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Fermionic Construction of Free-Fermionic Models

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Abstract

Although the Standard Model has long been considered as an amazingly predictive model in describing Nature, and is still reckoned as very accurate from many points of view, most of the physics community is now convinced that it cannot describe truly our world. Many theories have been developed to enhance the Standard Model in order to take into account new phenomena, out of which is the Minimal SuperSymmetric Model (often named as MSSM), but this new theory itself requires a larger context to account for the different values of its parameters. Different new contexts have been analyzed, and the one in which I will develop my work is string theory, and more precisely the heterotic superstring, which is one of the five consistent theories of strings. In the frame of the heterotic string, I will focus on the free fermionic construction. One important feature of this formalism is to achieve a very standard-like description. The aim of this study is to determine certain important phenomenological issues in the framework of the MSSM, namely the μ-problem, which is one problem of the theories beyond the Standard Model that the strings are very likely to solve.

Bien que le Modèle Standard soit considéré depuis longtemps comme un modèle extrêmement prédictif, et que sa précision soit reconnue sur de nombreuses expériences, la plupart de la communauté physicienne est convaincue qu'il ne peut décrire entièrement notre univers. De nombreuses théories ont été développées pour étendre ce modèle, tel le Minimal SuperSymmetric Model (MSSM). Cette théorie elle-même peut dériver de plusieurs contextes. Différents contextes ont été étudiés, et mon travail se situera dans celui de la théorie des cordes, plus précisement la supercorde hétérotique, une des cinq théories des cordes existantes. Je m'appuyerais sur une construction fermionique libre de cette supercorde, qui présente l'avantage de parvenir à un modèle très proche du Modèle Standard. Le but de ce travail est de résoudre certains des problèmes rencontrés par le MSSM, notamment le problème du terme μ, pour lequel les thèories des cordes semblent adaptées.

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Chapter 1

Introduction

Experimental values of all physical experiments are always compared to the expected value of the Standard Model, for it is currently the best predictive model physicists have. However there are crucial theoretical issues that this model cannot handle, this is the reason why we are looking for a theory beyond the Standard Model. First of all, the SM[1] is a quantum field theory, and does not include the notion of General Relativity, although we know that this latter is probably true. And moreover there are a number of parameters that need to be set ad-hoc – which weakens the predictive power of the model – and at last, the mass differences between the three generations of particles as the various coupling constants of the forces are likely to have a deeper explanation. The MSSM model was created as the nearest extension from the SM by including supersymmetry, which we believe is a necessity for beyond the Standard Model theories. However, we still have some parameters that must be fixed, and the MSSM does not solve all the problems that the SM has to face. We have then turned to string theory to get an explanation of these problem.

I will focus my work on the μ problem, that can be described shortly (I will come back upon it later) as the following : the superpotential of the MSSM contains the term $\mu h\overline{h}$ where $h, \overline{h}$ are the two Higgs states of the model. Experimentally the value of μ is fixed to be at a certain scale (the electroweak scale, because the Higgs are supposed to acquire a mass at the electroweak breaking), but there is no theoretical justification for this. A possible reason would be the existence of a new symmetry that forbids this term in the superpotential (we will see how), and that would be broken at the scale that matches with the experimental bounds. The search of such a symmetry will be the aim of the construction of our string model.

Although string theories were first investigated in relation to the strong interactions in the nuclei, their major appealing aspect is that they include naturally the gravitation interaction (whereas ordinary quantum field theory is incompatible with General Relativity). The other interesting fact is that, by combining string theory with Supersymmetry in an heterotic string, the underlying gauge group will be $SO(32)$ or $E_8 \times E_8$, which include the Standard Model gauge group

[1] Standard Model

(which is $SU(3)_C \times SU(2)_L \times U(1)_Y$) via the Grand Unification gauge group $SO(10)$. Understanding the way how mathematical consistency requirements (upon which I shall come back) would turn into realistic and accurate physical constraints was a major point for string theory.

Chapter 2

Free-fermionic construction

The aim of this theoretical part is not to give a lecture on string theory, which is far from the purpose of this report, but only to state the axioms and properties that I have considered as true and used in the description of my model. This being said, let us start with some bases of string theory.

The fundamental point in string theory is that particles are not any more seen as point particles, but as strings. The specific feature of the string is that, in addition to the movement of the center of mass, it gives rise to vibrational modes, and so one could imagine that these vibrational modes would be used as additional degrees of freedom of the particles, and could account for the quantum numbers for instance.

Here we describe the string by the variable $X^\mu(\sigma,\tau)$, where X^μ is the position of a point of the string parametrized by the time τ and the coordinate along the string σ. X is a point of a D-dimensional space, which index is μ, wherein the string is moving (normally we would expect it to be a four dimensional space as usual Riemann or Minkowski spaces, but we will see that more dimensions can be required). The part of the space swept by the string is called the *world sheet*. The next step is to write down the action of the string. There are different formulations (some of these being equivalent) and the one which is most often taken is the sigma model action, also called the Polyakov action :

$$S_\sigma = -\frac{1}{2}T\int d\sigma d\tau\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu$$

where $h^{\alpha\beta}$ is the metric of the world sheet, deduced from the metric of the space, h its determinant, and T is the tension of the string.

Using the invariance of the action under the Poincare transformations, the world sheet reparametrizations and the Weyl transformations (which are described in the Appendix A.1) we can rewrite the Polyakov action as

$$S = -\frac{T}{2}\int d\sigma d\tau\ \partial_\alpha X_\mu\partial^\alpha X^\mu$$

Note that we do not fix completely the invariance under these latter transformations, we will fix them completely by choosing two other gauges (namely the superconformal gauge and the light cone gauge, as we will see).

Now we must consider the fact that the theory we want to describe – the MSSM – introduces bosonic and fermionic fields that are related each other by a supersymmetric transformation. This is achieved by the action:

$$S = -\frac{T}{2}\int d\sigma d\tau\, (\partial_\alpha X_\mu \partial^\alpha X^\mu + \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu)$$

where X is the bosonic field and ψ the fermionic field, here represented by a Majorana spinor $\psi^\mu = \begin{pmatrix} \psi^\mu_+ \\ \psi^\mu_- \end{pmatrix}$. It can be shown that this action is invariant under a specific supersymmetric transformation (see Appendix A.2).

We can now derive from the action the equations of motion for both X^μ and ψ^μ in terms of light-cone coordinates which are $\sigma^\pm = \tau \pm \sigma$:

$$\partial_+\partial_- X^\mu = 0 \qquad \text{and} \qquad \partial_+\psi^\mu_- = \partial_-\psi^\mu_+ = 0.$$

The general solution for these types of equations is a decomposition in left and right-movers as follows:

$$X^\mu(\sigma,\tau) = X^\mu_R(\sigma-) + X^\mu_L(\sigma+).$$

Similarly ψ turns to have one component which is left-moving ψ_+ and one right-moving ψ_-. By taking the superconformal gauge, we have that each component of the energy-momentum tensor must vanish (see Appendix A.3), which leads to four more equations :

$$\partial_+ X^\mu \partial_+ X_\mu = \partial_- X^\mu \partial_- X_\mu = 0$$

$$\psi^\mu_+ \partial_+ \psi_{+\mu} = \psi^\mu_- \partial_- \psi_{-\mu} = 0$$

2.1 Closed string and mode expansion

We must now introduce the fact that we are dealing with closed string, which are characterized by periodical boundary conditions

$$X(\sigma,\tau) = X(\sigma+\pi,\tau),$$

if we take π to be the length of the string. The boundary conditions on ψ are derived from the supersymmetric action and include antiperiodicity :

$$\psi_+(\sigma) = \pm\psi_+(\sigma+\pi) \qquad \text{and} \qquad \psi_-(\sigma) = \pm\psi_-(\sigma+\pi)$$

Hence we see that this allows us to develop $X_\pm$ and $\psi_\pm$ in Fourier modes, as they are periodic or antiperiodic.

$$\text{(Periodicity)} \qquad \psi^\mu_\pm = \sum_{n\in Z} d^\mu_n e^{-2in\sigma^\pm}$$

$$\text{(AntiPeriodicity)} \qquad \psi^\mu_\pm = \sum_{r\in Z+1/2} b^\mu_r e^{-2in\sigma^\pm}$$

The periodic conditions are called Ramond, and antiperiodic Neveu-Schwarz. The expansion of X is slightly more complicated, since X_R and X_L are a priori not periodical, but only their derivatives as periodical and can be developed as $\partial_- X^\mu_R = l_s \sum_{n\in Z} \alpha^\mu_n e^{-2in\sigma^-}$ (and the same goes for the left-movers). The coefficients d_n and b_r are often called raising operators for $n < 0$ and lowering oscillators for $n > 0$.

It is useful to denote the right-movers by a bar ($\overline{d}_n$,$\overline{b}_r$), in order to avoid confusion. As $\psi^\pm$ are real (ψ is a Majorana spinor) we must have that $d_{-n} = d^*_n$ and $b_{-r} = b^*_r$, the same conditions being obtained for the bosonic fields by stating that X^μ is real.

2.2 Quantization

I will now consider only fermions, because the free-fermionic construction mainly makes use of fermions. However nearly the same results hold for a bosonic string (some slight differences as replacement of anticommutators by commutators must be taken into account).

The next step is to quantize the theory, which is done by assuming the canonical fermionic anticommutation relations $\{\psi^\mu_A(\sigma,\tau),\psi^\nu_B(\sigma',\tau)\} = \pi\eta^{\mu\nu}\delta_{A,B}\delta(\sigma-\sigma')$, which tuns into

$$\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n}$$
$$\{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s}$$

in terms of Fourier modes.

We have also here another constraint which arises from the vanishing of the energy-momentum tensor (see Appendix A.3), which states:

$$L_m|\phi\rangle = \overline{L}_m|\phi\rangle = 0 \qquad \text{for all } m > 0 \text{ and}$$
$$(L_0 - a)|\phi\rangle = (\overline{L}_0 - a)|\phi\rangle = 0$$

where $L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} : d_{m-n}d_n :$, the semi-columns (:) indicating the normal-ordering prescription. I have used here the periodic modes, but the equation holds as well for anti-periodic fermions. Here two specific requirements that arise from fixing the physical states to be of positive norm : first, the dimension of the space must be 26 if there is no supersymmetry and 10 otherwise. Secondly the parameter a must be a=0 for a periodic field and a=$\frac{1}{2}$ for an antiperiodic field in a supersymmetric case, and a=1 in a non supersymmetric case.

The last equation (on L_0) is known as the Virasoro constraint or the mass equation, and states that our states must be massless (to respect Lorentz invariance) :

$$\alpha' M^2 = -a + \sum_{n=1}^{\infty} n d_{-n} d_n + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r} b_r = -a + \sum_{n=1}^{\infty} n \overline{d}_{-n} \overline{d}_n + \sum_{r=\frac{1}{2}}^{\infty} r \overline{b}_{-r} \overline{b}_r$$

where $M^2 = -p^\mu p_\mu$ and $p^\mu = T\int_0^\pi d\sigma \dot{X}^\mu(\sigma)$ is the total momentum of the string.

2.3 The heterotic String

We have until now discussed a string theory with N=2 world-sheet supersymmetry, because we needed to add a fermionic field to our bosonic field. However, an interesting feature of the closed string is that the supersymmetric transformations decouple in the left and right-movers. Indeed, taking for instance the bosonic part of a infinitesimal transformation:

$$\delta X^\mu = \bar{\epsilon}\psi^\mu$$

We can write it as $\delta(X_R^\mu + X_R^\mu) = \bar{\epsilon}_A\psi_+^\mu + \bar{\epsilon}_B\psi_-^\mu$ which by derivation and integration turns into two equations

$$\delta X_R^\mu = \bar{\epsilon}_A\psi_+^\mu + C \qquad \text{and} \qquad \delta X_L^\mu = \bar{\epsilon}_B\psi_-^\mu - C$$

We get rid of the constant by assuming that X_R and X_L are defined with the same integration constant (remember that X_R is obtained from the series expansion of $\partial_- X_R$, and similarly for X_L). We end up with the two decoupled equations:

$$\delta X_R^\mu = \bar{\epsilon}_A\psi_+^\mu$$
$$\delta X_L^\mu = \bar{\epsilon}_B\psi_-^\mu$$

This correspond to two different supersymmetric transformations. With a similar work on the whole supersymmetric transformation, we can express our N=2 world-sheet supersymmetry as an N=1 world-sheet supersymmetry on the left-movers and an N=1 world-sheet supersymmetry on the right-movers.

The aim of this point is that we can construct a model where the string has a N=1 world-sheet supersymmetry on the left-movers and no supersymmetry on the right movers, which will at the end generates a N=1 space-time supersymmetry, one of the requirements of the model. Until now I had mostly been working on supersymmetric strings, but all the results apply to a non-supersymmetric string, the main point being that there are no fermionic field in that case.

This could seem quite weird as first glance because canceling the central anomaly require that the space-time dimension for right-movers is 26 and 10 for the left-movers, although one would expect to have the same space-time dimension. However, as we will see, we can reduce the two numbers of dimensions to the same value, which can furthermore be taken as equal to four in order to match with our real 4-dimensional space-time. This last step is achieved through compactification.

2.3.1 Compactification

We have seen that the left and right movers (in any superfield) were decoupled. Thus we can add to our model more fermions that would be only right or left moving. The aim is that they would contribute to cancel the central charge, and thus reduce the space-time critical dimension. Indeed if we add 44 right-moving fermions and 18 left-moving fermions, the conformal anomaly would become:

$$C_L = -26 + 11 + D_L + \frac{D_L}{2} + \frac{18}{2}$$

$$C_R = -26 + D_R + \frac{44}{2}$$

where D_L and D_R are the left and right space-time dimension. We see here that for the same 4-dimensional space-time for both movers, the conformal anomalies are canceled, and thus the theory can be conformally invariant.

We may ask ourselves what do these additional fermions mean, as we are only trying to describe one superfield Y with its bosonic and fermionic components. They can be seen as internal degrees of freedom (such as internal symmetries) or extra dimensions. Indeed this internal fermionic part of the action is as follow:

$$\frac{1}{\pi} \int d^2\sigma \; (i \sum_1^{18} \lambda_j^1 \partial_+ \lambda_j^1 + i \sum_1^{44} \lambda_j^2 \partial_+ \lambda_j^2).$$

There is a global $SO(18)_L \times SO(44)_R$ symmetry, under which the internal fermions transform as fundamental representations.

At this point we are left with 74 fields, which are the four space-time coordinates of X^μ, ψ^μ, and $\overline{X}^\mu$ (remember that we have a left and right moving bosonic field, but only a left moving fermionic field), and the 18 left and 44 right additional fermions. We can reduce this number of fields by imposing the light-cone gauge. The light-cone gauge eliminates the coordinate "$\pm$" of the lagrangian (with $X^\pm = X^0 \pm X^{D-1}$, where D is the space dimension, similarly we get $\psi^\pm$). So we are left with only the transverse coordinates of X^μ, ψ^μ, and $\overline{X}^\mu$, which gives a total of fields of 68.

The string states (which represent the particle present in the theory) are obtained by acting with the raising operators on the ground state, which is by definition annihilated by any lowering operator. The states form then an Hilbert space.
The set of boundary conditions from which we derived the mode expansions – remember that each fermion can have two possible boundary conditions – is called a sector (and thus each state belongs to one sector).

2.4 Partition function and consistency conditions

Instead of calculating the action of our string (which would be the next logical step), we can compute the one-loop partition function. Indeed it can be proved that this partition function includes all the physical states and is sufficient to derive some constraints on our model. This partition function is an integration over all possible world-sheets, in the case of the one-loop partition function the wold-sheets are tori, so we need to integrate over all different tori.

In this case, not only will the fermions which propagate around the string have a boundary condition around the string (i.e. towards the σ coordinate), but they also can pick up a phase by propagating along the τ dimension, as formulated in $f \rightarrow -e^{-i\alpha(f)\pi} f$ (hence we will have $\alpha = 1$ for periodicity and 0 for antiperiodicity). That means that we have boundary conditions on "time" direction and "space" directions.

We have now 68 fields to consider, our 62 internal fermions, the two transverse coordinates of X_L^μ, X_R^μ and ψ^μ. There is no choice of boundary conditions for the bosonic fields (which must be periodic), however there are some for the fermionic fields, we said that they could be periodic or anti-periodic (we will see later on that there are more general boundary conditions). Hence the partition function must include all possible combinations of 64 boundary conditions of the fermions, and this integrated over all inequivalent tori. We can thus write it as:

$$Z = \sum_{\alpha,\beta} C \binom{\alpha}{\beta} Z[\alpha, \beta]$$

where α and β are sets of 64 two-dimensional boundary conditions (one "space" boundary condition, while the other is "time").

I will not reproduce here the work on the partition function (however the main formulas are in the appendix A.4), but I will use directly its consequences. Indeed, requiring the partition function to be invariant under modular transformations gives us a specific set of rules to characterize the states, which will be described in the next chapter.

We end up by finding that there are several partition functions that abide by our constraints, each of them being specified by a set of boundary condition vector b_i (each of the component of a boundary condition vector being the boundary condition of a fermion) and with a set of coefficients $C(b_i, b_j)$ associated to these vectors in the partition function. Hence this construction will provide us with different models, each specified by a set of vectors and a set of associated coefficients.

Chapter 3

Building a free-fermionic model

3.1 The ABK rules

We have seen that for each consistent heterotic superstring model, there is a partition function defined by a set of boundary conditions vectors and a set of coefficients associated to each pair of these vectors. What we assume now is that for each set of boundary conditions vectors and set of associated coefficients verifying some specific rules, there is a consistent model of heterotic superstring. A definition of these rules was given by Antoniadis, Bachas and Kounnas [1]so I will refer to them as the ABK rules (however these rules were also developed by Kawai, Lewellen and Tye). We will see first what these rules are, and then how we derive the phenomenology of such a model from its boundary condition vectors and its coefficients.
For further convenience we will call the boundary conditions vectors used to define a model the basis vectors (as it will be explained later on) and the coefficients the one-loop phases (as they appear in the partition function).

3.1.1 Rules on the basis vectors

I start by recalling that all these rules come from the modular invariance of the partition function.

First, these vectors span a subgroup of $((\mathbb{R}/\{2\mathbb{Z}\})^{64},+)$ called Ξ, as we have 64 fermions and the boundary condition is defined modulo 2, as the phase is $-i\,\alpha(f)\,\pi$ ($\alpha = 1, 0$ for periodic and antiperiodic conditions). The reason why all the phases are not all in $\mathbb{Z}_2$ (i.e. periodic or antiperiodic) will be explained later on, and come from generalized boundary conditions. In our study we may however restrict to rational phases. Thus, because of this condition of rationality, there is for each basis vector b_i an integer N_i which is the smallest natural integer n such as $n\,b_i = 0$. Thus we can write each element ξ of Ξ as

[1] *Four-Dimensional Superstrings.* Published in Nucl.Phys.B289:87,1987

follows:

$$\xi = \sum_i n_i b_i$$

with

$$m_i \in [1, N_i]$$

Secondly the b_i must form a basis of Ξ which means that they are linearly independent.

We will use a scalar product over this group, which is slightly different from the usual one

$$a.b = \sum_{Left-movers}^{f} (a(f)b(f)) \quad - \sum_{Right-movers}^{f} (a(f)b(f))$$

that will help us to define two other rules:
For all basis vectors b_i, b_j

$$N_i \, b_i.b_i = 0 \; [8] \qquad \text{and} \qquad N_{i\vee}N_j \, b_i.b_j = 0 \; [4]$$

where $N_{i\vee}N_j$ is the least common multiple of N_i and N_j. Lastly, the vector 1 (that is to say each boundary condition equals 1), is always part of the basis.

3.1.2 Rules on the one-loop phases

Before talking about these coefficients I introduce the useful notation $\delta_{b_i} = e^{i\pi b_i(\psi^\mu)}$ where b_i is a basis vector. Here ψ_μ is a complex fermion that I will describe later.

The possible values of the one-loop coefficient $C(b_i, b_j) = \binom{b_i}{b_j}$ are:

$$\begin{pmatrix} b_i \\ b_j \end{pmatrix} = \delta_{b_i} e^{2i\pi \frac{n}{N_j}}$$

where n is an natural integer. Being taken with this form, the coefficients must then obey the two rules :

$$\begin{pmatrix} b_i \\ b_j \end{pmatrix} = e^{i\pi \frac{b_i \cdot b_j}{2}} \begin{pmatrix} b_j \\ b_i \end{pmatrix}^*$$

$$\begin{pmatrix} b_i \\ b_i \end{pmatrix} = -e^{i\pi \frac{b_i \cdot b_i}{4}} \begin{pmatrix} b_i \\ 1 \end{pmatrix}$$

We can extend the coefficients to $\Xi \times \Xi$ with the formula $\binom{b_i}{b_i+b_k} = \delta_{b_i} \binom{b_i}{b_i} \binom{b_i}{b_k}$, combined with the other rules this gives us a unique value for each $\binom{\alpha}{\beta}$ $(\alpha, \beta \in \Xi)$, the formula being given in Appendix B.1. Hence we have now characterized all the coefficients of the partition function.

3.2 Deriving the states

We know from our theoretical study that for each vector $\alpha \in \Xi$ there is an Hilbert space of states that are obtained by acting with raising operators on the vacuum. Such a state could be:

$$b^5_{-\frac{1}{2}} b^8_{-\frac{1}{2}} d^{12}_{-1} |0\rangle$$

which is the state obtained by acting on a ground state (i.e. annihilated by all lowering oscillators) with the oscillators $b_{-\frac{1}{2}}$ of the 5th and 8th fermions (I have assumed an implicit numeration of the 68 fields), they are here antiperiodic, and the d_{-1} oscillator of the 12th field which is periodic (and so it can be either a fermionic or a bosonic field).

3.2.1 Periodic fermions and complex fermions

Here we notice that as a d_0 oscillator either anticommute or commute with all lowering oscillators (whether they are bosonic of fermionic fields), so $d_0|0\rangle$ is also annihilated by all lowering oscillators and hence is a ground state. As $d_0 d_0$ is proportional to the identity (because of the anticommutation relation $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$), we see that each periodic field generates a doubly degenerated ground state. In the following the vacuum will be described by two states $|+\rangle_i$ and $|-\rangle_i$ where i is the number of the field and we will refer to them as Ramond vacua (as they come from periodic fermions).

I will also introduce the notation for complex fermions. Indeed each pair of our real fermions that have the same boundary conditions in each basis vector (and thus in all the possible sectors) can be paired as

$$\lambda_{ij} = \frac{1}{\sqrt{2}}(\lambda_i + i\lambda_j)$$

This fermion has a conjugate which is linearly independent:

$$\lambda_{ij}* = \frac{1}{\sqrt{2}}(\lambda_i - i\lambda_j)$$

as the λ_i are reals. Thus we have replaced two real fermions by a complex one and its conjugate. As λ_{ij} has the same boundary condition and verifies the same equations, it can be expanded in Fourier modes.

We may here pay attention to the fact that the coefficients of the expansion of $\lambda_{ij}*$ are not the conjugates of those of λ_{ij}. Indeed with the expansion $\lambda_{ij}* = \sum_{r\in Z+\frac{1}{2}} c_r^{ij} e^{-2in\sigma^\pm}$ we have

$$c_r^{ij} = b_r^i - ib_r^j$$

where b_r^i is the coefficient in the expansion of the real fermion i (for the purpose of the example I have taken an antiperiodic expansion, but this is analogous for a periodic expansion).

However if we call b^{ij} the coefficient of the expansion of λ_{ij}, we have $b_r^{ij} = b_r^i + ib_r^j \neq c_r^{ij*}$ since b_r^i are complex numbers.

There is a great difference between the conjugate of the oscillator of a complex fermion and the oscillator of the conjugate of a complex fermion, indeed if one is a raising operator the other will be a lowering operator.

The complex fermions allow us to have different boundary conditions : indeed the general boundary condition will be $\psi(\sigma+2\pi,\tau) = -e^{-\alpha\pi}\psi(\sigma,\tau)$, where α is a real in]-1,1] and the same goes for the time condition.

We can check that the oscillators from a complex fermions have the same anticommutation relation :

$$\{\lambda_r^{ij}, \lambda_s^{kl}\} = \frac{1}{2}\{\lambda_r^i + i\lambda_r^j, \lambda_s^k + i\lambda_s^l\} = \delta_{(i,j)=(k,l)}\delta_{r-s}$$

with the implicit rule that 2 complex fermions cannot share a real fermion (unless, of course, they are conjugate of each other).

As an example the superpartners $\psi^{\mu=1}, \psi^{\mu=2}$ of the bosonic field have always in our model the same boundary conditions and are paired in a complex fermion ψ^μ. We will see in our study that some of the fermions are always complex fermions. I give here a set of names for the real fermions, and a set of usual pairing, which will mainly be used after.
Real left fermions
$\{\psi^{\mu 1}, \psi^{\mu 2}, \chi^1, y^1, \omega^1, \chi^1, y^1, \omega^1, \chi^2, y^2, \omega^2, \chi^3, y^3, \omega^3, \chi^4, y^4, \omega^4, \chi^5, y^5, \omega^6\}$
Real right fermions
$\{\overline{y}^1, \overline{\omega}^1, \overline{y}^1, \overline{\omega}^2, \overline{y}^2, \overline{\omega}^3, \overline{y}^3, \overline{\omega}^4, \overline{y}^4, \overline{\omega}^5, \overline{y}^6, \overline{\omega}^6\}$

And the complex fermions are
Complex left fermions $\{\psi^\mu, \chi^{12}, \chi^{34}, \chi^{34}\}$
Complex right fermions $\{\overline{\psi}^1, \overline{\psi}^2, \overline{\psi}^3, \overline{\psi}^4, \overline{\psi}^5, \overline{\eta}^1, \overline{\eta}^2, \overline{\eta}^3, \overline{\phi}^1, \overline{\phi}^2, \overline{\phi}^3, \overline{\phi}^4, \overline{\phi}^5, \overline{\phi}^6, \overline{\phi}^7, \overline{\phi}^8\}$
As we notice here, the last 16 complex right fermions are always in a complexified form. Moreover some of the fermions are not paired (precisely the y and ω), which is due to the fact that their boundary conditions do not always allow a pairing.

3.2.2 The massless spectrum

We can focus on the mass of the states we are creating, indeed for each state the mass is given by the Virasoro constraint, so that acting with an oscillator d_{-n} (or b_{-r}) will raise the mass term of n (or r).
The Mass formula must take all fermions into account, and will hence be slightly different from the one we saw above.

$$M^2 = -\frac{1}{2} + \frac{\alpha_L.\alpha_L}{8} + N_L = -1 + \frac{\alpha_R.\alpha_R}{8} + N_R$$

where α_L is the left part of α (which is the sector we are looking at) and α_R the right part (the minus sign in the scalar product must not be counted in $\alpha_R.\alpha_R$). $N_L = \sum_{i,r,n} r b^i_{-r} b^i_r + n d^i_{-n} d^i_n$ is the sum of the oscillator numbers over all the left-moving fermions, N_R being defined similarly.

The states we will be interested in are the massless states (because we need the spectrum of the effective string theory) that are deduced from every vector $\alpha \in \Xi$. We will call the set of states generated by α a sector (as mentioned before), and thus the spectrum is the union of all the sectors.

3.2.3 The GSO projection

One of the consequences of the modular invariance of the partition function is that part of the possible states must be taken away, which is done by applying a GSO projection (from its authors Gliozzi, Scherk and Olive) which is the following:
in the Hilbert space generated by α the only states which survive are the $|s\rangle$ so that

$$\left(e^{i\pi b_j \bullet F} - \delta_\alpha \begin{pmatrix} \alpha \\ b_j \end{pmatrix}^*\right)|s\rangle = 0$$

for all the basis vectors b_j. Here we have to define a fermion number F =

- +1 for a real antiperiodic oscillator
- +1 for a complex antiperiodic oscillator
- -1 for the conjugate of a complex antiperiodic oscillator
- 0 for a Ramond vacua $|+\rangle$
- -1 for a Ramond vacua $|-\rangle$

and a new scalar product $\bullet$:

$$b_j \bullet F = \sum_{Left-movers} (b_j(f)F(f)) \quad - \sum_{Right-movers} (b_j(f)F(f))$$

where the sum runs on complex or reals fermions.

To have a little insight of how this projection arises (a complete insight simply needs to derive thoroughly the partition function), I may say that this partition function is the trace of an operator (constructed with the Hamiltonian operator and the oscillator number), which can be written as a product of operators. Thus, if a state is in the kernel of one of these operators, it will have no contribution to the partition function, and thus can be taken away (we say that it was projected out by the operator).

3.2.4 Charges

We are now able to produce the massless spectrum, as we know how to build the states in each sector and how to project into physical states with GSO projections, but we are also interested in the internal symmetries that were represented by the additional internal fermions, which will produce the gauge group.

The easiest way to construct the gauge group is to find the gauge bosons in the massless spectrum, and then finding the group structure which they can fit with. Then all the massless spectrum can be described in terms of representations of this group. It is a major point in our aim to obtain realistic model that our gauge groupe be of the form $SU(3) \times SU(2) \times U(1) \times G \times G_H$ where G is a group which is broken at the standard model scale and G_H an hidden group under which the hidden matter transforms, and also to find the generation of fermions with the same quantum number under $SU(3) \times SU(2) \times U(1)$ that in the standard model.

Another requirement for our model is to an have anomaly-free model, for instance with charges with a vanishing trace over the states. However there is often one U(1) symmetry that remains anomalous, and we have to break it by the Dine-Seiberg-Witten mechanism. I do not emphasize this calculation, which I have not studied in details, but the interesting point for our study is that this mechanism gives non vanishing VEV (vacuum expectation value) to some superfields, and thus can give heavy mass to some states.

3.3 The aim of this model

Let us remind the aim of our model. We want to find new symmetries that would be broken at the current experimental scale (that is why they would not have been already detected), but which would account for the phenomenological issues of the proton decay and the μ problem, two well-known problems of the MSSM. The first one is due to the presence in the superpotential of non-renormalizable terms that mediate the proton decay : these terms are not sufficiently suppressed to account for the long life-time (10^{31} years) of the proton. The second one deals with the term $\mu h_1 h_2$ in the superpotential (where h_1, h_2 are the two Higgs doublet of the MSSM) : the current experiments fix the value of μ at a certain scale (the electroweak scale), although there is nothing predicting this scale in the MSSM. An additional symmetry would be helpful in the first case by suppressing the unwanted operators (those who mediate a fast proton decay) of the superpotential; indeed, as long as the symmetry is not broken the non-vanishing terms of the superpotential must have a vanishing total charge under this symmetry (and so this suppresses the operators with a non-zero charge). In the second case this symmetry would suppress the term $\mu h_1 h_2$ by giving non-opposite charges to h_1 and h_2, up to a certain scale which would match with the experimental bounds.
To that extent we have to derive the whole massless spectrum, calculate the symmetries, veryfing if they are anomalous or not, and calculate the superpotential. This will be done in the study of every model.

Chapter 4

Analysing a model

I have first obtained the massless spectrum of a realistic model with a set of basis vectors and the associated phases given [1](I have reproduced these in Appendix B.2). The bibliographic researches I did upon the standard-like free fermionic models (as in *Minimal Standard Heterotic String Models.* Eur.Phys.J.C50:701-710,2007., *Phenomenological study of a minimal superstring standard model.* Nucl.Phys.B593:471-504,2001 and others) indicated that these types of models imply a high amount of simple calculations. Indeed there are usually 8 basis vectors that lead to about 512 different sectors, and although these vectors are composed mainly of 0 and 1 this leaves us with a huge amount of equations to verify, only to check the ABK rules. I decided then to implement a program to check the consistency rules, derive the massless spectrum, find the suitable gauge group and calculate the superpotential. Of course all of this was not obtained from the first try, but I improved the program as I my understanding of the model grew.

4.1 How to calculate the spectrum

4.1.1 ABK rules and sectors

The first thing to do, when given a set of basis vectors and theirs coefficients, is to make sure that they verify the ABK rules. This can be easily done by a program, and in the same time the program can find all the possible sectors.

4.1.2 Massless spectrum

As we are interested by the massless states only, we have to check the Virasoro condition for M=0, which give us a constraint on the oscillators. More precisely, we know how many different oscillators are in the expression of a state of a given sector. For example if we get the equation N=1, that means that we can have either 2 oscillators of the form $b_{-\frac{1}{2}}$ or one oscillator d_{-1}. More generally, as some of the fermions can have a rational boundary conditions, we

[1] *A New standard - like model in the four-dimensional free fermionic string formulation.* A.E. Faraggi, Phys. Lett. B 278 (1992)

may have several combinations as two $b_{-\frac{1}{4}}$ and one $b_{-\frac{1}{2}}$, and so on.

All the periodic fields are also generating doubly degenerated Ramond vacua with their operator d_0, as we said before, that multiplies in the same way the number of possible states (to be more precise, having n periodic fermions turn a single state into a spinorial representation of SO(n)). Here again, finding all the states that match the Virasoro condition and applying then the GSO projection is a work that can be done using a program.

There are however some specific features that I have used which will be useful later : first the fermions must be paired as much as possible. Indeed we will have Ramond vacua only for complex fermions (and not their conjugates). Furthermore if we want to write all the possible Ramond vacua of complex fermions, we need to have a complete pairing, but this is impossible in this model. We will then have another pairing, which is specific to Ramond vacua and which allows to pair a left-moving and a right-moving real fermion. Such a pairing would be $y^2\overline{y}^2$, because they have the same boundary conditions in each sectors. We call this pairing an Ising model, we will see them again when calculating the superpotential. In fact Ising models rise a small problem in the GSO projection : indeed there is a scalar product of the fermion number of a state with a basis vector of the form $\sum(Left - Right)$, whereas the Ising model operator is neither Left or Right : this is solved by transforming the scalar product to $\sum(Left - Right + \text{Ising-Model})$.
Here we must pay attention to the fact that there is actually two different pairing in our model, one which is used to determine the Ramond vacua (where we can have Ising model operators) and one which is used to determine the oscillators acting on the vacuum (where there are only left and right-moving oscillators but where we have also the conjugates of the complex fermions).

I will know describe the states I obtained with the first model. We can partially describe them by the sector where they come from, as an example if α is a sector, then $\alpha+S$ is the sector which contains the superpartners of the states of α. Here S is a basis vector given in Appendix B.2. This being said we only have to characterized half of the spectrum, the other being its superpartner. Thus it is often quite useful to characterized the states by the sector.

4.1.3 Description of the spectrum

We are now talking about the specific model I cited at the beginning of the chapter. The first point that I will make is about the space-time spin of the states. Indeed states that are in a sector where the fermion ψ_μ is antiperiodic will be bosons (either spin 1 if ψ_μ is acting as an oscillator or spin 0 scalar otherwise), whereas states where ψ_μ is periodic will be in a spinorial representation of SU(2) (because of the Ramond vacua) and thus have spin $\frac{1}{2}$. Acting with a bosonic oscillator X^μ will also raise the spin by one, so we can also have spin $\frac{3}{2}$ and spin 2 particles. Now let us analyze the different states

- The graviton/gravitino
  Its is a generic feature of heterotic strings that they always contain a gravi-

ton. Here we find it in the 0 sector (where all fermions are antiperiodic) in the form $\psi^{\mu}_{-\frac{1}{2}}\delta\overline{X}^{\nu}_{-1}|0\rangle$ (which is a spin 2 particle). Our study will not be linked any more with this graviton but it is important to realize that this permit to include the gravity in our model. The graviton will of course have a superpartner, in the S sector, the gravitino with spin $\frac{3}{2}$.

- The gauge group
The gauge bosons are the states that have a ψ^{μ} oscillator acting on the vacuum (this oscillator being different from a zero mode). We get a certain number (37) in the 0-sector, that we will call the NS sector, as all the fermions are antiperiodic in this sector. They are of the form $\psi^{\mu}\phi^{a}\phi^{b}|0\rangle$, from now on I do not write the $-\frac{1}{2}$ index of these oscillators as all the antiperiodic fermions will give only these oscillators, no higher modes being allowed because of the mass condition. We also get 16 in the $1+b_1+b_2+b_3$ sector. We have now to determine the structure of a gauge groupe that they could fit in. We do this by assuming that each complex fermion generates a current which leads to a Cartan generator of a specific algebra, which means that all the states of the form $\psi^{\mu}\phi^{a}\phi^{a*}|0\rangle$ where ϕ^{a} is a complex fermion correspond to the Cartan generator. The group structure is then obtained by calculating the roots associated to the others gauge bosons in respect to these Cartan generators.
We end up with the group $SU(3)\times SU(2)\times U(1)\times U(1)\times U(1)^{6}\times SU(5)\times SU(3)\times U(1)\times U(1)$. We can divide this group in the observable and the hidden sector assuming that certain fermions are related to the observable or the hidden sector by stating that the hidden group is the group under which the fermionic states (that we will see in the next paragraph). It turns out that the $\overline{\phi}^{i}$ generate the hidden group. and the hidden sector is $SU(5)\times SU(3)\times U(1)\times U(1)$. We will see later that the $U(1)\times U(1)^{6}$ is broken in this model by some fields acquiring VEV, which will leaves us with a gauge group perfectly compatible with the Standard Model.

- The fermions generation
After having discovered our group of internal symmetries, it would be a nice thing to get some fermionic states that would transform under this group. The sectors b_1, b_2, b_3 give us each 16 states that are in a 16 representation of SO(10) usual for one generation of the Standard Model (that means we get two triplets of SU(3) which correpond to the left-handed quarks of type up and down, 2 singlets that are the left-handed electron-type and neutrino leptons, one (3,2) (a triplet under SU(3) and a doublet under SU(2)) which are the right-handed quarks and a doublet of SU(2) for the right-handed leptons). All of these are spinors under the Ramond vacuum of ψ^{μ}, and so they are fermions. Getting three generations of these means that we have exactly all the fermions of the standard model. We begin to see here why these models are called standard-like realistic models, as it could not be easily expected from our 8 basis vectors to generate exactly the three generations with the right quantum numbers under the observable symmetries (these generations are detailed in Appendix B.2).

- The hidden matter
They are also approximately 40 sectors that give states with non-zero

quantum numbers under the hidden group. Hence these states correspond to hidden matter. I will be concise about these states, because their main utility is to acquire a VEV when canceling the anomalous symmetry which is discussed in the next paragraph, and by coupling with matter states in the superpotential give heavy mass to these states.

- The Higgs and other observable particles
  There are still some states that transform uniquely under the observable group that can be found in the NS sector and the $b_1 + b_2 + \alpha + \beta$ sectors. Some of them are doublets under SU(2) and neutral under the U(1), they will turn to be Higgs particles (in fact, they will couple to the fermions generations in the superpotential).

4.1.4 Anomalous symmetry

Out of the 8 U(1) symmetries, only one remains anomalous (i.e. with a non-vanishing trace among the states) after an orthogonal transformation. The breaking of this symmetry (which is described in A.E. Faraggi, Phys. Lett. B 278 (1992)) will give VEV to some fields, in respect to a few constraints describe by the Dine-Seiberg-Witten. The choice of the fields that get a VEV will detremine which symmetries will be broken, because a field with a non-zero charge under a U(1) symmetry (or in a non-trivial representation of a non abelian group) breaks this symmetry when acquiring a VEV. In my first model, we can choose these fields to break the observable group down to $SU(3) \times SU(2) \times U(1)$, which is the gauge group in the Standard Model.

Since we found that our model was generating a very realistic spectrum, we may now turn to calculate the superpotential to see how this model can help us with the two issues of the MSSM we are dealing with.

4.2 The superpotential

In a string theory model, one must take into account all the terms, whether they are renormalizable or not. In a standard model, the nonrenormalizable term are suppressed by the cutoff, but in our model they may have important value, all the more that some fields can acquire large VEV and thus compensate the $(\frac{1}{M_s})^{n-3}$ term for higher order (where M_s is the string scale).

It is quite interesting to focus on the first order terms, which are the 3 dimension operators. The expression of the third-level superpotential in this model and the way to obtain it are described in the Appendix B.3. We may only focus on the relevant terms:

As an example we get the term $\overline{h}_i u_i Q_i$ where $\overline{h}_i$ (i=1..3) is a scalar doublet of SU(2) in the NS sector, and u_i, Q_i the right and left-handed quark as mentioned before. We see here that these are Yukawa couplings between Higgs and the fermions, that will give mass to the fermions. However from the three Higgs $\overline{h}_i$, two of them will get heavy mass because of couplings with fields that acquire a VEV (at higher orders. Hence there is only one generation that acquires a mass term from the cubic superpotential, which is an explanation of the heaviness of the top quark.

Chapter 5

Phenomenological issues: The μ problem

5.1 The μ problem

We may now focus on some peculiarities of the physics given by the model. I will focus more on the μ problem, which was the topic of my work.

Our aim, as we said before, is to find one U(1) symmetry in respect to which $h_1 h_2$, where h_1, h_2 are the two light Higgs, is not an invariant, that is to say h_1 and h_2 do not have opposite charges under this U(1). Hence this term would be generated only when this U(1) is broken, and thus μ would be related to the scale at which this U(1) is broken, and to M$_{string}$ directly.

I have been looking for such a symmetry in another model, a left-right symmetric model (see *Flat directions in left-right symmetric string derived models.* Cleaver CLements Faraggi, Phys.Rev.D65:106003,2002.), where the $SU(2)_L \times U(1)_L$ is replaced by a $SU(2)_L \times SU(2)_R$. The first thing to do is to identify the Higgs, which is done by analyzing closely the superpotential. We only know that the Higgs are doublets under $SU(2)_{L/R}$, but to decide which doublets are relevant we must see their couplings in the superpotential.
The search for Higgs is hence a necessary step in order to solve the μ problem. However finding the Higgs require much more work that I thought during most part of my internship : indeed you start by taking all the doublets under SU(2) in the model that are neutral under $SU(3) \times U(1)$, which is an easy job, but then you have to calculate the couplings of these doublets to the other particles to see if they couple with the fermions generations, and at last you have to check if some of these will acquire a heavy mass through a coupling with a field with a large VEV. For instance in the first model (ref 1) we started with 8 Higgs, but by analyzing the superpotential term up to the 7th order, we see that only two combinations of these Higgs remain light. This problem was a major obstacle in solving the μ problem, because before looking for a symmetry that would differentiate the two Higgs doublets of the MSSM, I should have first found the combinations of Higgs that remain light and check that there are only two of them.

In the model of Ref[1], the 8 Higgs doublets are in the NS sector and have the following form
$h_1 = [(\mathbf{1}, 0); (\mathbf{2}, -1)]_{1,0,0,0,0,0}$
$h_2 = [(\mathbf{1}, 0); (\mathbf{2}, -1)]_{0,1,0,0,0,0}$
$h_3 = [(\mathbf{1}, 0); (\mathbf{2}, -1)]_{0,0,1,0,0,0}$
$h_{45} = [(\mathbf{1}, 0); (\mathbf{2}, 1)]_{-\frac{1}{2},-\frac{1}{2},0,0,0,0}$
where the notation stands for the representations of the gauge group $[(\mathbf{SU(3)}, U_C); (\mathbf{SU(2)}, U_L)]_{U_1,U_2,U_3,U_4,U_5,U_6}$. The four other doublets are called the complex conjugates of these 4 doublets, because they have opposite charges, but they are not exactly the conjugate of the states. Thus, we have:

$$h_1 = \chi^{12}\{\overline{\psi}^{4,5*}\}\overline{\eta}^1|0\rangle$$

and

$$\overline{h_1} = \chi^{12}\{\overline{\psi}^{4,5}\}\overline{\eta}^{1*}|0\rangle$$

My program was able to calculate the third level superpotential in approximately 2 minutes. However going one order higher would multiply the calculation duration by a factor equal to the number of states, i.e. around 200. With some few improvements I could have possibly go to the fourth and fifth order, but going higher would have demanded a specific study in itself. That was mostly the reason why I could not recognize properly the Higgs in my models, and so could not find constraints of any candidate of U(1) symmetry.

Chapter 6

Conclusion

The first point of this analysis is to show us how the femionic construction of the heterotic string can be a powerful method to access to MSSM-like models easily. Indeed we have found that many of the features of this model were to be derived from the spectrum of a heterotic string. But the analogy is not the only goal of our model, we have also found tools that could give us extra laws to describe the MSSM, as is shown in the study of the μ problem. These are two reasons that would lead us to carry out more concrete work on this kind of model. I will however have a more neutral judgment on the free-fermionic models : there are many different models, and each is very simple to define (what we need is just a set of basis vector and their coefficients), and at first sight there is no reasons to choose one model over an other (there are some specific constraints if we want a standard like-model, i.e. with 3 generations and a suitable gauge group, but that still leaves us with a lot of different models). In such circumstances what we do is analyzing thoroughly one first model, see how we can derive interesting features from it, and then turn to another model, but there is no such thing as a theoretic way to obtain the best (or more realistic) model. To that respect, I would be tempted to say (with the very little experience I have in this field) that this fermionic construction is powerful to raise interest in string derived theories, but is not likely to be the basis of the next Standard Model.

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Chapter 7

Bibliography

I found most of my formulas on the free-fermionic constructions in the papers
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Calculable Nonrenormalizable Terms In String Theory: A Guide For The Practitioner. S. Kalara, Jorge L. Lopez, Dimitri V. Nanopoulos Published in Nucl.Phys.B353:650-682,1991.
The Essential role of string derived symmetries in ensuring proton stability and light neutrino masses. Jogesh C. Pati Published in Phys.Lett.B388:532-542,1996.
The theoretical background was learned using the following books:

String theory and M-theory. A modern introduction by Becker, Becker, Schwarz.
Supersymmetry and String Theory by Dine.
Supersymmetry. An introduction with conceptual and calculational details by Muller-Kirsten, Wiedenmann.
Symmetries, Lie Algebras and Representations by Fuchs, Schweigert.

Chapter 8

Appendix

A.1 Invariance of the string action

The string action in invariant under the following transformations:

- *Poincaré transformations.* These are the world-sheet affine transformations:
$$X^\mu \to a^\mu_\nu X^\nu + b^\mu$$
where a^μ_ν stands for a lorentz transformation and b^μ a translation.

- *Reparametrizations* If we change the coordinates σ^i to σ'^i the action is unaltered and we have:
$$if\ \sigma^i \to \sigma'^i = f^i(\sigma) \qquad \text{then} \qquad h_{\alpha,\beta}(\sigma) = \frac{\partial f^\gamma}{\partial \sigma^\alpha}\frac{\partial f^\delta}{\partial \sigma^\beta} h_{\gamma,\delta}(\sigma')$$

- *Weyl transformations* We can also rescale the metric:
$$h_{\alpha,\beta} \to e^{\phi(\sigma,\tau))} h_{\alpha,\beta}$$

- *superconformal transformations and modular transformations.*

These invariance will give us the freedom to take different gauges of our fields. For instance the first thing we can do is reduce the metric $h_{\alpha,\beta}$ to the Minkowski metric, by using reparametrizations and Weyl rescaling. Weyl rescaling also accounts for the vanishing of the trace of the energy-momemtum tensor.

However the change of the metric does not fix the invariance completely, and we can add the light-cone gauge and superconformal gauge.

A.2 Supersymmetry over the world sheet

Using the world sheet super coordinates $\sigma, \tau, \theta_-, \theta_+$ the generators of the supersymmetric transformations are

$$Q_A = \partial_{\overline{\theta}^A} - (\rho^\alpha \theta)_A \partial_\alpha$$

and they are acting on the superfield Y related to the bosonic and fermionic fields X and ψ by $Y^\mu = X^\mu + \overline{\theta}\psi^\mu$. In order to have an invariant acion it is useful to introduce the supercovariant derivative $D_A = \partial_{\overline{\theta}^A} - (\rho^\alpha\theta)_A\partial_\alpha$

The action then have the form

$$S = \frac{i}{4\pi}\int d\sigma d\tau d^2\theta \overline{D} Y^\mu D Y_\mu$$

which is developed as:

$$S = -\frac{T}{2}\int d\sigma d\tau \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right)$$

A.3 Superconformal gauge

The energy-momemtum tensor has the expansion

$$T_{--} \propto \sum_m L_m e^{-im\sigma^-} \qquad \text{and} \qquad T_{++} \propto \sum_m \overline{L}_m e^{-im\sigma^+}$$

where $L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} : d_{m-n}d_n :$. The terms T_{+-}, T_{-+}, which also vanish are not relevant (because they are equivalent to the first equations of motion we found).

The L_m form a super-Virasoro algebra, and by quantization of the string we will require that all the positive L_m have to annihiliate the physical states, which turn in

$$L_m|\phi\rangle = 0 \qquad \text{for m¿0} \qquad (L_0 - a)|\phi\rangle = 0$$

and the same goes for righ-movers $\overline{L}_m$.
The a constant is there to parametrize the normal-ordering prescription of the L_m. a is determined by the constraint that all the states must have a positive norm.

A.4 Partition Function

We said that the partition function was a sum over all world-sheet of the topology of a torus.

Implicitely we have thought each world sheet as describing a different physical event when we have formulated the sum, but we have seen that two world-sheets that could be related by some transformations (as reparametrizations, Poincarré and Weyl transformations, and so on) would describe the same event. We then have to count each class of equivalence under these transformation once. That means that we will integrate only over unequivalent tori.

This paragraph is a part of the work on partition function, that I give only to have an insight on how the ABK rules can be derived from the invariance of the partition function. A complete work can be find in (*Construction of Fermionic String Models in Four-Dimensions.* by Kawai, Lewellen, Tye Nucl.Phys.B288:1,1987.) and my calculations come from it.

Every torus is parametrized by τ which is the rapport between the two dimensions of the torus (i.e. in our case the length of the space dimension is the length of the string 2π, and so the length of the time dimension will be $2\pi\tau$). The partition function of one fermionic field propagating around a torus of parameter τ can be computed as the sum of overlap between a state before and after one time-loop on the torus, that means for a time equals to $2\pi\tau$. This sum of overlap will hence be the trace of an operator, which is derived from the Hamiltonian

$$Z[u,v] = Tr\big(e^{i2\pi\tau H_u} e^{i2\pi(\frac{1}{2}-v)N_u}\big)$$

We recognize here the action of the Hamiltonian H_u (which depend from the space boundary condition u of the fermion) during a time $2\pi\tau$ and the action of the number operator N_u which count the number of oscillators ($N = \sum b_{-r}b_r$). Using the variable $q = e^{i2\pi\tau}$ will simplifies the equation :

$$Z[u,v] = Tr\big(q^{H_u} e^{i2\pi(\frac{1}{2}-v)N_u}\big)$$

We can now see how this expression transform under a modular transformation such as:

$$\tau \to -\frac{1}{\tau} \qquad Z[u,v] \to e^{i2\pi(u-\frac{1}{2})(v-\frac{1}{2})} Z[v,-u]$$

In taking account all the 64 fermions boundary conditions, we would have a similar equation, where the phase would be given by a scalar product of the two basis conditions vectors α, β. Assuming that the partition function is invariant under modular transformations would so ensure some relations between the coefficients of $Z[\alpha,\beta]$ and $Z[\beta,-\alpha]$ using scalar product of the vectors. We have hence obtain one ABK rule.

B.1 Formulae for the one-loop coefficients

From the last rule on the one loop phase,

$$\begin{pmatrix} b_i \\ b_i + b_k \end{pmatrix} = \delta_{b_i} \begin{pmatrix} b_i \\ b_i \end{pmatrix} \begin{pmatrix} b_i \\ b_k \end{pmatrix}$$

one can extract a formula to calculate $\binom{\alpha}{\beta}$, for any $\alpha, \beta \in \Xi$ but I will give here th eexpression of $\binom{\alpha}{b_i}$ where $\alpha = \sum n_j b_j$ and b_i one of the basis vector, because it is the formula used in the GSO projection:

$$\begin{pmatrix} \alpha \\ b_i \end{pmatrix} = e^{\frac{i}{2}\pi\alpha . b_i} \, \delta_{b_i}^{-1+\sum_j m_j} \prod_j \begin{pmatrix} b_i \\ b_j \end{pmatrix}^{*\, m_j}$$

B.2 First model

I write here explicitely the results of the model given in Ref:(A.E. Faraggi, Phys. Lett. B 278 (1992)).

I write here the basis vectors, as a set of fermions with the following notation: if a fermion is not in the set then its boundary condition is 0 (antiperiodic), if it is in it will be equal to 1, and if it is in with the comment :$\frac{1}{2}$, it will be equal

to $\frac{1}{2}$
1 : All fermions are periodic
S : $\{\psi^\mu, \chi^{12}, \chi^{34}, \chi^{56}\}$
b_1 : $\{\psi^\mu, \chi^{12}, y^3y^6, y^4\overline{y}^4, y^5\overline{y}^5, \overline{y}^3\overline{y}^6, \overline{\psi}^1, \overline{\psi}^2, \overline{\psi}^3, \overline{\psi}^4, \overline{\psi}^5, \overline{\eta}^1\}$
b_2 : $\{\psi^\mu, \chi^{34}, y^1\omega^5, y^2\overline{y}^2, \omega^6\overline{\omega}^6, \overline{y}^1\overline{\omega}^5, \overline{\psi}^1, \overline{\psi}^2, \overline{\psi}^3, \overline{\psi}^4, \overline{\psi}^5, \overline{\eta}^2\}$
b_3 : $\{\psi^\mu, \chi^{56}, \omega^1\overline{\omega}^1, \omega^2\omega^4, \omega^3\overline{\omega}^3, \overline{\omega}^2\overline{\omega}^4, \overline{\psi}^1, \overline{\psi}^2, \overline{\psi}^3, \overline{\psi}^4, \overline{\psi}^5, \overline{\eta}^3\}$
α : $\{y^3y^6, \omega^3\overline{\omega}^3, \omega^6\overline{\omega}^6, \overline{y}^1\overline{\omega}^5, \overline{\omega}^2\overline{\omega}^4, \overline{\psi}^1, \overline{\psi}^2, \overline{\psi}^3, \overline{\phi}^1, \overline{\phi}^2, \overline{\phi}^3, \overline{\phi}^4\}$
β : $\{y^1\omega^5, \omega^1\overline{\omega}^1, y^5\overline{y}^5, \overline{\omega}^2\overline{\omega}^4, \overline{y}^3\overline{y}^6, \overline{\psi}^1, \overline{\psi}^2, \overline{\psi}^3, \overline{\phi}^1, \overline{\phi}^2, \overline{\phi}^3, \overline{\phi}^4\}$
γ : $\{y^2\overline{y}^2, \omega^2\omega^4, y^4\overline{y}^4, \overline{y}^1\overline{\omega}^5, \overline{y}^3\overline{y}^6, \overline{\psi}^1 : \frac{1}{2}, \overline{\psi}^2 : \frac{1}{2}, \overline{\psi}^3 : \frac{1}{2}, \overline{\psi}^4 : \frac{1}{2}, \overline{\psi}^5 : \frac{1}{2}, \overline{\eta}^1 : \frac{1}{2}, \overline{\eta}^2 : \frac{1}{2}, \overline{\eta}^3 : \frac{1}{2}, \overline{\phi}^1 : \frac{1}{2}, \overline{\phi}^3, \overline{\phi}^4, \overline{\phi}^5 : \frac{1}{2}, \overline{\phi}^6 : \frac{1}{2}, \overline{\phi}^7 : \frac{1}{2}\}$

One loop phases (as the matrix $(\binom{b_i}{b_j})_{ij})$):

$$\begin{pmatrix} 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1i \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1i \\ 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1i \\ -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1i \end{pmatrix}$$

Gauge Group

The observable gauge group is given by the gauge bosons that do not involve the $\overline{\phi}^i$ fermions, there are all in the Neveu-Schwarz sector, and we can compute it easily to obtain a group $SU(3)\times SU(2)\times U(1)\times U(1)\times U(1)^6$, the U(1) charges being given by:
$U(1)_C$ $\overline{\psi}^1\hat{\overline{\psi}}^1 + \overline{\psi}^2\hat{\overline{\psi}}^2 + \overline{\psi}^3\hat{\overline{\psi}}^3$
$U(1)_L$ $\overline{\psi}^4\hat{\overline{\psi}}^4 + \overline{\psi}^5\hat{\overline{\psi}}^5$
$U(1)_1$ $\overline{\eta}^1\hat{\overline{\eta}}^1$
$U(1)_2$ $\overline{\eta}^2\hat{\overline{\eta}}^2$
$U(1)_3$ $\overline{\eta}^3\hat{\overline{\eta}}^3$
$U(1)_4$ $\overline{y^3y^6\hat{y^3y^6}}$
$U(1)_5$ $\overline{\omega^2y^4\hat{\omega^2y^4}}$
$U(1)_6$ $\overline{y^1\omega^5\hat{y^1\omega^5}}$

The hidden gauge group is much more complicated to obtain, indeed you have 18 gauge bosons (as a SO(4), a U(3) and three U(1)) from the NS sector that are mixed with 16 bosons of the $1+b_1+b_2+b_3$ sector. However we can separate the total group in a $SU(5)\times SU(3)\times U(1)\times U(1)$.

The matter states

We find in this model the three generations of fermions of the MSSM, each in a b_i (i=1..3) sector. We can specifiy them by their representation in the

gauge group. I write here the first generation.
$[(\mathbf{1},+\frac{3}{2});(\mathbf{1},+1)]_{\frac{1}{2},0,0,+\frac{1}{2},0,0} = e^1_L$ 1cm left-handed electron-type lepton.
$[(\mathbf{1},+\frac{3}{2});(\mathbf{1},-1)]_{\frac{1}{2},0,0,-\frac{1}{2},0,0} = N^1_L$ 1cm left-handed Neutrino.
$[(\overline{\mathbf{3}},-\frac{1}{2});(\mathbf{1},-1)]_{\frac{1}{2},0,0,+\frac{1}{2},0,0} = e^1_L$ 1cm left-handed up-type quark.
$[(\overline{\mathbf{3}},-\frac{1}{2});(\mathbf{1},+1)]_{\frac{1}{2},0,0,-\frac{1}{2},0,0} = e^1_L$ 1cm left-handed down-type quark.
$[(\mathbf{3},+\frac{1}{2});(\mathbf{2},\ \ 0)]_{\frac{1}{2},0,0,-\frac{1}{2},0,0} = e^1_L$ 1cm right-handed quark multiplet.
$[(\mathbf{1},-\frac{3}{2});(\mathbf{2},\ \ 0)]_{\frac{1}{2},0,0,+\frac{1}{2},0,0} = e^1_L$ 1cm right-handed lepton doublet.

B.3 Third level superpotential

Coefficients of the 3d operators

The third level superpotential is a sum over the product of any three superfields: $W = \sum_{i,j,k} A_{ijk} Y^i Y^j Y^k$, where the Y^i describe all the superfields in our spectrum. The way to calculate the amplitude A_{ijk} is the following.

in order to have A_{ijk} non-zero, the product $Y^i Y^j Y^k$ must be invariant under any gauge transformation. For instance, let us take an infinitesimal transformation under a U(1) ;

$$\delta\psi = iq\alpha\psi$$

where q is the charge of the ψ field under the U(1), and α being dependent or not from x, since we we have a global or local symmetry. If we apply this transformation to a product of two fields, we get

$$\delta(\psi_1\psi_2) = ie\alpha\ \psi_1\psi_2\ (q_1 + q_2)$$

So we can see with this simple example that all the products in the superpotential must have a vanishing total charge under each U(1).

If we turn to non-abelian symmetry the infinitesimal transformation is
$\delta(\psi_1\psi_2) = ig(\psi_1 \overrightarrow{\alpha}.\overrightarrow{T}\psi_2 + \psi_2 \overrightarrow{\alpha}.\overrightarrow{T}\psi_1)$
That means that the three supefield must be in representations that cancels each other, as for example a triplet in the adjoint representation, an anti-triplet (hence in the conjugate representation) and a singlet.

For all the products that are invariant under gauge transformations, the amplitude is determined using the correlators of the fermioic fields of each superfields, a long procedure that is well explained in the *Calculable Nonrenormalizable Terms In String Theory: A Guide For The Practitioner.*, from Kalara, Lopez and Nanopoulos (Published in Nucl.Phys.B353:650-682,1991).