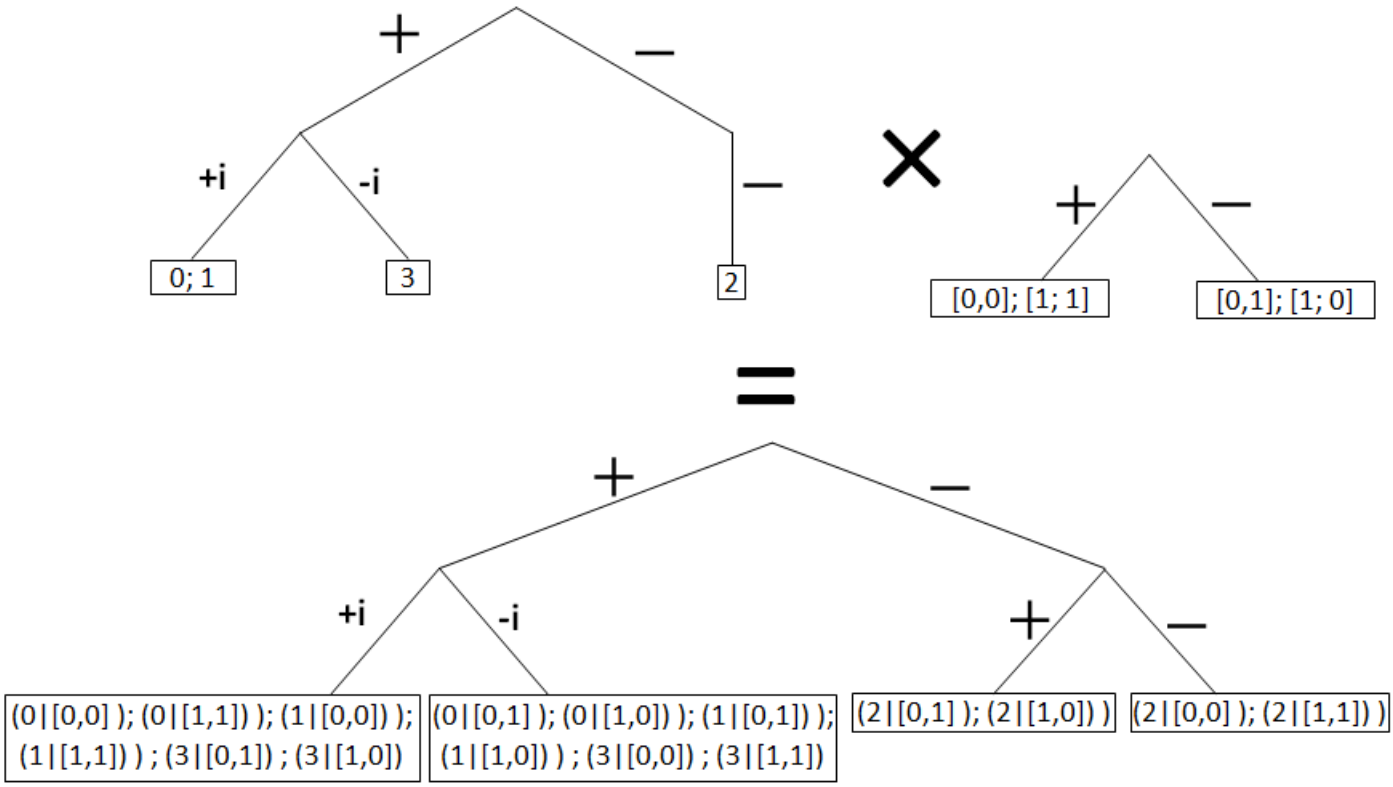
$$\beta.F_S^{nvac} = -(\frac{1}{2} \times 1 + \frac{1}{2} \times -1).$$

The first tree, `TreeVectorial`, is computed from the array of coordinates of the basis vectors, and the coordinates of K. The program takes each possible oscillator (if one oscillator) or pair of oscillators (if two oscillators) and constructs the tree of GSO projections of those oscillators. It is not difficult to do this: from the last array, we see straight away that the oscillator $\bar{\psi}$ is $+1$ on S, e_1, e_2, e_3, b, is -1 on z and is $-i$ on α. The (partial, because here the vacuum is not taken into account) GSOs are of course done in the order "non-vacuum basis vector first".

The second tree, `TreeSpinorial`, is done in the same way: The computer generates all the possible spinorials $|\pm,\pm,\pm,\pm,\pm,\pm\rangle$, encoded by vectors of 0 (for $+$) or 1 (for -), and adds them one by one to a `TreeSpinorial`. The depth of this tree is the number of "vacuum basis vector" (for our example K, it is 3).

The last step is just the "multiplication of the trees": we first reproduce `TreeVectorial` for the basis vectors which are "non vacuum". When comes one "vacuum basis vector" β, we take all the possible combinations of pairs of branches corresponding to β (one in each tree), multiply the numbers corresponding to these two branch (± 1 or $\pm i$) and add a new branch, labelled by this number, in the new tree.

For example:



Here `TreeSpinorial` has just a depth of 1, and TreeVectorial a depth of 2.

3.1.4 The use of the trees

GSO trees are the basic component of the program. They are quite general and their use can be adapted, depending on what we want to do. In our case, the goal is to determine quickly if, with a given matrix $\mathcal{C}$, some sectors have states in or not. Here, we are interested in the sectors leading to enhancement states, exotics states, generations and light Higgs. We will call them "enhancement sectors", "exotic sectors", "generation sectors" and "Higgs sectors".

The "enhancement sectors" are the vectors of Ξ with a mass $M_L = -\frac{1}{2}$ and a mass $M_R \leq 0$. The remaining sectors are defined when we have special vectors in our basis $\mathcal{B}$ (or, at least, in our group Ξ): we must have $S, e_1, e_2, e_3, e_4, e_5, e_6, b_1, b_2, b_3, X$, we recall their coordinates here:

- On the left sector:

	ψ_1	ψ_2	χ_1	y_1	ω_1	χ_2	y_2	ω_2	χ_3	y_3	ω_3	χ_4	y_4	ω_4	χ_5	y_5	ω_5	χ_6	y_6	ω_6
S	1	1	1			1			1			1			1			1		
e_1				1	1															
e_2							1	1												
e_3										1	1									
e_4													1	1						
e_5																1	1			
e_6																			1	1
b_1									1	1		1	1		1	1		1	1	
b_2			1	1		1	1								1	1		1	1	
b_3			1	1		1	1		1	1		1	1							
X																				

- On the right sector:

	$\bar{y}_1$	$\bar{\omega}_1$	$\bar{y}_2$	$\bar{\omega}_2$	$\bar{y}_3$	$\bar{\omega}_3$	$\bar{y}_4$	$\bar{\omega}_4$	$\bar{y}_5$	$\bar{\omega}_5$	$\bar{y}_6$	$\bar{\omega}_6$	$\bar{\psi}_1$	$\bar{\psi}_2$	$\bar{\psi}_3$	$\bar{\psi}_4$	$\bar{\psi}_5$	$\bar{\eta}_1$	$\bar{\eta}_2$	$\bar{\eta}_3$	$\bar{\varphi}_1$	$\bar{\varphi}_2$	$\bar{\varphi}_3$	$\bar{\varphi}_4$	$\bar{\varphi}_5$	$\bar{\varphi}_6$	$\bar{\varphi}_7$	$\bar{\varphi}_8$
S																												
e_1	1	1																										
e_2			1	1																								
e_3					1	1																						
e_4							1	1																				
e_5									1	1																		
e_6											1	1																
b_1					1		1		1		1		1	1	1	1	1	1										
b_2	1		1						1		1		1	1	1	1	1		1									
b_3	1		1		1		1						1	1	1	1	1			1								
X													1	1	1	1	1	1	1	1								

With the vectors $B^1_{pqrs} = S+b_1+pe_3+qe_4+re_5+se_6$, $B^2_{pqrs} = S+b_2+pe_1+qe_2+re_5+se_6$ and $B^3_{pqrs} = S + b_3 + pe_1 + qe_2 + re_3 + se_4$, where $p, q, r, s \in \{0, 1\}$:

- The "exotics sectors" are all the elements of Ξ of the form: $B^i_{pqrs} + K$, where $K \in \Xi/\{0\}$ has nonzero coordinates only in the right sector:
- The "generation sectors" are all the elements of Ξ of the form: B^i_{pqrs}.
- The "Higgs sectors" are all the elements of Ξ of the form: $B^i_{pqrs} + X$.

We are looking for a model such has we have no state in the enhancement and exotic sectors. We also want the generation and Higgs sectors to satisfy some conditions.

We will now describe how the program uses the trees, with a quick description of the main part of the program. In Java, its name is `_Main_Random_Matrix`.

The computer first generates all the GSO trees for the states that we are interested in, and stores them. Then, it launches a loop over the matrices that we want to test. In this loop:

1. The computer generates a random matrix $\mathcal{C}$.

2. For each "enhancement vectors" K, it computes the vector of coefficients $\left(\delta_K C\binom{K}{\beta}^*\right)_{\beta\in\mathcal{B}}$ on all $\beta \in \mathcal{B}$ from $\mathcal{C}$, with the rules of theorem 2. It compares it to the corresponding tree. If this vector is in the tree, we have an enhancement state: $\mathcal{C}$ is not good and we begin a new loop, with a new matrix. If not, we go to the next enhancement state.

3. The computer does the same task on "exotic sectors"

4. For each "generation sector" K, the computer computes the same vector of $\left(\delta_K C\binom{K}{\beta}^*\right)_{\beta\in\mathcal{B}}$. If this vector is in the corresponding vacuum tree, the states which are "in" are stored. At the end of this stage, the computer looks over the "generation states" in. If they do not satisfy some conditions (such as: those states correspond to 3 generations), we begin a new loop

5. The computer does the same task on "Higgs sectors".

3.2 A "user guide"

We will now describe a more exhaustive description of the main part of the program, `_Main_Random_Matrix`. The aim of this section is to go into the details of the way the program can be adapted to different bases. This section will follow the order of the code. The code also contains many indications in commentary, useful for the user.

3.2.1 The basic changes

The program is designed to alow the user to turn off some stages of the checking. This is done with the booleans `TestEnhancement`, `TestExotics`, `TestGenerations` and `TestHiggs`. If one of them is "true", the following test will be done. If `NbGenerationsFixed` is "true", the program will look for a fixed number of generations `NbGenerations`. `NbLoop` and `Random_Generator` act on the loop, ie the number of matrices that the program tests, and the random matrix that it tests. These are the only parameters that the users have to change when they wants to study a fixed model.

3.2.2 To change the model: before the loop

To change the model, the user must change more parameters, as indicated in the program as commentary: All the description of the basis and the decomposition of the enhancement, exotics sectors. Be careful: before the commentary `/*Don't touch the following*/`, every constant may have to be changed.

Depending on the basis, some other parts of the program will have to be changed. Before the loop, the program first computes `Decompo_Sect_Exot`, `Decompo_Sect_Gene` and `Decompo_Sect_Higgs`: these are arrays giving the decompositions of all Exotic, Generation and Higgs sectors in $\mathcal{B}$. For example, `Decompo_Sect_Gene` is an array of 48 lines, and each line gives the decomposition of all the 48 elements B^i_{pqrs}, $\forall p,q,r,s \in \{0,1\}, i \in \{0,1,2\}$. Depending on $\mathcal{B}$, those elements have a different decomposition: if $b_3 \notin \mathcal{B}$ for instance, one has to use the decomposition of b_3 in $\mathcal{B}$ to create the lines of the array.

From these arrays are created `Sect_Enhancement`, `Sect_Exot_L`, `Sect_Exot_R`, `Sect_Gene_L`, `Sect_Gene_R`, `Sect_Higgs_L` and `Sect_Higgs_R`, the coordinates of the corresponding sectors: this should never be changed.

`Gram_RegularBase` and `Gram_BaseWith_1_Without_z2_` are the matrices of $(\beta_i.\beta_j)_{i,j}$ for different bases. The first is the one on the basis with 1 as a first vector, and no z_2. The second is the one on the regular basis, $\mathcal{B}$. This will be important when $\mathcal{C}$ is computed, and may have to be changed: I will talk about it later, when we will describe how $\mathcal{C}$ is generated.

`MatScalProdBasSect_Enhanc` is the matrix of $(K.\beta)$ for K an enhancement sector and $\beta \in \mathcal{B}$. `MatScalProdBasSect_Enhanc_Transform` is a small transformation of it: this will be useful for the computation of the $\delta_K C\binom{K}{\beta}^*$. The same matrices are computed for exotic, generation and Higgs sectors: this should never be changed.

Under our gauge group, the 16 and $\bar{16}$ representation of $SO(10)$ decompose in some sum of irreducibles. The vectors `sixteen` and `sixteenbar` are vectors that will give the number of each irreducible representation we obtain with the states of the generation sectors. Thus, their size may have to be changed: for example, it is 3 (from 0 to 2) if the gauge group is the Pati-Salam gauge group ($16 = 1 + \bar{5} + 10$).

`Higgs_Psi` and `Higgs_Psistar` will be use to check that we have the light higgs. `ArrayT_Enhanc`, `ArrayT_Exot`, `ArrayT_Gene`, `ArrayT_Higgs` contains the GSO trees of each sectors.

3.2.3 To change the model: inside the loop

The first part of the loop generates the random matrix. It was not easy to write something general: what is done is not so general, and could be quicker. A good solution is perhaps to write a method which generates a random matrix $\mathcal{C}$ from the random numbers `rand.nextInt(2)`. This matrix must be called `Matrix_4C`, **and have coefficients equals to** 0 **(for** 1**),** 2 **(for** i**),** 4 **(for** -1**) or** 6 **(for** $-i$**)**: this could gain 5% in the speed of the program.

What is done here is: a matrix `K4` is created, which is the matrix $\mathcal{C}$ in an other ba-

sis: one which contains 1 as a first vector, and doesn't contain z_2. From this matrix, `Matrix_4C` is computed: this is the real matrix $\mathcal{C}$, corresponding to the basis $\mathcal{B}$. If in the new model, we still have $1 = S + e_1 + ... + e_6 + z_2 + 2\alpha_2$ then nothing has to be changed. If the expression of 1 is different, but always has $z2$ in it, then the line corresponding to the coefficients of `Matrix_4C` implying z_2 has to be changed. If not, then the user has to choose a vector which in the decomposition of 1 (for example: let us say it is e_6). The same lines have to be changed. Every time z_2 appears, it has to be changed in, for example, e_6. And `Gram_BaseWith_1_Without_z2_` has to be changed to correspond to the new modified basis, with 1, and not (in this case) e_6: this must be done in the file "ScalProd", for the function `public static int[][] MatScalProdMain_BaseWith_1_Without_z2_()`, where all z_2 have to be change into, in the example, e_6 (It is easier than it looks here. And the solution of writing a method which generates a random matrix $\mathcal{C}$ from the random numbers `rand.nextInt(2)` can be use instead of what I'm trying to explain here).

The second part of the loop does the GSOs:

1. On the enhancement sectors: for each $K \in$ `Sect_Enhancement`, the vector $\left(\delta_K C\binom{K}{\beta}^*\right)_{\beta\in\mathcal{B}}$ is computed, then compared to the corresponding tree. If this vector is in the tree, the loop is stopped and a new matrix is generated. If not, we go to the next enhancement state.

2. On the exotic sectors: the program does the same work. To gain time, between K and K' which differ only by an e_i, the vector $\left(\delta_K C\binom{K}{\beta}^*\right)_{\beta\in\mathcal{B}}$ is not erased but just modified. Here is an example: the first vector of exotics is $K = B^1_{0000} + \alpha$ and the second one $K' = B^1_{0001} + \alpha = K + e_6$. Then $\delta_{K'} C\binom{K'}{\beta}^* = \delta_K C\binom{K+e_6}{\beta}^* = \delta_K \delta_\beta C\binom{\beta}{e_6} C\binom{K}{\beta}^*$: the program uses this to compute the vectors $\left(\delta_K C\binom{K}{\beta}^*\right)_{\beta\in\mathcal{B}}$ quicker.

3. On the generation sector: the states in are, sectors after sectors, stored in `ListSpinIn_Gene`. When all the sectors have been done, the computer takes all the states one by one, and computes each time to which irreducible factor of 16 or $\bar{16}$ they correspond. Here the user has to change the way `sixteen` and `sixteenbar` are modified. After this, `sixteen` and `sixteenbar` contain the number of states in each irreducible factor of 16 or $\bar{16}$, ie the number of each irreducible factor times its dimension: each component of `sixteen` and `sixteenbar` must be divided by the dimension of the vector space that it represents.

4. On the Higgs sector: as for generation sectors, states (just the spinorial part of it) are stored in `ListOscIn_Higgs`. Afterwards, `Higgs_Psi` and `Higgs_Psistar`

compute the number of time an oscillator contributing to light Higgs appears: this depends on the model and may be changed. Again, `Higgs_Psi` and `Higgs_Psistar` must be divided by the right number.

At last, if the matrix $\mathcal{C}$ (`Matrix_4C` in the program) meets the conditions on the enhancement, exotic, generation and Higgs sectors, then a copy of it is stored in the list `ListMatrix_4CIn`. This list is printed on the screen after the loop.

Conclusion

After a quick introduction to the basis of string theory, we have seen how it was possible to introduce a formalism to study the heterotic string. Thanks to two simplifying hypothesis (the operators of the action of $\pi_1(M)$ on fermions commutes, and Ξ is finite), we found a way to create many different models, with different gauge group and matter content.

This method to create models, and analyse them, can be done by hand for the simplest cases but needs too much calculation as soon as the models are more complex. We then need the power of the computer. A first way to use it is to first derive by hand some algebraic expressions to then compute easily the gauge group and the matter content of a specific family of models (associated to one set of basis vectors). We have seen here that it was indeed possible to have this task made by the computer, thanks to the stockpiling in the memory of the computer of the GSO trees, while keeping a satisfying speed of the program. This has the advantage to study more easily different set of basis vectors.

Nevertheless, a computer will never be able to answer everything... Our program is designed for a specific use, but many questions remains. Why, with some set of basis vectors, we always find an even number of generations? It seems that some GSOs on some of the enhancement sectors never give enhancement: Why?

Those two observations are true on a big number of models: there may exist a mathematical demonstration of it, and also, maybe, a physical interpretation...

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