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An Analysis of a Standard-Like String Model in the Free Fermionic Formuation

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Abstract

A modification of the generalised GSO coefficients in the string model presented by Faraggi et al (1989) results in a string model with Standard-like properties including an observable gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ after symmetry breaking at the string level, an anomaly free massless spectrum except for a single anomalous $U(1)_A$, three generations (plus a partial fourth generation) of chiral fermions, and Higgs doublets that may break the symmetry in a realistic way. The key difference between the model presented and the 1989 model is the level at which the $U(1)_C \times U(1)_L$ symmetry breaks; in the modified model presented here the symmetry is required to break directly at the string level.

1 Introduction

The Standard Model is a mathematically consistent theory of all known matter, plus the weak, strong, and electromagnetic interactions. However, there are problems with the Standard Model that give us reason to believe that our knowledge of particle theory is far from complete. A particular example is the incompatibility of the Standard Model with general relativity. Supersymmetric string theories such as heterotic string theory may provide a solution to this problem, unifying all known matter and interactions into one anomaly free theory in ten dimensions via the Green-Schwarz Mechanism [1]. We stumble across another issue however, in that this unification occurs at high energy scales. For this reason critics of string theory argue that at present, string theory is untestable.

The free fermionic construction of heterotic string theory presented by Kawai, Lewellen, and Tye [2] and Antoniadas, Bachas, and Kounnas [3] allows us to build and probe string models directly in four dimensions and at low, experimentally observable energies such as those of the Standard Model. A method to derive a Standard-Like Model in the free fermionic formulation was presented by Faraggi, Nanopoulos and Yuan [4].

In this paper we begin with a brief discussion of the structure of the Standard Model, summarising the properties one hopes to achieve with a standard-like string model. We then proceed to introduce string theory, discussing the method to construct space states similar to those in particle physics from the quantum string, and eventually discussing the ABK rules for building a string model in the free fermionic formulation. Finally we modify the generalised GSO projection coefficients for the model presented by Faraggi et al [4], derive the massless spectrum of this modified model and, through calculation of the anomalies, present a difference in energy at which the standard model gauge group is acheived.

Due to anomalous trace $U(1)$'s in the observable sector, we have a requirement for the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ through a unique combination of the trace $U(1)$'s, $U(1)_Y = \frac{1}{3}U(1)_C + \frac{1}{2}U(1)_L$, with the remaining observable $U(1)$ charges anomaly free except for a single $U(1)_A$ after an orthogonal rotation. This differs from the model presented in Faraggi et al; in the unmodified model the trace $U(1)$'s are anomaly free, and hence the symmetry may *either* break directly to the standard model gauge group (as in the modified model presented here), or the trace $U(1)$'s may break in an orthogonal combination $U(1)_{Z'} = U(1)_C - U(1)_L$. The model presented in this paper is standard-like directly at the string level.

2 An Overview of The Structure of The Standard Model

We begin by discussing the structure of the Standard Model and the properties we ideally wish to obtain with a standard-like string model. This chapter is just a brief overview of the particles comprising the Standard Model, and as such does not contain information on particle interactions or such like. Our primary reference for this material is P. Langacker's book *The Standard Model and Beyond* [6], however chapters from references [5, 7, 16] among others were also relevant.

The Standard Model is a Quantum Field Theory that includes all matter particles in the form of spinor fermion fields ψ, plus three of the four fundamental interactions (strong, weak, and electromagnetism) in the form of vector boson fields; the weak fields $W^{\pm}$ and Z, the photon field A and gluon fields $G^{1,\dots,8}$, and finally the Higgs boson in the form of a scalar Higgs field ϕ. Each particle also has an anti-particle, with equal mass but opposite charge.

The Standard Model is a gauge theory, with gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. Each group consists of generators and basis vectors (group theory is discussed in more detail and these generators and basis vectors are derived explicitly in Appendix D). Physically, the generators correspond to gauge bosons whilst the basis vectors correspond to the fermions on which these bosons act. The $SU(3)_C$ gives rise to Quantum Chromodynamics and couples with the eight gluon fields. The $SU(2)_L \times U(1)_Y$ gives rise to the electroweak interaction; the broken $SU(2)$ coupling to the three weak fields, and an unbroken $U(1)_Q$ coupling with the A field. It is an anomaly free theory.

2.1 Fermions

The Standard Model contains three generations of chiral fermions. The defining property of a fermion is that it has half integer spin, and all Standard Model fermions are spin 1/2. The fermion field is a four spinor, such as a Dirac or Majorana spinor. We may split the fermions into two groups; there are six leptons and six quarks, two of each per generation. Quarks carry a colour charge, and electric charge of either $+2/3$ or $-1/3$. Quarks are not found free, instead being found either in groups of three or as a quark-antiquark pair, bonded by the strong interaction of Quantum Chromodynamics. Leptons do not carry a colour charge and as such are found free in nature. Leptons with electric charge -1 are the electron, muon, and tau, whilst leptons with charge 0 are three neutrinos.

Furthermore, our fermions are chiral. For each field ψ, we can define left and right chiral projections using the chiral representation, in which we write

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{1}$$

where our left and right projections are defined by

$$\psi_L = P_L\psi = \frac{1-\gamma^5}{2}\psi, \qquad \psi_R = P_R\psi = \frac{1+\gamma^5}{2}\psi, \tag{2}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the fifth Dirac gamma matrix, and $P_{L,R}$ satify

$$P_L P_R = P_R P_L = 0, \qquad P_L + P_R = I. \tag{3}$$

We obtain

$$P_L = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{4}$$

Thus the two component representation of the four components of ψ is a Weyl spinor.

The $SU(2)_L$ coupling is chiral, as the gauge bosons $W^{1,2,3}$ act only on the flavour indices of L-chiral fermions. This results in L-chiral quark and lepton doublets in each family, and R-chiral singlets. For each family $m = 1, 2, 3$, we have the L-chiral fermions

$$Q_m = \begin{pmatrix} u_m \\ d_m \end{pmatrix}, \qquad L_m = \begin{pmatrix} e_m \\ \nu_m \end{pmatrix}. \tag{5}$$

We now note that the charges of an R-chiral fermion f_R determine the charges of the corresponding L-chiral antifermion, f_L^c. Hence we now list our L-chiral antifermions (denoted by the superscript c), which are determined by R-chiral fermions. Our L-chiral antifermions are

$$u_m^c, \qquad d_m^c, \qquad e_m^c, \qquad \nu_m^c \tag{6}$$

where u and d represent quarks with charge $+2/3$ and $-1/3$ respectively, e represents a lepton with charge -1, and ν represents a neutral lepton, or neutrino. Finally it is worth noting that each quark singlet or doublet is found with a colour charge of $\alpha = 1, 2, 3$, also called red, green, or blue. Hence there are three quark doublets and six quark singlets per family.

2.2 Gauge Bosons

The Standard Model gauge group is $SU(3)_C \times SU(2)_L \times U(1)_Y$, where the individual gauge symmetries couple with gauge bosons that give rise to the strong and electroweak interactions. All bosons have integer spin; the Standard Model gauge bosons have spin 1.

The non-chiral $SU(3)_C$ group has eight generators, so gives rise to eight gluons $G^{1,\cdots,8}$ which act as mediators for the strong interaction between quarks. Here the subscript C refers to the colour index of quarks. Since the $SU(3)_C$ is not spontaneously broken, gluons remain massless in the Higgs mechanism (see chapter 2.3).

The $SU(2)_L \times U(1)_Y$ group is the electroweak factor. This group is chiral, with the $SU(2)_L$ acting only on L-chiral fermions (hence the subscript L), whilst the $U(1)_Y$ acts on both L- and R-chiral fermions with different charges. The $SU(2)_L$ has three generators, so gives rise to three bosons $W^{1,2,3}$, whilst the $U(1)_Y$ has only one generator so gives rise to a single field denoted the B field.

In the Higgs mechanism, spontaneous symmetry breaking (SSB) breaks the $SU(2)_L \times U(1)_Y$ into a single unbroken $U(1)_Q$, resulting in the theory of electromagnetism. As a result of the Higgs mechanism, we obtain the orthogonal linear combinations

$$A = \cos\theta_W B + \sin\theta_W W^3 \tag{7}$$

$$Z = -\sin\theta_W B + \cos\theta_W W^3, \tag{8}$$

both of which are neutral. Here θ_W is the Weinberg angle. The A field, which is the photon field, remains massless, whilst the Z field obtains mass. We also define the two massive complex charged gauge boson fields

$$W^{\pm} = \frac{1}{\sqrt{2}}(W^1 \mp iW^2) \tag{9}$$

The photon acts as the mediator for the electromagnetic interaction, whilst the $W^{\pm}$ and Z bosons act as mediators for the weak interaction.

2.3 The Higgs Boson

It can be shown that gauge theories do not allow elementary mass terms for gauge bosons. However, we saw above that the $W^{\pm}$ and Z bosons are massive. The solution to this problem lies in another problem that arises from the Nambu-Goldstone theorem, which states that for every generator of global symmetry that is spontaneously broken there exists a massless, scalar particle, called a Nambu-Goldstone boson. In the Standard Model we observe no massless Nambu-Goldstone bosons.

The two problems actually resolve each other in the case that the broken symmetry is a gauge symmetry [8]. The degrees of freedom that are carried by the Goldstone boson exist not as a massless spin 0 particle, but as the longitudinal spin component of a massive gauge boson. This is the Higgs mechanism under which the gauge bosons acquire mass.

We introduce the Higgs field, the doublet $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$, transforming as an $SU(3)_C$ singlet and an $SU(2)_L$ doublet. If ϕ^0 acquires a non-zero vacuum expectation value (v.e.v), or

$$\langle 0|\phi|0\rangle = v = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \nu \end{pmatrix} \tag{10}$$

the $SU(2)_L \times U(1)_Y$ gauge symmetry breaks to an unbroken $U(1)_Q$. Since the $SU(2)_C$ and $U(1)_Y$ are broken, the appearence of Nambu-Goldstone bosons would be expected for these generators, and hence through the Higgs mechanism the degrees of freedom manifest themselves as longitudinal components of massive $W^{\pm}$ and Z bosons. These bosons aquire mass. Since the $SU(3)_C$ and $U(1)_Q$ are unbroken, the corresponding gluons and photons remain massless. Expanding the field around the non-zero v.e.v, and quantising in the unitary gauge, we obtain

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \nu + H \end{pmatrix} \tag{11}$$

where H is a Hermitian scalar field. This field represents the Higgs boson, which is a byproduct of the Higgs mechanism and exists as a massive spin 0 boson.

2.4 Extensions of the Standard Model

Supersymmetry is a theoretical property that relates fermions and bosons [9]. For each boson, there is one or more fermionic superpartner, and likewise for each fermion there is one or more bosonic superpartner, with, in an unbroken supersymmetric theory, the same mass and internal quantum numbers except, of course, spin, which differs by 1/2 since all fermions have half integer spin whilst bosons have integer spin. Since no light sparticles (supersymmetric particles) have been observed, the supersymmetry must be broken, resulting in sparticles being much heavier than their partners. The Minimal Supersymmetric Standard Model (MSSM) is the supersymmetric version of the standard model with the minimal number of fields and the minimal number of couplings.

For a string theory involving fermions, we require the involvement of a supersymmetric version of the bosonic string, the superstring. Hence when deriving a standard-like string model, we require supersymmetry to be present in the model. A requirement we will place on our string model is that we want $N = 1$ spacetime supersymmetry in four dimensions.

Another extension to the standard model we must introduce is a quantum particle to act as a mediator to gravity, called the graviton. This is a theoretical spin 2 particle that can be compared to the W and Z bosons of the weak interaction, the photons of electromagnetism, and the gluons of the strong interaction; it is the mediator of the gravitational interaction.

Since one of the primary successes of string theory is as a quantum theory of gravity, another constraint we place on our string model is that we require the presence of a spin 2 graviton.

3 From Particle Physics to String Theory

String Phenomenology is all about relating the results of string theory to particle physics. String theory models elementary particles as vibrational modes on a string. We can use the analogy of a guitar string vibrating in different modes to result different musical notes. In this analogy the musical notes are particles, and the guitar string is the fundamental string of string theory. We will now construct mathematically the relation between string theory and particle theory, beginning with a discussion of particle theory and how particle states arise from the use of creation and annihilation operators. We then introduce string theory and discuss how particle states arise from strings. In particular we shall focus on closed strings (strings which have no end points, forming a loop), since the heterotic string, which is the string theory used to construct the free fermionic formulation in which our string model is built, is a closed string theory. Our primary reference for Chapter 3.1 is the book *A Modern Introduction to Quantum Field Theory* by Michael Maggiore [7], whilst the primary reference for Chapters 3.2 - 3.5 is the book *A First Course in String Theory* by Barton Zwiebach [5].

3.1 The Foundations of Particle Physics and QFT

Elementary particle physics is described by quantum field theory (QFT). In QFT, our dynamical variables are described by operator valued fields. Much like non-relativistic quantum mechanics, examples of key operators in QFT are the position and momentum operators, which act on a particle state to measure the position and momentum of the particle respectively, and the annihilation and creation operators, which act on an n-particle state to form an $(n + 1)$-particle state.

A plane wave solution to the Klein-Gordon equation takes the form

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}e^{-ip\cdot x} + a^*_{\vec{p}}e^{ip\cdot x})\Big|_{p_0=E_{\vec{p}}}. \tag{12}$$

Upon quantisation of this field, we exchange the dynamic variables a_p and a_p^* with operators a_p and $a_p^\dagger$, where a superscript † represents the Hermitian conjugate. The operator $a_p^\dagger$ is the creation operator, whilst a_p is the annihilation operator. They have the commutation relations

$$[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{p}') \tag{13a}$$

$$[a_{\vec{p}}, a_{\vec{p}'}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}] = 0 \tag{13b}$$

The classical field (12) then becomes the quantum field

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}e^{-ip\cdot x} + a^\dagger_{\vec{p}}e^{ip\cdot x}) \tag{14}$$

Acting with a creation operator on a vacuum state $|0\rangle$ results in a one particle state, or

$$\sqrt{2E_{\vec{p}}}a^\dagger_{\vec{p}}|0\rangle = |\vec{p}\rangle \tag{15}$$

where $|\vec{p}\rangle$ is a one particle state for a particle with momentum $\vec{p}$, and $\sqrt{2E_{\vec{p}}}$ is a renormalization constant. We can define an n-particle state as

$$|\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n\rangle = \sqrt{2E_{\vec{p}_1}}\sqrt{2E_{\vec{p}_2}}\ldots\sqrt{2E_{\vec{p}_n}}a^\dagger_{\vec{p}_1}a^\dagger_{\vec{p}_2}\ldots a^\dagger_{\vec{p}_n}|0\rangle. \tag{16}$$

The annihilation operator has the property

$$a_{\vec{p}}|0\rangle = 0. \tag{17}$$

We may also define the Hamiltionian operator H and momentum operator P^i which measure the energy and momentum of a system respectively. The form of the Hamiltonian is

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3}E_{\vec{p}}a^\dagger_{\vec{p}}a_{\vec{p}} \tag{18}$$

and the momentum operator is

$$P^i = \int \frac{d^3\vec{p}}{(2\pi)^3}p^i a^\dagger_{\vec{p}}a_{\vec{p}}. \tag{19}$$

We will now verify that these operators measure the energy and momentum of a particle in the state $|\vec{p}'\rangle$. We begin with the Hamiltonian.

$$\begin{aligned}
H|\vec{p}'\rangle &= \int \frac{d^3\vec{p}}{(2\pi)^3}E_{\vec{p}}\sqrt{2E_{\vec{p}'}}a^\dagger_{\vec{p}}a_{\vec{p}}a^\dagger_{\vec{p}'}|0\rangle \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3}E_{\vec{p}}\sqrt{2E_{\vec{p}'}}a^\dagger_{\vec{p}}[a_{\vec{p}}, a^\dagger_{\vec{p}'}]|0\rangle \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3}E_{\vec{p}}\sqrt{2E_{\vec{p}'}}a^\dagger_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p} - \vec{p}')|0\rangle \\
&= E_{\vec{p}'}|\vec{p}'\rangle
\end{aligned}$$

Similarly it is possible to show that

$$P^i|\vec{p}'\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} p^i \sqrt{2E_{\vec{p}'}} a^\dagger_{\vec{p}}[a_{\vec{p}}, a^\dagger_{\vec{p}'}]|0\rangle = \vec{p}'^i|\vec{p}'\rangle.$$

Thus we have shown that H and P^i measure energy and momentum of a particle respectively.

We can write equations (15), (18), and (19) in light-cone coordinates[1] [5]. We write $\vec{p}$ in terms of the transverse momenta $\vec{p}_T = (p^2, p^3)$ and the light-cone momentum p^+. The light-cone energy p^- is fixed and specified by these values. We obtain

$$a^\dagger_{p^+,\vec{p}_T}|0\rangle = |p^+, \vec{p}_T\rangle \tag{20a}$$

$$p^+ = \int \frac{d\vec{p}_T dp^+}{(2\pi)^3} p^+ a^\dagger_{p^+,\vec{p}_T} a_{p^+,\vec{p}_T} \tag{20b}$$

$$p^{2,3} = \int \frac{d\vec{p}_T dp^+}{(2\pi)^3} p^{2,3} a^\dagger_{p^+,\vec{p}_T} a_{p^+,\vec{p}_T} \tag{20c}$$

$$p^- = \int \frac{d\vec{p}_T dp^+}{(2\pi)^3} \frac{1}{2p^+}(p^j p^j + m^2) a^\dagger_{p^+,\vec{p}_T} a_{p^+,\vec{p}_T} \tag{20d}$$

where (20a) is the light-cone equivalent of (15) (where we have dropped the renormalization factor for convinience), (20b) and (20c) are the light-cone equivalents of (19), and (20d) is the light-cone equivalent of (18) where $E_{\vec{p}}$ is the energy determined from the mass shell condition $p^2 + m^2 = 0$.

It follows that we can show vibrational modes of a string are equivalent to elementary particles if we can derive creation and annihilation operators with the properties of those above. Furthermore, in order to make any form of useful measurement on the particles resulting from acting with the creation operator on a vacuum state, we also wish to find operators for observables such as position, energy and momentum.

3.2 An Introduction to String Theory

We previously discussed how string theory models particles as vibrational modes on an elementary string. As a result, when working in string theory we are no longer studying a zero dimensional point particle which traces a one dimensional *world-line* through space-time, as we are in relativistic particle theory, but a one dimensional string which traces a two dimensional *world-sheet* through spacetime, as is demonstrated in Figure 1. We can discuss particle interacions in the same way. The left hand side of Figure 2 demonstrates the world-line of a particle P_1 decaying into particles P_2 and P_3, whilst the right hand side demonstrates the world-sheet of the same process in string theory.

We may describe the world-sheet in D-dimensional space-time as a function of each space-time coordinate $x^0, \ldots, x^D$. However, it is much more convenient to describe the world-sheet in a two dimensional parameter space, and then map the parameter space onto our D-dimensional space-time. For this we require two parameters, which we call σ and τ. We then define our target space-time by

$$\vec{x}(\sigma, \tau) = \left(x^0(\sigma, \tau), x^1(\sigma, \tau), \ldots, x^D(\sigma, \tau)\right) \tag{21}$$

Now, we define the string coordinates $X^\mu(\sigma, \tau)$ by this mapping of parameter space to target space. We capitalise the mapping functions so that we can distinguish between our space-time coordinate

[1] The light-cone coordinates are explained in Appendix B.

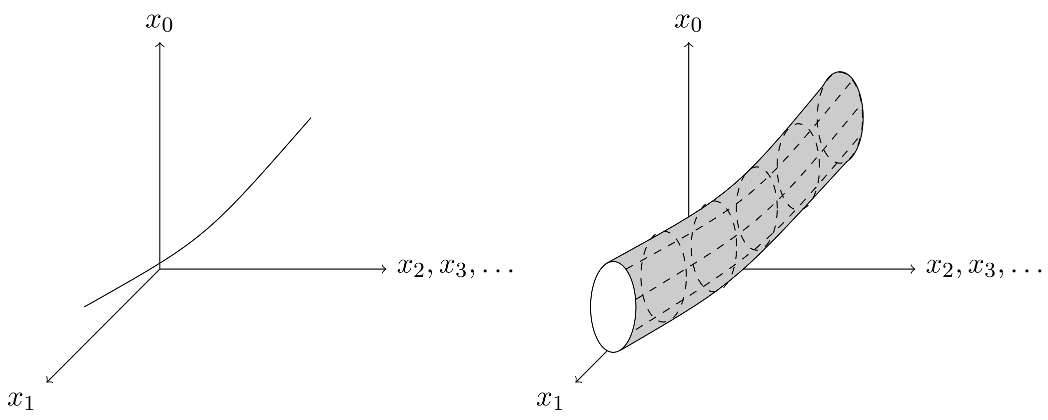

Figure 1: **Left**: The world-line of a point particle. **Right**: The world-sheet of a closed string.

Figure 2: **Left**: Particle decay. **Right**: String decay.

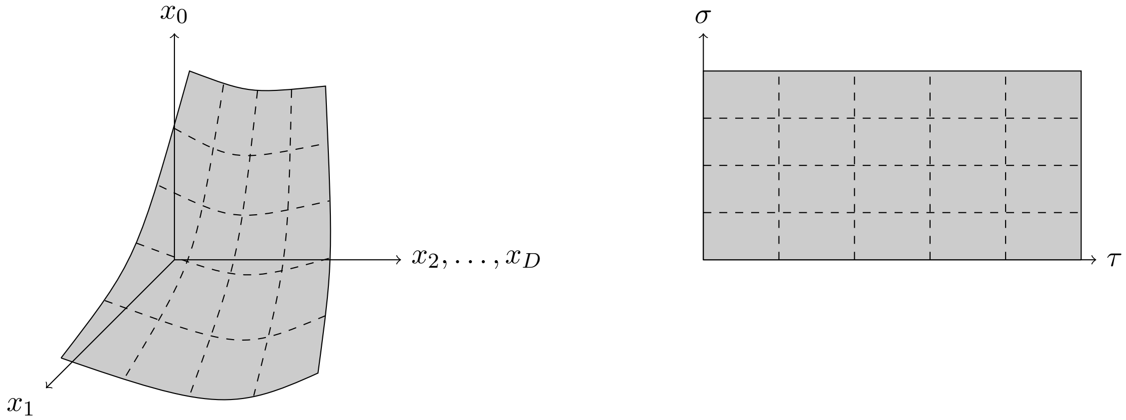

Figure 3: **Left**: Open string world-sheet in D-dimensional spacetime. **Right**: The same world-sheet in two dimensional parameter space. Dotted lines represent lines of constant σ and τ.

system and our string coordinates. Figure 3 demonstrates this mapping for an open string world-sheet.[2]

3.3 Parameterisation of the World-Sheet

We now wish to fix the parameterisation of the world sheet. Setting τ as a linear combination of string coordinates, we obtain

$$n \cdot X(\tau, \sigma) = \lambda(n \cdot p)\tau. \tag{22}$$

where p^μ is the momentum of the string, and we have chosen n such that $(n \cdot p)$ is a constant. For example, setting $n = (1, 0, \ldots, 0)$ and $\lambda(n \cdot p) = c$, we obtain

$$X^0(\tau, \sigma) = c\tau,$$

and hence in this gauge τ is the world-sheet time coordinate.

We chose to include the vector n^μ on both sides of (22) as opposed to simply writing $\lambda(n \cdot p)$ as one constant to demonstrate that the magnitude of n^μ is irrelevant. All that is important is the direction.

We now consider the constant λ. By comparing units in (22), we obtain $\lambda = \alpha'$, where α' is the slope parameter defined by

$$\alpha' = \frac{1}{2\pi T_0} \tag{23}$$

where T_0 is the string tension. Note here we are working in natural units, and only considering closed strings, since the heterotic string is a closed string. Hence we have fixed our τ parameterisation as

$$n \cdot X(\tau, \sigma) = \alpha'(n \cdot p)\tau. \tag{24}$$

[2]This is no different than for a closed string. In Figure 3 we choose an open-string world sheet as it is easier to visualise the mapping.

In order to set the parameterisation of σ, we introduce the following notation. We begin with the derivatives

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \tag{25a}$$

$$X^{\mu\prime} = \frac{\partial X^\mu}{\partial \sigma}. \tag{25b}$$

We also define our momentum densities as derivatives of the Lagrangian density, $\mathcal{L}$, with respect to $\dot{X}^\mu$ and $X^{\mu\prime}$, which are

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{1}{2\pi\alpha'}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} \tag{26a}$$

$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{1}{2\pi\alpha'}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}. \tag{26b}$$

These are calculated from the Nambu-Goto string action, which is derived and discussed in Appendix C.

Since $\mathcal{P}^{\tau\mu}$ has two σ derivatives in the numerator and effectively only one in the denominator, then under a reparameterisation $\mathcal{P}^{\tau\mu}$ varies as

$$\mathcal{P}^{\tau\mu}(\tau,\sigma) = \frac{d\tilde{\sigma}}{d\sigma}\mathcal{P}^{\tau\mu}(\tau,\tilde{\sigma}). \tag{27}$$

Dotting this equation with n, we can see that if $n\cdot\mathcal{P}^{\tau\mu}(\tau,\tilde{\sigma})$ depends on $\tilde{\sigma}$, we can adjust $\frac{d\tilde{\sigma}}{d\sigma}$ so that

$$\frac{d\tilde{\sigma}}{d\sigma} n\cdot\mathcal{P}^{\tau\mu}(\tau,\tilde{\sigma}) = f(\tau) \tag{28}$$

Combining the above equations and integrating over the string, we obtain

$$f(\tau) = \frac{n\cdot p}{2\pi} \tag{29}$$

and hence

$$n\cdot\mathcal{P}^\tau(\tau,\sigma) = \frac{n\cdot p}{2\pi}. \tag{30}$$

This is a world-sheet constant. Dotting the equation of motion $\partial_\tau(n\cdot\mathcal{P}^\tau)+\partial_\sigma(n\cdot\mathcal{P}^\sigma) = 0$, and noting that the first term on the left hand side is zero due to (30), we see that $n\cdot\mathcal{P}^\sigma$ is independent of σ.

Consider the world-sheet of the closed string as a collection of strings of constant τ. On one of these strings, we arbitrarily choose a point σ_0 to be $\sigma = 0$. The point $\sigma = 0$ on each of the other strings is then defined by $n\cdot\mathcal{P}^\sigma = 0$. Finally, since $n\cdot\mathcal{P}^\sigma$ is independent of σ, then

$$n\cdot\mathcal{P}^\sigma = 0 \tag{31}$$

at any point on the world-sheet. Equations (24, 30, 31) define the parameterisation of the world-sheet.

3.4 The Wave Equation

Equations (31) and (26b) lead to

$$(\dot{X} \cdot X')\partial_\tau(n \cdot X) - (\dot{X})^2\partial_\sigma(n \cdot X) = 0 \tag{32}$$

Equation (24) leads to the second term being zero, and $\partial_\tau(n \cdot X) = \alpha'(n \cdot p) =$ constant. Hence we obtain the condition

$$\dot{X} \cdot X' = 0. \tag{33}$$

Substituting into (26a), we obtain

$$\mathcal{P}^\tau_\mu = \frac{1}{2\pi\alpha'}\frac{(X')^2\dot{X}_\mu}{\sqrt{-(\dot{X})^2(X')^2}} \tag{34}$$

We then substitute this result into (30) to arrive at

$$n \cdot p = \frac{1}{\alpha'}\frac{(X')^2\partial_\tau(n \cdot X)}{\sqrt{-(\dot{X})^2(X')^2}} \tag{35}$$

We stated previously that $\partial_\tau(n \cdot X) = \alpha'(n \cdot p)$, so we obtain

$$1 = \frac{(X')^2}{\sqrt{-(\dot{X})^2(X')^2}} \tag{36}$$

Rearranging, we obtain

$$\dot{X}^2 + X'^2 = 0 \tag{37}$$

We can state (33) and (37) as

$$(\dot{X} \pm X')^2 = 0 \tag{38}$$

Equation (37) also allows us to rewrite (26a) and (26b) as

$$\mathcal{P}^\tau_\mu = \frac{1}{2\pi\alpha'}\dot{X}_\mu \tag{39a}$$

$$\mathcal{P}^\sigma_\mu = -\frac{1}{2\pi\alpha'}X'_\mu \tag{39b}$$

The field equation $\partial_\tau\mathcal{P}^{\tau\mu} + \partial_\sigma\mathcal{P}^{\sigma\mu} = 0$ can then be states as

$$\ddot{X}^\mu - X^{\mu\prime\prime} = 0 \tag{40}$$

This is the wave equation. The general solution to the wave equation is

$$X^\mu(\tau, \sigma) = X^\mu_L(\tau + \sigma) + X^\mu_R(\tau - \sigma) \tag{41}$$

where X_L is a left moving wave (moving in the negative σ direction) and X_R is a right moving wave (moving in the positive σ direction). Due to the 2π periodicity of σ ($\sigma \sim \sigma + 2\pi$), we can write

$$X^\mu_L(\tau + \sigma + 2\pi) - X^\mu_L(\tau + \sigma) = X^\mu_R(\tau - \sigma) - X^\mu_R(\tau - \sigma - 2\pi). \tag{42}$$

This shows that the periodicity of $X^\mu_L(\tau + \sigma)$ and $X^\mu_R(\tau - \sigma)$ are dependent on each other. We also note that

$$\frac{\partial}{\partial(\tau - \sigma)}\big(X^\mu_L(\tau + \sigma + 2\pi) - X^\mu_L(\tau + \sigma)\big) = \frac{\partial}{\partial(\tau + \sigma)}\big(X^\mu_R(\tau - \sigma) - X^\mu_R(\tau - \sigma - 2\pi)\big) = 0 \tag{43}$$

Then $X_L^{\mu\prime}(\tau+\sigma) = \frac{\partial}{\partial(\tau+\sigma)} X_L^\mu(\tau+\sigma)$ and $X_R^{\mu\prime}(\tau-\sigma) = \frac{\partial}{\partial(\tau-\sigma)} X_R^\mu(\tau-\sigma)$ are functions with a periodicity of 2π. We can expand in the form of mode expansions, integrate, and setting $\alpha_0^\mu = \bar{\alpha}_0^\mu$, obtain

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n}(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}) \tag{44}$$

where the α modes are simply the coefficients of expansion; barred α modes arise from the expansion of the left moving wave, whilst unbarred α modes arise from the expansion of the right moving wave. Substituting our mode expansions into (39a) and integrating over the string, we obtain the momentum

$$p^\mu = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu \tag{45}$$

3.5 Quantization of the String

We are now in a position to quantize the string. We shall work in the light cone gauge, and we wish to specify operators that measure position, momentum and energy. Namely, we must specify the Heisenberg operators $X^i(\tau,\sigma), x_0^-(\tau), \mathcal{P}^{\tau i}(\tau,\sigma)$ and $p^+(\tau)$, as well as their time independent Schrodinger equivalents. The remaining operators are set by the light cone gauge conditions. Furthermore we shall construct the Hamiltonian, plus the relevant creation and annihilation operators that allow us to set up a state space equivalent to the one defined in (16) for a quantum field theory.

Our first step is to set up the equal time canonical commutation relations. For the Schrodinger operators, there is no time dependence, but there is a dependence on σ. Firstly we consider simultaneous measurements of position and momentum. According to the Heisenberg Uncertainty Principle, these observables are not simultaneously measurable [10] and hence the operators $X^i(\sigma)$ and $\mathcal{P}^{\tau i}(\sigma)$ have a non-zero commutator. However, simultaneous measurements at different points along the string do not affect each other, so the commutator is zero for measurements at two different values of σ, giving rise to a delta function in the commutator. It is clear also that x_0^- and p^+ also do not commute. All other commutators are zero. Hence our CCRs are

$$[X^i(\sigma), \mathcal{P}^{\tau j}(\sigma')] = i\eta^{ij}\delta(\sigma-\sigma') \tag{46a}$$

$$[X^i(\sigma), X^j(\sigma')] = [\mathcal{P}^{\tau i}(\sigma), \mathcal{P}^{\tau j}(\sigma')] = 0 \tag{46b}$$

$$[x_0^-, p^+] = -i \tag{46c}$$

$$[x_0^-, X^i(\sigma)] = [x_0^-, \mathcal{P}^{\tau i}(\sigma)] = 0 \tag{46d}$$

$$[p^+, X^i(\sigma)] = [p^+, \mathcal{P}^{\tau i}(\sigma)] = 0 \tag{46e}$$

Our equal time CCRs in the Heisenberg picture are identical.

To construct the Hamiltonian, we consider the gauge condition (24) for the light cone gauge. We obtain

$$X^+ = \alpha' p^+ \tau. \tag{47}$$

We expect our light-cone energy, p^-, to generate an X^+ translation. Then $\partial_{X^+} \to p^-$. Since we expect our Hamiltonian to generate a τ translation,

$$H = \partial_\tau = \partial_\tau X^+ \partial_{X^+} = \alpha' p^+ p^- \tag{48}$$

Now, we wish to find the explicit form of p^-. To do this we consider (38) in the light cone gauge. We write

$$(\dot{X} \pm X')^2 = -2(\dot{X}^- \pm X^{-\prime})(\dot{X}^+ \pm X^{+\prime}) + (\dot{X}^i \pm X^{i\prime})^2 = 0 \tag{49}$$

From our definition of X^+ in (47), we obtain

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{2\alpha' p^+}(\dot{X}^i \pm X^{i\prime})^2 \tag{50}$$

Using this result, we obtain

$$\dot{X}^- = \frac{1}{2}(\dot{X}^- + X^{-\prime}) + (\dot{X}^- - X^{-\prime}) = \frac{1}{2\alpha' p^+}(\dot{X}^i \dot{X}^i + X^{i\prime} X^{i\prime}) \tag{51}$$

Now, from (39a) we can obtain $\mathcal{P}^{\tau -}$.

$$\mathcal{P}^{\tau -} = \frac{1}{2\pi\alpha'}\dot{X}^\mu = \frac{1}{2\pi\alpha'}\frac{1}{2\alpha' p^+}\left((2\pi\alpha')^2 \mathcal{P}^{\tau i}\mathcal{P}^{\tau i} + X^{i\prime}X^{i\prime}\right) = \frac{\pi}{p^+}\left(\mathcal{P}^{\tau i}\mathcal{P}^{\tau i} + \frac{X^{i\prime}X^{i\prime}}{(2\pi\alpha')^2}\right) \tag{52}$$

At last we can write our Hamiltonian in terms of our independent dynamic operators as

$$H(\tau) = \alpha' p^+ \int_0^{2\pi} d\sigma \mathcal{P}^{\tau -} = \alpha' \pi \int_0^{2\pi} d\sigma \left(\mathcal{P}^{\tau i}\mathcal{P}^{\tau i} + \frac{X^{i\prime}X^{i\prime}}{(2\pi\alpha')^2}\right) \tag{53}$$

Finally we said we would construct creation and annihilation operators in order to define a state space such as the one in (20a). To do this, we take our α modes and make them operator valued, quantising equation (44). We now wish to find expressions for the commutators of these operators. Consider the derivatives of equation (44) with respect to τ and σ. These are

$$\dot{X}(\tau,\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0} e^{-in\tau}(\bar{\alpha}_n^\mu e^{-in\sigma} + \alpha_n^\mu e^{in\sigma}) \tag{54a}$$

$$X'(\tau,\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0} e^{-in\tau}(\bar{\alpha}_n^\mu e^{-in\sigma} - \alpha_n^\mu e^{in\sigma}) \tag{54b}$$

Then we can write

$$\dot{X}^\mu(\tau,\sigma) + X^{\mu\prime}(\tau,\sigma) = \sqrt{2\alpha'}\sum_{n\neq 0}\bar{\alpha}_n^\mu e^{-in(\tau+\sigma)} \tag{55a}$$

$$\dot{X}^\mu(\tau,\sigma) - X^{\mu\prime}(\tau,\sigma) = \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu e^{-in(\tau-\sigma)} \tag{55b}$$

Calculating explicitly the commutator $[\dot{X}^i(\tau,\sigma) + X^{i\prime}(\tau,\sigma), \dot{X}^j(\tau,\sigma) + X^{j\prime}(\tau,\sigma')]$, we obtain

$$[\dot{X}^i(\tau,\sigma) + X^{i\prime}(\tau,\sigma), \dot{X}^j(\tau,\sigma) + X^{j\prime}(\tau,\sigma')] = 2\alpha'\sum_{n\neq 0}\sum_{m\neq 0} e^{-in(\tau+\sigma)}e^{-im(\tau+\sigma')}[\bar{\alpha}_n^i, \bar{\alpha}_m^j] \tag{56}$$

Similarly, we can obtain

$$[\dot{X}^i(\tau,\sigma) - X^{i\prime}(\tau,\sigma), \dot{X}^j(\tau,\sigma) - X^{j\prime}(\tau,\sigma')] = 2\alpha'\sum_{n\neq 0}\sum_{m\neq 0} e^{-in(\tau-\sigma)}e^{-im(\tau-\sigma')}[\alpha_n^i, \alpha_m^j] \tag{57}$$

$$[\dot{X}^i(\tau,\sigma) + X^{i\prime}(\tau,\sigma), \dot{X}^j(\tau,\sigma) - X^{j\prime}(\tau,\sigma')] = [\dot{X}^i(\tau,\sigma) - X^{i\prime}(\tau,\sigma), \dot{X}^j(\tau,\sigma) + X^{j\prime}(\tau,\sigma')] = 0 \tag{58}$$

Now, let us consider these commutators from a different point of view. Operators $A^i_H(\tau)$ in the Heisenberg Picture, such as the time dependent operators $X^\mu(\tau,\sigma)$ and its derivatives, follow the Heisenberg equation of motion

$$i\dot{A}^i_H(\tau) = [A^i_H(\tau), H]. \tag{59}$$

Taking $X^\mu(\tau,\sigma)$ as our operator, using (53) as our Hamiltonian, and considering our CCRs, we obtain

$$\begin{aligned}
i\dot{X}^i(\tau,\sigma) = [X^i(\tau,\sigma), H] &= \alpha'\pi\left[X^i(\tau,\sigma), \int_0^{2\pi} d\sigma\left(\mathcal{P}^{\tau j}(\tau,\sigma')\mathcal{P}^{\tau j}(\tau,\sigma') + \frac{X^{j\prime}(\tau,\sigma')X^{j\prime}(\tau,\sigma')}{(2\pi\alpha')^2}\right)\right] \\
&= \alpha'\pi\left[X^i(\tau,\sigma), \int_0^{2\pi} d\sigma\mathcal{P}^{\tau j}(\tau,\sigma')\mathcal{P}^{\tau j}(\tau,\sigma')\right] + \frac{1}{4\pi\alpha'}\left[X^i(\tau,\sigma), \frac{X^{j\prime}X^{j\prime}}{(2\pi\alpha')^2}\right] \\
&= \alpha'\pi\int_0^{2\pi} d\sigma[X^i(\tau,\sigma), \mathcal{P}^{\tau j}(\tau,\sigma')\mathcal{P}^{\tau j}(\tau,\sigma')] \\
&= 2\pi i\alpha'\int_0^{2\pi} d\sigma\mathcal{P}^{\tau j}(\tau,\sigma')\eta^{ij}\delta(\sigma-\sigma')
\end{aligned}$$

Integrating over the string, we obtain

$$\dot{X}^i(\tau,\sigma) = 2\pi\alpha'\mathcal{P}^{\tau i}(\tau,\sigma) \tag{60}$$

Then the commutator calculated in (56) can be written as (dropping the arguments for brevity):

$$\begin{aligned}
[(\dot{X}^i + X^{i\prime})(\tau,\sigma), (\dot{X}^j + X^{j\prime})(\tau,\sigma')] &= [\dot{X}^i, \dot{X}^j] + [\dot{X}^i, X^{j\prime}] + [X^{i\prime}, \dot{X}^j] + [X^{i\prime}, X^{j\prime}] \\
&= (2\pi\alpha)^2[\mathcal{P}^{\tau i}, \mathcal{P}^{\tau j}] + \partial_{\sigma'}[\dot{X}^i, X^j] + \partial_\sigma[X^i, \dot{X}^j] + \partial_\sigma\partial_{\sigma'}[X^i, X^j]
\end{aligned}$$

From our CCRs, the first and fourth term are zero.

$$\begin{aligned}
[(\dot{X}^i + X^{i\prime})(\tau,\sigma), (\dot{X}^j + X^{j\prime})(\tau,\sigma')] &= \partial_{\sigma'}[\dot{X}^i, X^j] + \partial_\sigma[X^i, \dot{X}^j] \\
&= 2\pi\alpha'\partial_{\sigma'}[\mathcal{P}^{\tau i}, X^j] + 2\pi\alpha'\partial_\sigma[X^i, \mathcal{P}^{\tau j}] \\
&= -2\pi\alpha' i\eta^{ji}\frac{d}{d\sigma'}\delta(\sigma'-\sigma) + 2\pi\alpha' i\eta^{ij}\frac{d}{d\sigma}\delta(\sigma-\sigma')
\end{aligned}$$

This leads to the result

$$[(\dot{X}^i + X^{i\prime})(\tau,\sigma), (\dot{X}^j + X^{j\prime})(\tau,\sigma')] = 4\pi\alpha' i\eta^{ij}\frac{d}{d\sigma}\delta(\sigma-\sigma') \tag{61}$$

Comparing (56) with (61), we have

$$\sum_{n\neq 0}\sum_{m\neq 0} e^{-in(\tau+\sigma)}e^{-im(\tau'_\sigma)}[\bar{\alpha}^i_n, \bar{\alpha}^j_m] = 2\pi i\eta^{ij}\frac{d}{d\sigma}\delta(\sigma-\sigma') \tag{62}$$

Integrate both sides over σ and σ',

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_0^{2\pi} d\sigma e^{in'\sigma}\int_0^{2\pi} d\sigma' e^{im'\sigma'}\sum_{n\neq 0}\sum_{m\neq 0} e^{-in(\tau+\sigma)}e^{-im(\tau'_\sigma)}[\bar{\alpha}^i_n, \bar{\alpha}^j_m] \\
&\qquad = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\sigma e^{in'\sigma}\int_0^{2\pi} d\sigma' e^{im'\sigma'}2\pi i\eta^{ij}\frac{d}{d\sigma}\delta(\sigma-\sigma')
\end{aligned} \tag{63}$$

which reduces to

$$[\bar{\alpha}_n^i, \bar{\alpha}_m^j] = n\eta^{ij}\delta_{m+n,0} \tag{64a}$$

Through similar derivations, it is possible to show that

$$[\alpha_n^i, \alpha_m^j] = n\eta^{ij}\delta_{m+n,0} \tag{64b}$$

$$[\alpha_n^i, \bar{\alpha}_m^j] = 0 \tag{64c}$$

We can then define our creation and annihilation operators as

$$a_n^i = \frac{\alpha_n^i}{\sqrt{n}}, \qquad a_n^{i\dagger} = \frac{\alpha_{-n}^i}{\sqrt{n}} \tag{65a}$$

$$\bar{a}_n^i = \frac{\bar{\alpha}_n^i}{\sqrt{n}}, \qquad \bar{a}_n^{i\dagger} = \frac{\bar{\alpha}_{-n}^i}{\sqrt{n}} \tag{65b}$$

where $n \geq 1$. We have the non vanishing commutation relations for the annihilation and creation operators

$$[\bar{a}_m^i, \bar{a}_n^{j\dagger}] = \delta_{m,n}\eta^{ij}, \qquad [a_m^i, a_n^{j\dagger}] = \delta_{m,n}\eta^{ij} \tag{66}$$

We have now briefly demonstrated how a theory of vibrating strings is related to the standard model of particle physics. Choosing creation operators correctly and acting on the ground state, it is possible to create a state space similar to those seen in particle physics.

4 The Heterotic String and The Free Fermionic Formulation, The ABK Rules

4.1 The Heterotic String

Our string model was built in the free fermionic formulation of heterotic string theory. We now take a look at what is meant by the heterotic string in order to see how the free fermionic formulation may be constructed. We begin by tracing string theory back to its origins, and look at the bosonic string. The bosonic string is a relativistic quantum open string theory that exists in $D = 26$ spacetime dimensions and accounts only for bosons. The construction of the bosonic string is covered in detail in Part I of Zweibach's book, and is done by considering first the motion of a classical string, then taking into account relativistic effects, then finally using the quantization procedure we outlined in chapter 3.5. All open string theories must also contain closed strings, and through constructing the closed bosonic string arises the graviton states that perked interest in string theory as a quantum theory of gravity.

What bosonic string theory is lacking, however, is the presence of fermionic states. We discussed in chapter 2.4 how a theoretical phenomenon called supersymmetry relates the fermionic and bosonic quantum states of a particle theory. Introducing supersymmetry to the bosonic string would allow for both bosonic and fermionic states. This is done by introducing an anti-commuting[3]dynamic variable $\psi_\alpha^\mu(\tau,\sigma)$ $(\alpha = 1, 2)$. This is the fermionic equivalent to the commuting (bosonic) dynamic variable $X^\mu(\tau,\sigma)$, and just as for each value of μ the variable $X^\mu(\tau,\sigma)$ is a world sheet boson, then for each value of μ the components $\psi_\alpha^\mu(\tau,\sigma)$ construct a world sheet fermion. The introduction of this variable reduces the number of spacetime dimensions from twenty-six to $D = 10$.

[3]Since we know that a quantum state of identical fermions is antisymmetric under exchange of two of the fermions.

The quantization procedure results in a Ramond (R) sector in which the fermions are periodic over one time like loop, and Neveu-Schwarz (NS) sector in which the fermions are antiperiodic. We can then construct a closed superstring by combining left and right moving copies of the open superstring theory. These closed superstring theories are called type II superstring theories. Furthermore combining these in different ways results in two versions of type II superstring theory called type IIA and type IIB superstring theory.

Instead of combining left and right moving superstrings, the heterotic string combines a left moving bosonic string with a right moving superstring. Since only ten of the twenty-six bosonic coordinates of the left moving string are matched by bosonic coordinates from the right moving string, heterotic string theory effectively exists in $D = 10$ spacetime dimensions.

4.2 The ABK Formulation

We arrive at an issue in that our heterotic string exists in ten dimensions, whilst in order to arrive at a realistic theory we wish to have four flat spacetime dimensions and $N = 1$ supersymmetry. A first approach to solving this issue was to compactify six of the ten dimensions of the superstring on a compact, six dimensional Calabi-Yau manifold [11], or an orbifold [12]. A simpler formulation was developed by Antoniadas, Bachas, Kounnas [3] and Kawai, Lewellen, Tye [2] in which the string theory is built directly in four dimensions with fermionic internal degrees of freedom. In this formulation, all extra quantum numbers of the string are carried by free fermions on the world sheet, which provide a non-linear realisation of world sheet supersymmetry [13]. We summarise here the free fermionic formulation as presented by Antoniadas et. al [3, 13, 14].

The fermionization of the internal degrees of freedom on an $E_8 \times E_8$ heterotic string results in 44 left moving and 20 right moving real internal free world sheet fermions. We denote these real fermions as follows.

$$\text{Right Movers} : \{\psi_1^\mu, \psi_2^\mu, \chi^{1\ldots6}, y^{1\ldots6}, \omega^{1\ldots6}\} \tag{67}$$

$$\text{Left Movers} : \{\bar{y}^{1\ldots6}, \bar{\omega}^{1\ldots6}, \lambda^{1\ldots32}\} \tag{68}$$

Furthermore, we shall complexify the fermions by pairing them as follows

$$\phi = \frac{1}{\sqrt{2}}(f^n + if^m) \tag{69}$$

where ϕ is a complex fermion and $f^{n,m}$ are real fermions. The internal free fermions can then be denoted

$$\text{Right Movers} : \{\psi^\mu, \chi^{12}, \chi^{34}, \chi^{56}, y^{1\ldots6}, \omega^{1\ldots6}\} \tag{70}$$

$$\text{Left Movers} : \{\bar{y}^{1\ldots6}, \bar{\omega}^{1\ldots6}, \bar{\psi}^{1\ldots5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1\ldots8}\} \tag{71}$$

where $\{y^{1\ldots6}, \omega^{1\ldots6}, \bar{y}^{1\ldots6}, \bar{\omega}^{1\ldots6}\}$ remain real whilst all other fermions are complex. The remaining real fermions will later be paired based on boundary conditions in the sectors of the model in this free fermionic formulation. Antoniadas, Bachas and Kounnas formulated a set of rules required to build a heterotic string model. To define a model, we must specify two things:

1. A set of basis vectors that specify boundary conditions for the internal free fermions.
2. Coefficients (one loop phases) for each intersection of basis vectors.

There are rules on both the basis vectors and the one loop phases, summarised in the following sections.

4.2.1 Rules on the Basis Vectors

The model is generated by a set of basis vectors a_i, denoted

$$a_i = \{f^j, \ldots, f^n\}, \tag{72}$$

where fermions f present in the vector have a periodic boundary condition, whilst those not present have anti-periodic boundary conditions. Fermions written as $\frac{1}{2}f$ are twisted by a phase $-i$. For example, for the sector S given in (78a), the fermions $\psi^\mu, \chi^{12}, \chi^{34}$, and χ^{56} are periodic, and all other fermions are antiperiodic. We write our boundary conditions as

$$a(f) = \begin{cases} 1 & \text{periodic} \\ 0 & \text{antiperiodic} \\ 1/2 & \text{twisted} \end{cases} \tag{73}$$

for each fermion f. The additive group Ξ is given by

$$\Xi = \sum_i m_i a_i, \qquad m_i = 0, \ldots, N_i - 1 \tag{74}$$

where $N_i a_i = 0 \mod 2$. We also define N_{ij} as the least common multiple of N_i and N_j. With these definitions, we write the following rules

1. The basis vector $\vec{1}$, in which all fermions are periodic, is present in the model, or equivalently $\vec{1} \in \Xi$ [4].
2. There are an even number of real fermions.
3. $\sum_i m_i a_i = 0$ if and only if for all i, $m_i = 0 \mod N_i$.
4. $N_{ij} a_i a_j = 0 \mod 4$, or equivalently $a_i a_j = 0 \mod 2$.
5. $N_i a_i a_i = 0 \mod 8$, or equivalently $a_i a_i = 0 \mod 4$.

4.2.2 One Loop Phases

The rules on the one loop phases for the intersection of each sector a_i are

1. $c\begin{pmatrix} a_i \\ a_j \end{pmatrix} = \delta_{a_i} e^{\frac{2i\pi n}{N_j}} = \delta_{a_j} e^{\frac{\pi i}{2} a_i \cdot a_j} e^{\frac{2i\pi m}{N_i}}$
2. $-c\begin{pmatrix} a_i \\ a_i \end{pmatrix} = e^{-\frac{i\pi}{4} a_i \cdot a_i} c\begin{pmatrix} a_i \\ 1 \end{pmatrix}$
3. $c\begin{pmatrix} a_i \\ a_j \end{pmatrix} = e^{\frac{i\pi}{2} a_i \cdot a_j} c\begin{pmatrix} a_j \\ a_i \end{pmatrix}^*$
4. $c\begin{pmatrix} a_i \\ a_j + a_k \end{pmatrix} = \delta_{a_i} c\begin{pmatrix} a_i \\ a_j \end{pmatrix} c\begin{pmatrix} a_i \\ a_k \end{pmatrix}$

These rules are particularly important for the GSO projections [15], which specify the surviving states in the massless spectrum. The GSO projections are given by

$$e^{i\pi(a_i \cdot F_A)} |s\rangle_A = \delta_A c\begin{pmatrix} A \\ a_i \end{pmatrix}^* |s\rangle_A \tag{75}$$

[4] Because of this constraint, we do not write $\vec{1}$ in our set of basis vectors given in (78), as it is given that the vector is present in the model for consistency.

where A is the sector being analysed, a_i are the eight basis vectors, and $|s\rangle_A$ is a state in the sector A. We define δ_A for any sector A as

$$\delta_A = \begin{cases} 1 & \text{if} \quad A(\psi^\mu) = 0 \\ -1 & \text{if} \quad A(\psi^\mu) = 1. \end{cases} \tag{76}$$

5 The String Model

The string model we use was constructed using the free fermionic formulated and presented by Faraggi, Nanopoulos, and Yuan (1989) [4]. It is generated by eight basis vectors. Real fermions are paired based on having the same boundary conditions in each sector to form a complex fermions as follows:

$$f^n f^m = \frac{1}{\sqrt{2}}(f^n + i f^m). \tag{77}$$

Finally a bar above a fermion represents a right mover, whilst an unbarred fermion is a left mover. The basis vectors are

$$S = \{\psi^\mu, \chi^{12}, \chi^{34}, \chi^{56}\} \tag{78a}$$
$$b_1 = \{\psi^\mu, \chi^{12}, y^3y^6, y^4\bar{y}^4, y^5\bar{y}^5, \bar{y}^3\bar{y}^6, \bar{\psi}^{1,\dots,5}, \bar{\eta}^1\} \tag{78b}$$
$$b_2 = \{\psi^\mu, \chi^{34}, y^1\omega^6, y^2\bar{y}^2, \omega^5\bar{\omega}^5, \bar{y}^1\bar{\omega}^6, \bar{\psi}^{1,\dots,5}, \bar{\eta}^2\} \tag{78c}$$
$$b_3 = \{\psi^\mu, \chi^{56}, \omega^1\omega^3, \omega^2\bar{\omega}^2, \omega^4\bar{\omega}^4, \bar{\omega}^1\bar{\omega}^3, \bar{\psi}^{1,\dots,5}, \bar{\eta}^3\} \tag{78d}$$
$$b_4 = \{\psi^\mu, \chi^{12}, y^3y^6, \omega^4\bar{\omega}^4, \omega^5\bar{\omega}^5, \bar{y}^3\bar{y}^6, \bar{\psi}^{1,\dots,5}, \bar{\eta}^1\} \tag{78e}$$
$$\alpha = \{\psi^\mu, \chi^{56}, \omega^1\omega^3, y^2\bar{y}^2, \omega^4\bar{\omega}^4, \bar{y}^1\bar{\omega}^6, \bar{y}^3\bar{y}^6, \bar{\psi}^{1,\dots,3}, \bar{\eta}^{1,2}, \bar{\phi}^{1,\dots,4}\} \tag{78f}$$
$$\beta = \{\psi^\mu, \chi^{34}, y^1\omega^6, \omega^2\bar{\omega}^2, \bar{y}^3\bar{y}^6, y^5\bar{y}^5, \bar{y}^1\bar{\omega}^6, \bar{y}^3\bar{y}^6, \frac{1}{2}(\bar{\psi}^{1,\dots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,5,6,7}), \bar{\phi}^{3,4}\} \tag{78g}$$

plus the vector 1 in which all fermions are periodic. We make the following choices of generalised GSO projection coefficients.

$$c\binom{\beta}{b_2} = c\binom{b_i}{b_j} = c\binom{\alpha}{b_i} = c\binom{1}{b_i} = c\binom{b_4}{b_i} = c\binom{b_4}{1} = -1 \qquad i, j = 1\dots3, \tag{79a}$$
$$c\binom{S}{b_j} = c\binom{S}{1} = c\binom{S}{\alpha} = c\binom{S}{\beta} = +1 \tag{79b}$$
$$c\binom{\alpha}{\beta} = c\binom{\alpha}{\alpha} = c\binom{\beta}{b_1} = c\binom{\beta}{b_3} = c\binom{\beta}{b_4} = c\binom{b_4}{b_1} = +1. \tag{79c}$$

All other projection coefficients are specified by the modular invariance constraints given in Section 4.2.2. Our model differs from that presented by Faraggi et al by the choice of GSO coefficient

$$c\binom{b_4}{1} = -1. \tag{80}$$

As we shall derive below, this model results in a $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)^6 \times SO(4) \times SU(3) \times U(1)^4$ gauge group and $N = 1$ space-time supersymmetry.

6 Deriving the Massless Spectrum

6.1 Eliminating Sectors Contain No Massless States

We begin by calculating which sectors contain massless states. This is done using the matching Virasoro condition

$$M_L^2 = -\frac{1}{2} + \frac{a_L \cdot a_L}{8} + N_L = -1 + \frac{a_R \cdot a_R}{8} + N_R = M_R^2 \tag{81}$$

where $N_{L,R}$ is given by

$$N_{L,R} = \sum_f (\nu_{L,R}) \tag{82}$$

where $\nu_{L,R}$ is the frequency of the left and right moving oscillators respectively. $N_{L,R}$ are non-negative. It can be seen from (81) that for a state to be massless, we require

$$\frac{a_L \cdot a_L}{8} \leq \frac{1}{2}, \qquad \text{and} \qquad \frac{a_R \cdot a_R}{8} \leq 1. \tag{83}$$

Hence for each sector we calculate these value using the formulae

$$a \cdot b = \left(\frac{1}{2} \sum_{\text{Real}} + \sum_{\text{Complex}} \right) a(f) b(f) \tag{84}$$

for each fermion f, and the summations are the sums over real and complex fermions respectively. The sectors containing massless states are those listed in Appendix A, plus the Neveu-Schwarz sector which is analysed in section 8.1. In addition, each sector A will have a sector $S + A$ which generates superpartners to the particles generated in A, and for any sector $B + \beta$ containing massless states there is a sector $B + 3\beta = B - \beta$ also containing massless states.

The number of left and right moving oscillators applied to the ground state is calculated using (82) and the equations

$$\nu_f = \frac{1 + a(f)}{2}, \qquad \nu_{f^*} = \frac{1 - a(f)}{2}. \tag{85}$$

6.2 GSO Projections

Once we have found sectors containing massless states, we must apply the GSO projections in order to derive the massless states that are part of the spectrum of the model. The GSO projections are given by equation (75), where F_A are the boundary conditions of the fermions in the sector A, and $a_i \cdot F_A$ is the dot product as defined in (84), essentially summing over each fermion present in both the sector A and the basis vector a_i. The states that satisfy (75) for each of the eight basis vectors are present in the massless spectrum of the model.

7 An example sector

Here we proceed to analyse a sector to give an example of the method used to analyse the spectrum. I have chosen the sector $b_3 + \alpha + \beta$, since this is a sector that, as we will see below, requires the application of an oscillator in order for the sector to be massless, so we are getting a full example of the method.

The internal fermions present in the sector $b_3+\alpha+\beta$ are

$$b_3+\alpha+\beta:\{\psi^\mu,\chi^{34},y^1\omega^6,y^{2\bar{2}},y^{5\bar{5}},\bar{\omega}^{13},\bar{\phi}^2,\frac{1}{2}[\bar{\psi}^{1,2,3},\bar{\phi}^{5,6,7}],-\frac{1}{2}[\bar{\psi}^{4,5},\bar{\eta}^{1,2,3},\bar{\phi}^1]\}.$$

Our first step is to analyse the number of oscillators that are required to act on the degenerate states in order for the sector to be massless. We begin by analysing the left moving part.

$$\begin{aligned}a_L\cdot a_L &= a^2(\psi^\mu)+a^2(\chi^{34})+a^2(y^1\omega^6)+\frac{1}{2}a^2(y^2)+\frac{1}{2}a^2(y^5)\\ &= 1+1+1+1/2+1/2=4\end{aligned}$$

For the right moving part,

$$\begin{aligned}a_R\cdot a_R &= a^2(\bar{\omega}^{13})+\frac{1}{2}a^2(\bar{y}^2)+\frac{1}{2}a^2(\bar{y}^5)+a^2(\bar{\psi}^{1,2,3})+a^2(\bar{\psi}^{4,5})+a^2(\bar{\eta}^{1,2,3})+a^2(\bar{\phi}^1)+a^2(\bar{\phi}^2)+a^2(\bar{\phi}^{5,6,7})\\ &= 1+1/2+1/2+3/4+2/4+3/4+1/4+1+3/4=6\end{aligned}$$

Using (81) we can now caclulate the number of left and right moving oscillators required for the sector to be massless. Setting $M_L^2=M_R^2=0$, we obtain

$$N_L=0,\qquad N_R=\frac{1}{4}.$$

There are no left moving oscillators. We must now calculate the number of right moving oscillators. Combining (82) and (85), we obtain the result that we can apply one right moving oscillator with boundary condition $a(f)=-1/2$, since

$$\nu_f=\frac{1+a(f)}{2}=\frac{1}{4},$$

or one complex conjugate oscillator with boundary condition $a(f)=1/2$, since

$$\nu_{f^*}=\frac{1-a(f)}{2}=\frac{1}{4}.$$

Hence the oscillators that may be applied are $\bar{\psi}^{1,2,3},\bar{\psi}^{4,5*},\bar{\eta}^{1,2,3*},\bar{\phi}^{1*},\bar{\phi}^{5,6,7}$.

The next step is to calculate the GSO projections. For each vector $a_i=1,S,b_1,b_2,b_3,b_4,\alpha,\beta$ we calculate the projection using the rules given above. We know from (75) that we must calculate

$$\delta_{(b_3+\alpha+\beta)}c\begin{pmatrix}b_3+\alpha+\beta\\ a_i\end{pmatrix}^*.$$

To do this we use the rules on the one loop phases above.

$$\text{GSO}=\delta_{(b_3+\alpha+\beta)}c\begin{pmatrix}b_3+\alpha+\beta\\ a_i\end{pmatrix}^*=-\left[\exp\left(\frac{i\pi}{2}(a_i\cdot(b_3+\alpha+\beta))\right)\delta_{a_i}c\begin{pmatrix}a_i\\ b_3\end{pmatrix}c\begin{pmatrix}a_i\\ \alpha\end{pmatrix}c\begin{pmatrix}a_i\\ \beta\end{pmatrix}\right]^*$$

For example, for the sector 1,

$$\text{GSO}=-\left[\exp\left(\frac{9i\pi}{2}\right)\delta_1 c\begin{pmatrix}1\\ b_3\end{pmatrix}c\begin{pmatrix}1\\ \alpha\end{pmatrix}c\begin{pmatrix}1\\ \beta\end{pmatrix}\right]^*=-\Big[i\cdot -1\cdot -1\cdot -1\cdot i\Big]^*=-1$$

We now consider those states that are degenerate. Each internal fermion that is periodic can occupy one of two states, $|+\rangle$ or $|-\rangle$, which are

$$|+\rangle = e^{i\pi 1\cdot a(f)} = +1, \qquad |-\rangle = e^{i\pi 1\cdot a(f)} = -1. \tag{86}$$

We introduce the notation $\binom{i}{j}$ where i is the total number of states and j is the number of $|-\rangle$ states. For example

$$\binom{3}{2} = |+\rangle|-\rangle|-\rangle, \quad |-\rangle|+\rangle|-\rangle, \quad \text{or} \quad |-\rangle|-\rangle|+\rangle.$$

Ramond Sector Projection

Let us label our oscillators $\bar{\lambda}_i$. Our GSO projection is

$$\begin{aligned} e^{i\pi(1\cdot a(\bar{\lambda}_i))}e^{i\pi(1\cdot F_{b_3+\alpha+\beta})}|s\rangle &= e^{i\pi(1\cdot a(\bar{\lambda}_i))}|\pm\rangle_{\psi^\mu}|\pm\rangle_{\chi^{34}}|\pm\rangle_{y^1\omega^6}|\pm\rangle_{y^2\bar{y}^2}|\pm\rangle_{y^5\bar{y}^5}|\pm\rangle_{\bar{\omega}^1\bar{\omega}^3}|\pm\rangle_{\bar{\phi}^2} \\ &= -|s\rangle. \end{aligned}$$

For all oscillators,

$$e^{i\pi(1\cdot a(\bar{\lambda}_i))} = -1,$$

hence we require our degenerate states to have an even number of negative states. The states that survive the sector 1 projection are

$$\{\bar{\psi}^{1,2,3}, \bar{\psi}^{4,5*}, \bar{\eta}^{1,2,3*}, \bar{\phi}^{1*}, \bar{\phi}^{5,6,7}\}\left[\binom{7}{0}+\binom{7}{2}+\binom{7}{4}+\binom{7}{6}\right]_{\psi^\mu,\chi^{34},y^1\omega^6,y^2\bar{y}^2,y^5\bar{y}^5,\bar{\omega}^1\bar{\omega}^3,\bar{\phi}^2}$$

S Projection

Our GSO projection for the S sector is

$$e^{i\pi(S\cdot a(\bar{\lambda}_i))}e^{i\pi(S\cdot F_{b_3+\alpha+\beta})}|s\rangle = |\pm\rangle_{\psi^\mu}|\pm\rangle_{\chi^{34}} = +|s\rangle.$$

since for all oscillators

$$e^{i\pi(S\cdot a(\bar{\lambda}_i))} = +1.$$

Hence we know that the ψ^μ and χ^{34} must either both be in the $|+\rangle$ or both in the $|-\rangle$ states. The states that survive this projection are

$$\{\bar{\psi}^{1,2,3}, \bar{\psi}^{4,5*}, \bar{\eta}^{1,2,3*}, \bar{\phi}^{1*}, \bar{\phi}^{5,6,7}\}\left[\binom{2}{0}+\binom{2}{2}\right]_{\psi^\mu,\chi^{34}}\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right]_{y^1\omega^6,y^2\bar{y}^2,y^5\bar{y}^5,\bar{\omega}^1\bar{\omega}^3,\bar{\phi}^2}$$

The $\left[\binom{2}{0}+\binom{2}{2}\right]$ correspond to CPT conjugate states. For brevity we shall consider only the $\binom{2}{0}$ state. We can reconstruct the conjugate states after all projections. Our surviving states are

$$\{\bar{\psi}^{1,2,3*}, \bar{\psi}^{4,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1}, \bar{\phi}^{5,6,7*}\}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right]_{y^1\omega^6,y^2\bar{y}^2,y^5\bar{y}^5,\bar{\omega}^1\bar{\omega}^3,\bar{\phi}^2}$$

b_4 Projection

For convenience, we do the b_4 projection next, since the order of projections does not matter. The GSO projection for this sector is

$$e^{i\pi(b_4\cdot a(\bar{\lambda}_i))}e^{i\pi(b_4\cdot F_{b_3+\alpha+\beta})}|s\rangle = e^{i\pi(b_4\cdot a(\bar{\lambda}_i)}|+\rangle_{\psi^\mu} = -|s\rangle.$$

Hence the only states that survive are the ones with oscillators that satisfy

$$e^{i\pi(b_4\cdot a(\bar{\lambda}_i))} = -1,$$

or, equivalently those that are periodic in the sector b_4. These are $\bar{\psi}^{1,2,3}, \bar{\psi}^{4,5*}$ and $\bar{\eta}^{1*}$. The surviving states are

$$\{\bar{\psi}^{1,2,3}, \bar{\psi}^{4,5*}, \bar{\eta}^{1*}\}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right]_{y^1\omega^6,y^2\bar{y}^2,y^5\bar{y}^5,\bar{\omega}^1\bar{\omega}^3,\bar{\phi}^2}$$

b_2 Projection

We do the b_2 projection next. Our GSO projection for this sector is

$$e^{i\pi(b_2\cdot a(\bar{\lambda}_i))}e^{i\pi(b_2\cdot F_{b_3+\alpha+\beta})}|s\rangle = e^{i\pi(b_2\cdot a(\bar{\lambda}_i)}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}|\pm\rangle_{y^1\omega^6}|\pm\rangle_{y^2\bar{y}^2} = +|s\rangle.$$

The oscillators $\bar{\psi}^{1,2,3}, \bar{\psi}^{4,5*}$ satisfy

$$e^{i\pi(b_2\cdot a(\bar{\psi}^{1,2,3},\bar{\psi}^{4,5*}))} = -1,$$

whereas the oscillator $\bar{\eta}^{1*}$ satisfies

$$e^{i\pi(b_2\cdot a(\bar{\eta}^{1*}))} = +1.$$

Hence applying these oscillators result in different states. The states that survive this projection are

$$\{\bar{\psi}^{1,2,3}, \bar{\psi}^{4,5*}\}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}\binom{2}{1}_{y^1\omega^6,y^2\bar{y}^2}\left[\binom{3}{1}+\binom{3}{3}\right]_{y^5\bar{y}^5,\bar{\omega}^1\bar{\omega}^3,\bar{\phi}^2}$$
$$\bar{\eta}^{1*}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}\left[\binom{2}{0}+\binom{2}{2}\right]_{y^1\omega^6,y^2\bar{y}^2}\left[\binom{3}{1}+\binom{3}{3}\right]_{y^5\bar{y}^5,\bar{\omega}^1\bar{\omega}^3,\bar{\phi}^2}$$

b_1 Projection

Our GSO projection for this sector is

$$e^{i\pi(b_1\cdot a(\bar{\lambda}_i))}e^{i\pi(b_1\cdot F_{b_3+\alpha+\beta})}|s\rangle = e^{i\pi(b_1\cdot a(\bar{\lambda}_i)}|+\rangle_{\psi^\mu}|\pm\rangle_{y^5\bar{y}^5} = -|s\rangle.$$

The states that survive this projection are

$$\{\bar{\psi}^{1,2,3}, \bar{\psi}^{4,5*}\}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}|+\rangle_{y^5\bar{y}^5}\binom{2}{1}_{y^1\omega^6,y^2\bar{y}^2}\binom{2}{1}_{\bar{\omega}^1\bar{\omega}^3,\bar{\phi}^2}$$
$$\bar{\eta}^{1*}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}|+\rangle_{y^5\bar{y}^5}\left[\binom{2}{0}+\binom{2}{2}\right]_{y^1\omega^6,y^2\bar{y}^2}\left[\binom{2}{0}+\binom{2}{2}\right]_{\bar{\omega}^1\bar{\omega}^3,\bar{\phi}^2}$$

b_3 **Projection**

Our GSO projection for this sector is

$$e^{i\pi(b_3\cdot a(\bar{\lambda}_i))}e^{i\pi(b_3\cdot F_{b_3+\alpha+\beta})}|s\rangle = e^{i\pi(b_3\cdot a(\bar{\lambda}_i)}|+\rangle_{\psi^\mu}|\pm\rangle_{\bar{\omega}^1\bar{\omega}^3} = +|s\rangle.$$

The states that survive this projection are

$$\begin{array}{c}\{\bar{\psi}^{1,2,3},\bar{\psi}^{4,5*}\}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}|+\rangle_{y^5\bar{y}^5}|-\rangle_{\bar{\omega}^1\bar{\omega}^3}|+\rangle_{\bar{\phi}^2}\begin{pmatrix}2\\1\end{pmatrix}_{y^1\omega^6,y^2\bar{y}^2}\\ \bar{\eta}^{1*}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}|+\rangle_{y^5\bar{y}^5}|+\rangle_{\bar{\omega}^1\bar{\omega}^3}|+\rangle_{\bar{\phi}^2}\left[\begin{pmatrix}2\\0\end{pmatrix}+\begin{pmatrix}2\\2\end{pmatrix}\right]_{y^1\omega^6,y^2\bar{y}^2}\end{array}$$

α **Projection**

Our GSO projection for this sector is

$$e^{i\pi(\alpha\cdot a(\bar{\lambda}_i))}e^{i\pi(\alpha\cdot F_{b_3+\alpha+\beta})}|s\rangle = e^{i\pi(\alpha\cdot a(\bar{\lambda}_i)}|+\rangle_{\psi^\mu}|\pm\rangle_{y^2\bar{y}^2}|+\rangle_{\bar{\phi}^2} = +|s\rangle.$$

The states that survive this projection are

$$\begin{array}{r}\bar{\psi}^{1,2,3}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}|+\rangle_{y^1\omega^6}|-\rangle_{y^2\bar{y}^2}|+\rangle_{y^5\bar{y}^5}|-\rangle_{\bar{\omega}^1\bar{\omega}^3}|+\rangle_{\bar{\phi}^2}\\ \bar{\psi}^{4,5*}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}|-\rangle_{y^1\omega^6}|+\rangle_{y^2\bar{y}^2}|+\rangle_{y^5\bar{y}^5}|-\rangle_{\bar{\omega}^1\bar{\omega}^3}|+\rangle_{\bar{\phi}^2}\\ \bar{\eta}^{1*}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}|-\rangle_{y^1\omega^6}|-\rangle_{y^2\bar{y}^2}|+\rangle_{y^5\bar{y}^5}|+\rangle_{\bar{\omega}^1\bar{\omega}^3}|+\rangle_{\bar{\phi}^2}\end{array}$$

β **Projection**

The GSO projection for the β sector is

$$e^{i\pi(\beta\cdot a(\bar{\lambda}_i))}e^{i\pi(\beta\cdot F_{b_3+\alpha+\beta})}|s\rangle = e^{i\pi(\beta\cdot a(\bar{\lambda}_i)}|+\rangle_{\psi^\mu}|+\rangle_{\chi^{34}}|\pm\rangle_{y^1\omega^6}|+\rangle_{y^5\bar{y}^5} = i|s\rangle.$$

All states in the above sector satisfy this condition.

8 The Massless Spectrum

8.1 The Neveu-Schwarz Sector

The Neveu-Schwarz sector, with all fermions antiperiodic (i.e. the sector 0), requires one left moving fermionic oscillator and either two right moving fermionic oscillators or one right moving bosonic oscillator, since $N_L = 1/2$ and $N_R = 1$, and $\nu_{f,f^*} = 1/2$ for a fermionic oscillator, and 1 for a bosonic oscillator. The tachyonic state that may be achieved through applying one right moving fermionic oscillator is eliminated by the GSO projections. We apply our GSO projections for each vector a_i, where any state is projected unless the condition

$$e^{i\pi a_i\cdot F_0}|0\rangle_0 = -|0\rangle_0 \tag{87}$$

is satisfied.

1. We obtain spin 2 fields (graviton) with

$$\psi^\mu \partial \bar{X}^\nu |0\rangle_0$$

where $\partial \bar{X}^\nu$ is the bosonic creation operator with frequency $\nu_f = 1$.

2. We obtain gauge bosons of the observable Standard Model gauge group

$$SU(3)_C \times SU(2)_L \times U(1)_C \times U(1)_L$$

with

$$\psi^\mu \bar{\psi}^a \bar{\psi}^{b*} |0\rangle_0 \qquad a, b = 1, 2, 3; 4, 5.$$

These are generated as follows:

- $\psi^\mu \bar{\psi}^{1,2,3} \bar{\psi}^{1,2,3*} |0\rangle_0$ generates $SU(3)_C$, and the trace of this generates $U(1)_C$.
- $\psi^\mu \bar{\psi}^{4,5} \bar{\psi}^{4,5*} |0\rangle_0$ generates $SU(2)_L$, and the trace of this generates $U(1)_L$.

The $U(1)_C \times U(1)_L$ symmetry breaks to the weak hypercharge $U(1)_Y$ as the combination

$$U(1)_Y = \frac{1}{3} U(1)_C + \frac{1}{2} U(1)_L \tag{88}$$

We show later that this symmetry breaking necessarily occurs at the string level in order for the model to be anomaly free.

3. Three extra observable $U(1)$'s arising from

$$\psi^\mu \bar{\eta}^a \bar{\eta}^{a*} |0\rangle_0 \qquad a = 1, 2, 3.$$

denoted $U(1)_{1,2,3}$ for $a = 1, 2, 3$ respectively.

4. Three extra observable $U(1)$'s arising from

$$\psi^\mu \bar{\zeta}^a \bar{\zeta}^{a*} |0\rangle_0 \qquad a = 1, 2, 3$$

denoted $U(1)_{4,5,6}$ for $a = 1, 2, 3$ respectively. Here $\bar{\zeta}^1 = (1/\sqrt{2})(\bar{y}^3 + i\bar{y}^6)$, $\bar{\zeta}^2 = (1/\sqrt{2})(\bar{y}^1 + i\bar{\omega}^6)$ and $\bar{\zeta}^3 = (1/\sqrt{2})(\bar{\omega}^1 + i\bar{\omega}^3)$.

5. We obtain gauge bosons of the hidden gauge group

$$SO(4)_H \times SU(3)_H \times U(1)^4$$

with

$$\{\psi^\mu \bar{\phi}^a \bar{\phi}^b + \psi^\mu \bar{\phi}^c \bar{\phi}^{d*}\} |0\rangle_0 \qquad a, b = 3, 4 \qquad c, d = 1; 2; 5, 6, 7; 8.$$

These are generated as follows:

- $\psi^\mu \bar{\phi}^{3,4} \bar{\phi}^{3,4} |0\rangle_0$ generates $SO(4)_H$.
- $\psi^\mu \bar{\phi}^1 \bar{\phi}^{1*} |0\rangle_0$ generates $U(1)_7$.
- $\psi^\mu \bar{\phi}^2 \bar{\phi}^{2*} |0\rangle_0$ generates $U(1)_8$.
- $\psi^\mu \bar{\phi}^{5,6,7} \bar{\phi}^{5,6,7*} |0\rangle_0$ generates $SU(3)_H$, and the trace of this generates $U(1)_H$.
- $\psi^\mu \bar{\phi}^8 \bar{\phi}^{8*} |0\rangle_0$ generates $U(1)_9$.

6. Scalars, including singlets which are neutral under $U(1)$ symmetries,

- $\chi^{12} \bar{y}^2 \bar{\omega}^2 |0\rangle_0$
- $\chi^{34} \bar{y}^4 \bar{\omega}^4 |0\rangle_0$

		$SU(3)_C$	$U(1)_C$	$SU(2)_L$	$U(1)_L$
Quark Doublet	$Q = \begin{pmatrix} u \\ d \end{pmatrix}$	3	1/2	2	0
Lepton Doublet	$L = \begin{pmatrix} e \\ \nu \end{pmatrix}$	1	$-3/2$	2	0
Anti-Quarks	u^c	$\bar{3}$	$-1/2$	1	-1
	d^c	$\bar{3}$	$-1/2$	1	1
Positron	e^c	1	3/2	1	1
Anti-Neutrino	ν^c	1	3/2	1	-1

Table 1: Properties of Chiral Fermions found in Sectors $b_{1...4}$

- $\chi^{56}\bar{y}^5\bar{\omega}^5|0\rangle_0$

7. Singlets, for each of which we denote the charges under the observable gauge group as

$$[(SU(3)_C, U(1)_C); SU(2)_L, U(1)_L)]_{U(1)_{1...6}}.$$

These are

$$\begin{aligned}
\phi_{12} + \bar{\phi}_{12} &= \{\chi^{56}\bar{\eta}^1\bar{\eta}^2 + \chi^{56}\bar{\eta}^{1*}\bar{\eta}^{2*}\}|0\rangle_0 &&: [(1,0);(1,0)]_{110000} + [(1,0);(1,0)]_{-1-10000} \\
\phi_{13} + \bar{\phi}_{13} &= \{\chi^{12}\bar{\eta}^{1*}\bar{\eta}^3 + \chi^{12}\bar{\eta}^1\bar{\eta}^{3*}\}|0\rangle_0 &&: [(1,0);(1,0)]_{-101000} + [(1,0);(1,0)]_{10-1000} \\
\phi_{23} + \bar{\phi}_{23} &= \{\chi^{12}\bar{\eta}^2\bar{\eta}^3 + \chi^{12}\bar{\eta}^{2*}\bar{\eta}^{3*}\}|0\rangle_0 &&: [(1,0);(1,0)]_{011000} + [(1,0);(1,0)]_{0-1-1000} \\
\phi_4 + \bar{\phi}_4 &= \{\chi^{34}\bar{\zeta}^1\bar{\omega}^2 + \chi^{34}\bar{\zeta}^{1*}\bar{\omega}^2\}|0\rangle_0 &&: [(1,0);(1,0)]_{000100} + [(1,0);(1,0)]_{000-100} \\
\phi'_4 + \bar{\phi}'_4 &= \{\chi^{56}\bar{\zeta}^1\bar{y}^2 + \chi^{56}\bar{\zeta}^{1*}\bar{y}^2\}|0\rangle_0 &&: [(1,0);(1,0)]_{000100} + [(1,0);(1,0)]_{0000-100} \\
\phi_{56} + \bar{\phi}_{56} &= \{\chi^{12}\bar{\zeta}^2\bar{\zeta}^3 + \chi^{12}\bar{\zeta}^{2*}\bar{\zeta}^{3*}\}|0\rangle_0 &&: [(1,0);(1,0)]_{000011} + [(1,0);(1,0)]_{0000-1-1} \\
\phi'_{56} + \bar{\phi}'_{56} &= \{\chi^{12}\bar{\zeta}^{2*}\bar{\zeta}^3 + \chi^{12}\bar{\zeta}^2\bar{\zeta}^{3*}\}|0\rangle_0 &&: [(1,0);(1,0)]_{0000-11} + [(1,0);(1,0)]_{00001-1}
\end{aligned}$$

8. We also obtain states that are doublets under $SU(2)_L$ but singlets under $SU(3)_C$. As such, these states are potentially Higgs doublets should they acquire a non-zero v.e.v.

$$\begin{aligned}
\bar{h}_1 + h_1 &= \{\chi^{12}\bar{\psi}^{4,5}\bar{\eta}^1 + \chi^{12}\bar{\psi}^{4,5*}\bar{\eta}^{1*}\}|0\rangle_0 &&: [(1,0);(2,1)]_{100000} + [(1,0);(2,-1)]_{-100000} \\
\bar{h}_2 + h_2 &= \{\chi^{34}\bar{\psi}^{4,5}\bar{\eta}^{2*} + \chi^{34}\bar{\psi}^{4,5*}\bar{\eta}^2\}|0\rangle_0 &&: [(1,0);(2,1)]_{0-10000} + [(1,0);(2,-1)]_{010000} \\
\bar{h}_3 + h_3 &= \{\chi^{56}\bar{\psi}^{4,5}\bar{\eta}^3 + \chi^{56}\bar{\psi}^{4,5*}\bar{\eta}^{3*}\}|0\rangle_0 &&: [(1,0);(2,1)]_{001000} + [(1,0);(2,-1)]_{00-1000}
\end{aligned}$$

8.2 Sectors b_i, $i = 1 \dots 4$

From the sectors b_i ($i = 1 \dots 4$), we obtain three generations of left moving chiral fermions, with the charges shown in Table 1. Our particles are $SU(2)$ doublets, whilst antiparticles are $SU(2)$ singlets. These are the only chiral states that transform under the observable gauge group only. A full list of states and their quantum numbers is given in Appendix A.1.

1. From sector b_1, we obtain

$$(Q)_{-1/2,0,0,1/2,0,0} + (L)_{-1/2,0,0,1/2,0,0} + (u^c)_{-1/2,0,0,-1/2,0,0}+ (d^c)_{-1/2,0,0,-1/2,0,0} + (e^c)_{-1/2,0,0,-1/2,0,0} + (\nu^c)_{-1/2,0,0,-1/2,0,0}$$

2. From sector b_2,

$$(Q)_{0,-1/2,0,0,1/2,0} + (L)_{0,-1/2,0,0,1/2,0} + (u^c)_{-0,1/2,0,0,-1/2,0}+ (d^c)_{0,-1/2,0,0,-1/2,0} + (e^c)_{0,-1/2,0,0,-1/2,0} + (\nu^c)_{0,-1/2,0,0,-1/2,0}$$

3. From sector b_3,

$$(Q)_{0,0,-1/2,0,0,1/2} + (L)_{0,0,-1/2,0,0,1/2} + (u^c)_{0,0,-1/2,0,0,-1/2}+ (d^c)_{0,0,-1/2,0,0,-1/2} + (e^c)_{0,0,-1/2,0,0,-1/2} + (\nu^c)_{0,0,-1/2,0,0,-1/2}$$

4. From sector b_4,

$$(L)_{-1/2,0,0,1/2,0,0} + (d^c)_{-1/2,0,0,-1/2,0,0} + (\nu^c)_{-1/2,0,0,-1/2,0,0} +[(\bar{3},-1/2),(2,0)]_{1/2,0,0,-1/2,0,0}+[(3,1/2),(1,1)]_{1/2,0,0,1/2,0,0}+[(1,-3/2),(1,-1)]_{1/2,0,0,1/2,0,0}$$

   This shows similarities to the first generation obtained in sector b_1. The lepton doublet, anti-down and anti-neutrino have the same charges under the six extra $U(1)$'s as their first generation equivalents. We also have states that show similarities to the quark doublet, anti-up and positron, but have negative charges under the observable $U(1)$'s in comparison to those obtained in sector b_1. The full spectrum for b_i is given in the tables in the appendix.[5] We have obtained a partial fourth generation.

8.3 Remaining Sectors

The remaining sectors are split into one of two groups. These are sectors that give states which are vector representations of the hidden gauge group, and sectors that give states which are chiral representations of the hidden gauge group.

Appendix A.2 contains all sectors that give vector representations of the hidden gauge group. These arise from the sectors I, $b_i+2\beta$ and $I+b_j+2\beta$, where $I=1+b_1+b_2+b_3$, $i=1\dots4$ and $j=1\dots5$ (where b_5 is the sector α). All states in the massless sector $1+b_1+b_2+\alpha+\beta$ are projected by the choice of GSO projection coefficient

$$c\begin{pmatrix} b_4 \\ 1 \end{pmatrix} = -1,$$

whilst those in the sector $1+b_1+b_2+\alpha-\beta$ survive. All other sectors either have no massless states or have all states projected by the GSO projections.

We have now fully derived the massless spectrum. In the following chapters we analyse the spectrum, ensuring it is anomaly free and seeing how the standard model gauge group $SU(3)_C\times SU(2)_L\times U(1)_Y$ is obtained.

[5] Note in the appendix we have labelled the final three states as $\tilde{Q}_4$, $\tilde{u}^c_4$ and $\tilde{e}^c_4$ respectively. The tildes above the labels of the states represent that although they are non-physical particles, they show similarities to the quark double, up quark and electron but with negative charges under the trace $U(1)$'s.

8.4 $U(1)$ anomalies

We calculate the traces of each of the twelve $U(1)$ symmetries. They are

$$\begin{array}{cccc} \mathrm{Tr}[U(1)_C] = -6 & \mathrm{Tr}[U(1)_L] = 4 & \mathrm{Tr}[U(1)_1] = -20 & \mathrm{Tr}[U(1)_2] = -22 \\ \mathrm{Tr}[U(1)_3] = 26 & \mathrm{Tr}[U(1)_4] = -12 & \mathrm{Tr}[U(1)_5] = 6 & \mathrm{Tr}[U(1)_6] = 6 \\ \mathrm{Tr}[U(1)_H] = 12 & \mathrm{Tr}[U(1)_7] = -2 & \mathrm{Tr}[U(1)_8] = 12 & \mathrm{Tr}[U(1)_9] = 0. \end{array} \quad (89)$$

Of the twelve, $U(1)_9$ is anomaly free, and we can combine $U(1)_C$ and $U(1)_L$ as given in (88) to show that the weak hypercharge $U(1)_Y$ is also anomaly free. The orthogonal combination of the trace $U(1)$'s, that is

$$U(1)_{Z'} = U(1)_C - U(1)_L \quad (90)$$

is not anomaly free, with $\mathrm{Tr}[U(1)_{Z'}] = -10$. Hence the $U(1)_C \times U(1)_L$ symmetry must break immediately to $U(1)_Y$, and this symmetry breaking must occur at the string level. Of the remaining nine symmetries, we may rotate eight and only eight in such a way that the orthogonal combinations are anomaly free. These combinations are not unique. The remaining anomalous combination is uniquely given by

$$U(1)_A = k \sum_j (\mathrm{Tr}[U(1)_j]) U(1)_j .$$

Selecting $k = 1/2$,

$$U(1)_A = -10U(1)_1 - 11U(1)_2 + 13U(1)_3 - 6U(1)_4 + 3U(1)_5 + 3U(1)_6 + 6U(1)_H - U(1)_7 + 6U(1)_8 \quad (91)$$

We have reduced our eleven anomalous $U(1)$'s down to a single anomalous $U(1)_A$, and our gauge group down to the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

9 Results and Conclusions

We have derived fully the massless spectrum of a string model in the four dimensional free fermionic ABK formulation. The observable gauge group of the model is $SU(3)_C \times SU(2)_L \times U(1)_C \times U(1)_L \times U(1)^6$. Calculation of anomalies resulted in an anomaly free weak hypercharge $U(1)_Y$, represented by the combination

$$U(1)_Y = \frac{1}{3} U(1)_C + \frac{1}{2} U(1)_Y$$

with $\mathrm{Tr}[U(1)_Y] = 0$, while the orthogonal combination

$$U(1)_{Z'} = U(1)_C - U(1)_L$$

is anomalous with $\mathrm{Tr}[U(1)_{Z'}] = -10$.

The Dine-Seiberg-Witten (DSW) mechanism [18] can be used to cancel the anomalies. Any anomalous $U(1)_{A_i}$ gives rise to a Fayet-Iliopoulos term

$$\mathrm{Tr}[U(1)_A] = A \neq 0 \quad (92)$$

We define the D and F terms as

$$\begin{aligned} D_A &= A + \sum_j Q_{A,j} |\langle \phi_{A,j} \rangle|^2 \\ D_i &= \sum_j Q_{i,j} |\langle \phi_{i,j} \rangle|^2 \\ F_i &= \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 \end{aligned} \tag{93}$$

where D_A is the D term arising from the anomalous $U(1)$ and D_i are D terms arising from anomaly free $U(1)$'s. W is the superpotential of the model. Requiring D and F flatness, we have the constraint

$$\langle D \rangle = \langle F \rangle = 0. \tag{94}$$

Equations (93) satisfy this constraint when the fields ϕ obtain vevs. Since all low energy effective field theories are free of gauge anomalies, then $U(1)_{Z'}$ must be broken. Our model breaks symmetry *directly at the string level* as

$$SO(10) \rightarrow SU(3)_C \times SU(2)_L \times U(1)_C \times U(1)_L \rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y.$$

We also calculated that the remaining $U(1)$'s were anomaly free except for a single anomalous $U(1)_A$.

This requirement of symmetry breaking to the standard model gauge group at the string level is a result of the modification of the GSO projections. The original 1989 model with

$$c \begin{pmatrix} b_4 \\ 1 \end{pmatrix} = +1$$

resulted in both trace $U(1)$'s being anomaly free. Hence the DSW Mechanism allows for the $U(1)_C \times U(1)_L$ symmetry to break to either $U(1)_Y$ or $U(1)_{Z'}$, since both are anomaly free and hence have the property of D flatness. Then we may have symmetry breaking to the standard model gauge group at either the string level *or* an intermediate level.

The requirement for the standard model gauge group means the spectrum includes the gauge bosons of the standard model. The model has $N = 1$ spacetime supersymmetry, and includes three generations of chiral fermions, plus a fourth partial generation showing similarities under the extra $U(1)$'s to the first generation. The spectrum also includes Higgs doublets that may produce realistic gauge symmetry breaking, and hence allows for the acquisition of mass of particles via the Higgs mechanism. Finally the spectrum includes a spin 2 graviton state, and is free of tachyons. The massless spectrum is representitive of the standard model as described in Chapter 2.

The modification of GSO projection coefficients results in a difference in the level at which the $U(1)_C \times U(1)_L$ symmetry breaks to $U(1)_Y$. In the Faraggi et al model, both of the trace $U(1)$'s are anomaly free, and hence there is no requirement for the gauge group $SU(3)_C \times SU(2)_L \times U(1)_C \times U(1)_L$ to break to the standard model gauge group at the string level. This symmetry may instead be broken at an intermediate level. In the modified model, $SU(3)_C \times SU(2)_L \times U(1)_C \times U(1)_L$ has a non-zero trace anomaly, and so we require that this symmetry is broken at the string level.

A Massless States and Their Quantum Numbers

The following tables contain massless states and their quantum numbers for each massless sector excluding the Neveu-Schwarz sector, which is analysed in 8.1.

A.1 Sectors that Transform Under the Observable Gauge Group Only

Sector	F	$SU(3)_C$ $\times SU(2)_L$	Q_C	Q_L	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	$SO(4)_H$ $\times SU(3)_H$	Q_H	Q_7	Q_8	Q_9
b_1	Q_1	$(3,2)$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	(1,1)	0	0	0	0
	L_1	$(1,2)$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	(1,1)	0	0	0	0
	u_1^c	$(\bar{3},1)$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	(1,1)	0	0	0	0
	d_1^c	$(\bar{3},1)$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	(1,1)	0	0	0	0
	e_1^c	$(1,1)$	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	(1,1)	0	0	0	0
	ν_1^c	$(1,1)$	$\frac{3}{2}$	-1	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	(1,1)	0	0	0	0
b_2	Q_2	$(3,2)$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	(1,1)	0	0	0	0
	L_2	$(1,2)$	$-\frac{3}{2}$	0	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	(1,1)	0	0	0	0
	u_2^c	$(\bar{3},1)$	$-\frac{1}{2}$	-1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	(1,1)	0	0	0	0
	d_2^c	$(\bar{3},1)$	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	(1,1)	0	0	0	0
	e_2^c	$(1,1)$	$\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	(1,1)	0	0	0	0
	ν_2^c	$(1,1)$	$\frac{3}{2}$	-1	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	(1,1)	0	0	0	0
b_3	Q_3	$(3,2)$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	(1,1)	0	0	0	0
	L_3	$(1,2)$	$-\frac{3}{2}$	0	0	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	(1,1)	0	0	0	0
	u_3^c	$(\bar{3},1)$	$-\frac{1}{2}$	-1	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	(1,1)	0	0	0	0
	d_3^c	$(\bar{3},1)$	$-\frac{1}{2}$	1	0	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	(1,1)	0	0	0	0
	e_3^c	$(1,1)$	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	(1,1)	0	0	0	0
	ν_3^c	$(1,1)$	$\frac{3}{2}$	-1	0	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	(1,1)	0	0	0	0
b_4	$\tilde{Q}_4$	$(\bar{3},2)$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	(1,1)	0	0	0	0
	L_4	$(1,2)$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	(1,1)	0	0	0	0
	$\tilde{u}_4^c$	$(3,1)$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	(1,1)	0	0	0	0
	d_4^c	$(\bar{3},1)$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	(1,1)	0	0	0	0
	$\tilde{e}_4^c$	$(1,1)$	$-\frac{3}{2}$	-1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	(1,1)	0	0	0	0
	ν_4^c	$(1,1)$	$\frac{3}{2}$	-1	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	(1,1)	0	0	0	0

A.2 Sectors that Give Vector Representations of the Hidden Gauge Group

Sector	F	$SU(3)_C$ $\times SU(2)_L$	Q_C	Q_L	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	$SO(4)_H$ $\times SU(3)_H$	Q_H	Q_7	Q_8	Q_9
I	V_1	(1,1)	0	0	0	0	0	0	0	0	(1,3)	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	V_2	(1,1)	0	0	0	0	0	0	0	0	(1,3)	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	V_3	(1,1)	0	0	0	0	0	0	0	0	$(1,\bar{3})$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	V_4	(1,1)	0	0	0	0	0	0	0	0	$(1,\bar{3})$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	V_5	(1,1)	0	0	0	0	0	0	0	0	(2,1)	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	V_6	(1,1)	0	0	0	0	0	0	0	0	(2,1)	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$b_1+2\beta$	V_7	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	(1,1)	$\frac{3}{2}$	$\frac{1}{2}$	0	0
	V_8	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,1)	$-\frac{3}{2}$	$-\frac{1}{2}$	0	0
	V_9	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$(1,\bar{3})$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
	V_{10}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,3)	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$b_2+2\beta$	V_{11}	(1,1)	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	(1,1)	$\frac{3}{2}$	$\frac{1}{2}$	0	0
	V_{12}	(1,1)	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	(1,1)	$-\frac{3}{2}$	$-\frac{1}{2}$	0	0
	V_{13}	(1,1)	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$(1,\bar{3})$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
	V_{14}	(1,1)	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	(1,3)	$\frac{1}{2}$	$-\frac{1}{2}$	0	0

Sector	F	$SU(3)_C$ $\times SU(2)_L$	Q_C	Q_L	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	$SO(4)_H$ $\times SU(3)_H$	Q_H	Q_7	Q_8	Q_9
$b_3 + 2\beta$	V_{15}	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	(1,1)	$\frac{3}{2}$	$\frac{1}{2}$	0	0
	V_{16}	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	(1,1)	$-\frac{3}{2}$	$-\frac{1}{2}$	0	0
	V_{17}	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$(1,\bar{3})$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
	V_{18}	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	(1,3)	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$b_4 + 2\beta$	V_{19}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	(1,1)	$\frac{3}{2}$	$\frac{1}{2}$	0	0
	V_{20}	(1,1)	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	(1,1)	$\frac{3}{2}$	$-\frac{1}{2}$	0	0
	V_{21}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,3)	$\frac{1}{2}$	$\frac{1}{2}$	0	0
	V_{22}	(1,1)	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,3)	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$I + b_1 + 2\beta$	V_{23}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,1)	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	V_{24}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,1)	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	V_{25}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
	V_{26}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
	V_{27}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(2,1)	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
	V_{28}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	(2,1)	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
$I + b_2 + 2\beta$	V_{29}	(1,1)	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	(1,1)	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	V_{30}	(1,1)	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	(1,1)	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	V_{31}	(1,1)	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
	V_{32}	(1,1)	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
	V_{33}	(1,1)	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	(2,1)	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
	V_{34}	(1,1)	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	(2,1)	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
$I + b_3 + 2\beta$	V_{35}	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	(1,1)	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	V_{36}	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	(1,1)	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	V_{37}	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
	V_{38}	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
	V_{39}	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	(2,1)	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
	V_{40}	(1,1)	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	(2,1)	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
$I + b_4 + 2\beta$	V_{41}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,1)	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	V_{42}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,1)	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	V_{43}	(1,1)	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,1)	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	V_{44}	(1,1)	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	(1,1)	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	V_{45}	(1,1)	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	(2,1)	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
	V_{46}	(1,1)	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	(2,1)	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
$I + \alpha + 2\beta$	V_{47}	(1,1)	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	(1,1)	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
	V_{48}	(1,1)	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	(1,1)	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
	V_{49}	(1,1)	0	-1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	(1,1)	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
	V_{50}	(1,1)	0	-1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	(1,1)	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
	V_{51}	(1,2)	0	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	(1,1)	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
	V_{52}	(1,2)	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	(1,1)	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
$1 + b_1 + \alpha$ $+2\beta$	V_{53}	(1,1)	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	(1,1)	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
	V_{54}	(1,1)	0	-1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	(1,1)	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
	V_{55}	(1,1)	0	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	(1,1)	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
	V_{56}	(1,1)	0	-1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	(1,1)	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
	V_{57}	(1,2)	0	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	(1,1)	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
	V_{58}	(1,2)	0	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	(1,1)	0	$-\frac{1}{2}$	0	$\frac{1}{2}$

A.3 Sectors that Give Chiral Representations of the Hidden Gauge Group

Sector	F	$SU(3)_C \times SU(2)_L$	Q_C	Q_L	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	$SO(4)_H \times SU(3)_H$	Q_H	Q_7	Q_8	Q_9
$\pm\beta$	H_1	(1,1)	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	(2,1)	$\frac{3}{4}$	$\frac{1}{4}$	0	0
	H_2	(1,1)	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	(2,1)	$-\frac{3}{4}$	$-\frac{1}{4}$	0	0
$I \pm \beta$	H_3	(1,1)	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	(1,1)	$-\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$
	H_4	(1,1)	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	(1,1)	$\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$
	H_5	(1,1)	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	(1,1)	$-\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$
	H_6	(1,1)	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	(1,1)	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$
$b_3 + \alpha \pm \beta$	H_7	(1,1)	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
	H_8	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	0	0	$-\frac{1}{2}$	(1,1)	$-\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0
	H_9	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$	(1,1)	$-\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0
	H_{10}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$	(1,1)	$-\frac{3}{4}$	$-\frac{3}{4}$	$\frac{1}{2}$	0
	H_{11}	(3,1)	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$-\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
	H_{12}	(1,2)	$\frac{3}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$-\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
	H_{13}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$	$(1,\bar{3})$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0
$b_1 + b_2 + b_3$ $+\alpha \pm \beta$	H_{14}	(1,1)	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
	H_{15}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	(1,1)	$-\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0
	H_{16}	(1,1)	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
	H_{17}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	(1,1)	$-\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0
$b_1 + b_2 + b_4$ $+\alpha \pm \beta$	H_{18}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	0	$\frac{1}{2}$	0	(1,1)	$-\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0
	H_{19}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	0	$-\frac{1}{2}$	0	(1,1)	$-\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0
	H_{20}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{2}$	0	(1,1)	$-\frac{3}{4}$	$-\frac{3}{4}$	$\frac{1}{2}$	0
	H_{21}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{2}$	0	$(1,\bar{3})$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0
	H_{22}	(3,1)	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	0	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
	H_{23}	$(1,\bar{2})$	$\frac{3}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	0	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
	H_{24}	(1,1)	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	0	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
$b_2 + b_3 + b_4$ $+\alpha \pm \beta$	H_{25}	(1,1)	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
	H_{26}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	(1,1)	$-\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0
	H_{27}	(1,1)	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
	H_{28}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	(1,1)	$-\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0
$1 + I + b_4$ $\pm\beta$	H_{29}	(1,1)	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	(2,1)	$\frac{3}{4}$	$\frac{1}{4}$	0	0
	H_{30}	(1,1)	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	(2,1)	$-\frac{3}{4}$	$-\frac{1}{4}$	0	0
$1 + b_4 \pm \beta$	H_{31}	(1,1)	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	(1,1)	$-\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$
	H_{32}	(1,1)	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	(1,1)	$-\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$
	H_{33}	(1,1)	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$
	H_{34}	(1,1)	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$
$1 + b_1 + b_2$ $+\alpha - \beta$	H_{35}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$	(2,1)	$\frac{3}{4}$	$-\frac{1}{4}$	0	$\frac{1}{2}$
	H_{36}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	0	$-\frac{1}{2}$
	H_{37}	(1,1)	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$	(1,1)	$\frac{3}{4}$	$-\frac{1}{4}$	0	$-\frac{1}{2}$
$1 + b_3 + b_4$ $+\alpha \pm \beta$	H_{38}	(1,1)	$\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	0	(2,1)	$-\frac{3}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$
	H_{39}	(1,1)	$\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	0	(1,1)	$-\frac{3}{4}$	$\frac{1}{4}$	0	$-\frac{1}{2}$
	H_{40}	(1,1)	$\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	0	(1,1)	$-\frac{3}{4}$	$\frac{1}{4}$	0	$-\frac{1}{2}$

B Light-Cone Coordinates

Light-cone coordinates are a special set of coordinates used to quantize the string simply and directly. In the light-cone coordinate system, we replace x^0 and x^1 with coordinates defined as the world-line of a beam of light in the positive and negative x_1 directions. For a beam of light travelling along the $+x^1$ direction, we have $x^1 = ct = x^0$; this defines the x^+ axis. Similarly, the x^- axis is defined as the worldline $x^1 = -ct = -x^0$. We can summarise these definitions as

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \tag{95}$$

and where x^2 and x^3 are unmodified from their usual definitions of a Lorentz four vector. The complete set of light-cone coordinates are (x^+, x^-, x^2, x^3). We can take either x^+ or x^- to be a light-cone time coordinate, so we choose x^+. Therefore (x^-, x^2, x^3) are the light-cone space coordinates.

In Minkowski spacetime coordinates, we can define the Lorentz invariant spacetime interval

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$$

From (95) we can see that

$$2dx^+ dx^- = (dx^0)^2 - (dx^1)^2,$$

which leads to

$$ds^2 = 2dx^+ dx^- - (dx^2)^2 - (dx^3)^2 = -\hat{\eta}_{\mu\nu} dx^\mu dx^\nu \tag{96}$$

where $\hat{\eta}_{\mu\nu}$ is the light-cone metric, where the indices run over $(+, -, 2, 3)$.

Under our new coordinate system, we must also calculate the components of the light-cone momentum vector. It can be seen from (95) that we can write

$$p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^1). \tag{97}$$

The light-cone energy is p^-.

C The Nambu-Goto String Action

The Nambu-Goto string action is a simple invariant action, which is the action of a relativistic string. Consider an infinitesimal parallelogram on the world-sheet demonstrated in Figure 3, with sides du^μ and dv^μ, which is the area element of the infinitesimal rectangle with sides $d\sigma$ and $d\tau$. Hence,

$$du^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau = \dot{X}^\mu d\tau, \qquad dv^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma = X^{\mu\prime} d\sigma. \tag{98}$$

Then the area of this parallelogram is

$$dA = |du^\mu||dv^\mu||\sin\theta| = \sqrt{|du|^2|dv|^2 - |du|^2|dv|^2\cos^2\theta} \tag{99}$$

Now, we come across the problem that the terms under the square root are negative. It does not, however, affect Lorentz invariance for us to change the sign under the square root, so we may write

$$dA = \sqrt{(du \cdot dv)^2 - (du \cdot du)(dv \cdot dv)}, \tag{100}$$

where we have changed the sign under the square root and written in terms of the dot product. Substituting (98) into (100) and integrating, we obtain our area

$$A = \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \tag{101}$$

Then our string action is

$$S = -\frac{1}{2\pi\alpha'} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \tag{102}$$

where we have introduced the constant at the beginning so that the units are the units of action. This is the Nambu-Goto string action. Furthermore, from our definition of string action,

$$S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \mathcal{L}, \tag{103}$$

we obtain the Lagrangian density

$$\mathcal{L} = -\frac{1}{2\pi\alpha'}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}. \tag{104}$$

It is with this result we obtain the equations (26a, 26b).

D Group Theory in Particle Physics

We discussed in chapter 2 the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, and a brief description of the physical meaning of these groups within particle physics, particularly with the interaction for which they are responsible. In this appendix I expand upon these descriptions, giving first a background on group theory and the symmetries Lie Groups give rise to, and then discussing in detail the key groups of the standard model. For this appendix our key references are [16, 6, 7] among others.

D.1 A brief overview of Lie Groups

We begin with a definition of a group. A group $(G, *)$ is a set G of elements $g_i \in G$ that act with an operator $*$ with the following properties:

1. **Closure**: $g_i * g_j = g_k \in G$.
2. **Associativity**: $(g_i * g_j) * g_k = g_i * (g_j * g_k)$.
3. **Identity Element**: There exists an identity element $g_I \in G$ such that $g_I * g_i = g_i * g_I = g_i$ for all elements g_i.
4. **Inverse Element**: For each g_i there exists an inverse element $g_i^{-1} \in G$ that satisfies $g_i^{-1} * g_i = g_i * g_i^{-1} = g_I$.

For example, the group $(\mathbb{Z}, +)$ is a group. We check that it satisfies the constraints.

1. **Closure**: Any integer added to another is an integer, so closure is a property of $(\mathbb{Z}, +)$.
2. **Associativity**: $(a + b) + c = a + (b + c)$ for $a, b, c \in \mathbb{Z}$, so associativity is a property of $(\mathbb{Z}, +)$.
3. **Identity Element**: $0 + a = a + 0 = a$, for $a \in \mathbb{Z}$. The number 0 is also an integer, so the identity element of the group $(\mathbb{Z}, +)$ is $g_I = 0$.
4. **Inverse Element**: $a - a = 0$, so since $g_I = 0$, for an element $g_i = a$ the inverse is $g_i^{-1} = -a \in \mathbb{Z}$.

However $(\mathbb{R}, \times)$ is not a group. Let us consider the constraints for $(\mathbb{R}, \times)$.

1. **Closure**: Any real number multiplied by another real number is real, so closure is a property of $(\mathbb{R}, \times)$.
2. **Associativity**: $(a \times b) \times c = a \times (b \times c)$ for $a, b, c \in \mathbb{R}$, so associativity is a property of $(\mathbb{R}, \times)$.
3. **Identity Element**: $1 \times a = a \times 1 = a$ for $a \in \mathbb{R}$. Since $1 \in \mathbb{R}$, then the identity element of the group $(\mathbb{R}, \times)$ is 1.

So far, we have had no issues with $(\mathbb{R}, \times)$ as a group. However, when we consider the inverse element, we come across an issue.

4. **Inverse Element**: $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$, with $a, \frac{1}{a} \in \mathbb{R}$, and since $g_I = 1$, this satisfies the inverse element, *if* $\frac{1}{a}$ exists for all $a \in \mathbb{R}$. However, $\frac{1}{a}$ does not exist for $a = 0$, so there is no inverse element for $(\mathbb{R}, \times)$, so it is *not* a group.

It is clear however that if we define $R = \mathbb{R} \setminus \{0\}$, then $(R, \times)$ is a group. Groups are important in physics since they give rise to symmetries. For example, in the group $(\mathbb{Z}, +)$ described above, the use of our operator $+$ between two elements will change the element. However there is a conserved property, and that is the fact that the new element will still be an integer. Essentially it is a statement that adding any two integers results in another integer, so the property of the elements being integers is preserved, even if the actual value of the integers has changed after the operation. It is these symmetries that are of interest in particle theory.

The most general Lie Group (and the one for which all of the standard model gauge groups are a subset) is the General Linear group in n dimensions, including the complex plane. This is denoted $GL(n, \mathbb{C})$. This group consists of any invertible $n \times n$ matrix. Consider a vector in $\mathbb{C}^n$. Acting upon it with any $n \times n$ invertible matrix may result in translation, reflection, rotation, or a change in magnitude, but it will always result in another vector in $\mathbb{C}^n$. The preserved property of $GL(n, \mathbb{C})$ is the space the vector occupies. A subgroup is $GL(n, \mathbb{R})$, which only acts upon the real dimensions $\mathbb{R}^n$. This group conserves the real space a vector occupies.

Consider a solid in $\mathbb{C}^n$. A matrix $\mathbf{A}$ from the group $GL(n, \mathbb{C})$ transforms the solid in some way. If we set $\det \mathbf{A} = +1$, we know that whilst the solid may be rotated or translated, it must occupy the same volume as before the transformation. Then the Special Linear group, $SL(n, \mathbb{C})$, which is the subgroup of $GL(n, \mathbb{C})$ with the matrices of $GL(n, \mathbb{C})$ that have $\det \mathbf{A} = +1$, preserves volume *and* space in which the solid exists. Similarly $SL(n, \mathbb{R})$ is the subgroup of $GL(n, \mathbb{R})$ for which $\det \mathbf{A} = +1$.

Another set of subgroups of the general linear group are the orthogonal group $O(n)$ and the unitary group $U(n)$. Consider a vector $\vec{r}$ in $\mathbb{R}^n$. Its magnitude is given by $\vec{r}^2 = \vec{r}^T \vec{r}$. Then if we act upon this vector with a matrix $\mathbf{A} \in GL(n, \mathbb{R})$, we obtain

$$\vec{r}^2 \to \vec{r}^T \mathbf{A}^T \mathbf{A} \vec{r}. \tag{105}$$

In the case that $\mathbf{A}^T \mathbf{A} = I$, where I is the identity matrix, the magnitude of the vector remains unchanged. In this case, the matrices correspond to a rotation. Similarly we could show that this matrix would only rotate a solid in n dimensions. Since

$$(\det \mathbf{A})^2 = \det(\mathbf{A}^T \mathbf{A}) = \det I = 1, \tag{106}$$

we know that $\det \mathbf{A} = \pm 1$, so the volume of the solid also remains unchanged. The subgroup of $GL(n, \mathbb{R})$ with $\mathbf{A}^T \mathbf{A} = I$ and (therefore) $\det \mathbf{A} = \pm 1$ is denoted $O(n)$ since these are the *orthogonal* matrices of $GL(n, \mathbb{R})$. This group conserves the space occupied, volume, and radius. $O(n)$ corresponds to rotations in $\mathbb{R}^n$.

Similarly we can construct a subgroup of $GL(n, \mathbb{C})$ that corresponds to rotations in $\mathbb{C}^n$. The only difference is that in the complex plane, $\vec{r}^2 = \vec{r}^\dagger \vec{r}$, where $\vec{r}^\dagger = (\vec{r}^*)^T$ is the Hermitian conjugate (and $\vec{r}^*$ is the complex conjugate). Then the subgroup that conserves radius are those with $\mathbf{A}^\dagger \mathbf{A} = I$. This subgroup is denoted $U(n)$ since these are *unitary* matrices. Once again the matrices of $U(n)$ have $\det \mathbf{A} = \pm 1$.

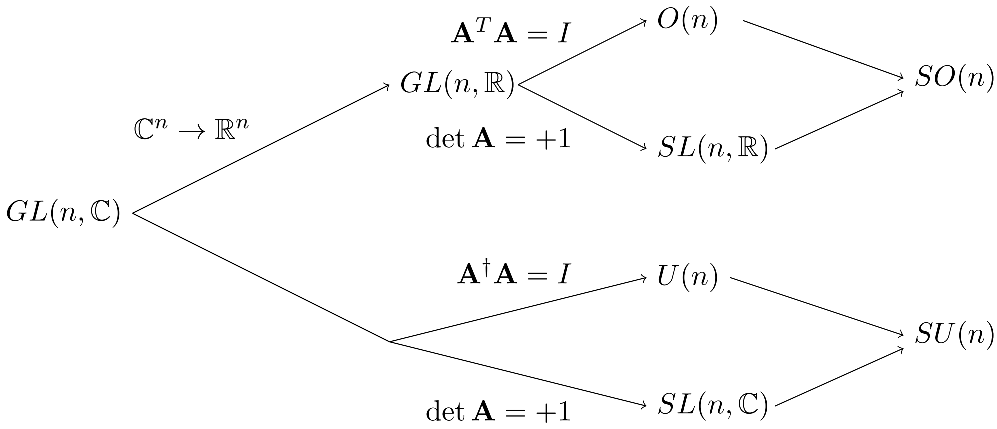

Figure 4: Decomposition of the general linear group in $\mathbb{C}^n$. $G_1 \to G_2$ represents $G_1 \supset G_2$, and hence $G_1 \to G_3 \leftarrow G_2$ represents that G_3 is a subset of both G_1 and G_2, or $G_3 \subset G_1 \cap G_2$.

The final decomposition we shall make are the subgroups $SU(n)$ and $SO(n)$, which contain the matrices of $U(n)$ and $O(n)$ respectively with $\det \mathbf{A} = +1$. Similarly we could write $SU(n) = U(n) \cap SL(n,\mathbb{C})$ and $SO(n) = O(n) \cap SL(n,\mathbb{C})$. These groups preserve handedness on top of everything that is preserved by the unitary and orthogonal groups.

Figure 4 is a graphic representation of these Lie groups and how they relate to their sub- and super- groups. Whilst our discussion shall primarily be limited to the gauge groups of the standard model, it is worth mentioning the orthogonal Lie Groups as they also play important roles in particle theory. For example the group $SO(m,n)$ is the special orthogonal matrix in $m+n$ dimensions, with the metric $x_i = x^i$ for $i = 1,\ldots,m$ and $x_i = -x^i$ for $i = m+1,\ldots,n$. Then $SO(1,3)$ is the Lorentz group of special relativity.

D.2 The Unitary Group $U(1)$ and the Electromagnetic Interaction

A unitary group of degree n may be written $U(n)$. Limiting our discussion to degree $n = 1$, we can see clearly that $U(1)$ is the set of numbers in the complex plane with magnitude 1. The proof of this is as follows. Euler's formula allows us to write any complex number as $z = re^{i\alpha}$, where $r = |z|$ and $\alpha = \arg z$ (hence α is real). The complex conjugate is $z^* = re^{-i\alpha}$. Multiplying a complex number and it's conjugate gives the square of the magnitude, or $zz^* = r^2 = |z|^2$. If we now introduce a matrix of degree 1, U, and impose a unitary condition $UU^\dagger = 1$, then it is clear that $U = e^{i\alpha}$, the set of complex numbers with $r = |z| = 1$. We can define the group $U(1)$ by

$$U(1) = \{z \in \mathbb{C} : |z| = 1\}. \tag{107}$$

Note here that since $U = e^{i\alpha}$ $(\alpha \neq \alpha(x))$, we have a global $U(1)$ symmetry.

In particle physics the $U(1)$ group represents the local gauge symmetry in the electromagnetic interaction. Classically, electromagnetism is described by Maxwell's equations [17],

$$\nabla \cdot \vec{E} = \rho \tag{108a}$$

$$\nabla \cdot \vec{B} = 0 \tag{108b}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{108c}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} \tag{108d}$$

Using equation (108b) and the vector calculus identity $\nabla \cdot (\nabla \times \vec{a}) = 0$, where $\vec{a}$ is any vector, we can write

$$\vec{B} = \nabla \times \vec{A} \tag{109}$$

where $\vec{A}$ is a vector potential. Substituting this into (108c), we obtain

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0, \tag{110}$$

from which we can deduce that the terms in the brackets are the gradient of a scalar potential V, or

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V. \tag{111}$$

Then we can write the electric field as

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}. \tag{112}$$

Equations (109,112) describe our electric and magnetic field in terms of scalar and vector potentials, and these, by definition, satisfy (108b, 108c). If we introduce the notation $A^\mu = (V, \vec{A})$ and $J^\mu = (\rho, \vec{J})$, we can define the Electromagnetic field strength tensor $F^{\mu\nu}$ as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{113}$$

Substituting in values of μ and ν, and noting that $F^{\mu\nu}$ is antisymmetric under exchange of indices, it is possible to construct the field strength tensor explicity.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{114}$$

By substituting in values of μ and ν, it is simple to show that

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{115}$$

satisfies the remaining Maxwell equations (108a , 108d).

The $U(1)$ symmetry arises in electromagnetism when we consider gauge invariance of the Lagrangian. The free field Lagrangian density of the electromagnetic interaction is

$$\mathcal{L}_Q = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{116}$$

If the Lagrangian is invariant under a transformation

$$A^\mu \to A'^\mu + \partial^\mu \Lambda(t, \vec{x}), \tag{117}$$

then it is clear that the electromagnetic interaction has a *local* gauge $U(1)$ symmetry[6]. Proof that the Lagrangian remains invariant under this transformation involves showing that the field strength tensor is invariant under the transformation, then it is clear from (116) that the Lagrangian remains invariant.

$$\begin{aligned} F'^{\mu\nu} &= \partial^\mu A'^\nu - \partial^\nu A'^\mu \\ &= \partial^\mu (A^\nu + \partial^\nu \Lambda(t, \vec{x})) - \partial^\nu (A^\mu + \partial^\mu \Lambda(t, \vec{x})) \\ &= F^{\mu\nu} + \partial^\mu \partial^\nu \Lambda(t, \vec{x}) - \partial^\nu \partial^\mu \Lambda(t, \vec{x}) \\ &= F^{\mu\nu}. \end{aligned} \tag{118}$$

From this, it is clear that $\mathcal{L}_Q(A'^\mu) = \mathcal{L}_Q(A^\mu)$. Hence electromagnetism gives rise to a local $U(1)$ gauge symmetry, that we label $U(1)_Q$. This $U(1)_Q$ is incorportated into the standard model gauge group, since electroweak symmetry breaks to electromagnetic symmetry as $SU(2)_L \times U(1)_Y \to U(1)_Q$.

D.3 The Special Unitary Group $SU(2)$ and Spin

The groups $SU(n)$ are the special unitary groups of degree n, or $SU(n) \subset U(n)$ with $\det U = 1$. We will construct first the group $SU(2)$, then consequently construct $SU(3)$.

D.3.1 The Fundamental Representation of $SU(2)$

The fundamental representation of $SU(2)$ is the group of 2×2 unitary matrices with determinant 1. Let us first write these matrices explicitly, as

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \qquad u_{ij} \in \mathbb{C}, \quad i, j = 1, 2$$

Imposing the unitary condition, we obtain $U^\dagger = U^{-1}$. The inverse of a 2×2 matrix is given by

$$U^{-1} = \frac{1}{|\det U|} \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix}$$

and the hermitian conjugate of a matrix with complex components is

$$U^\dagger = \begin{pmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{pmatrix}$$

Equating the above matrices, and imposing $\det U = 1$, we obtain the relations

$$u_{11}^* = u_{22}, \qquad u_{12}^* = -u_{21}$$

Then we can write our matrix U as

$$U = \begin{pmatrix} u_{11} & -u_{21}^* \\ u_{21} & u_{11}^* \end{pmatrix}$$

[6]The symmetry is local (as opposed to global) since $\Lambda = \Lambda(\vec{x})$. The symmetry is a $U(1)$ symmetry since there is only one function $\Lambda(t, \vec{x})$.

Finally, to ensure $\det U = 1$, we impose $\det U = u_{11}u_{11}^* + u_{21}u_{21}^* = |u_{11}|^2 + |u_{21}|^2 = 1$. We can now write $SU(2)$ as the set

$$SU(2) = \left\{ \begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix} : |A|^2 + |B|^2 = 1 \right\} \tag{119}$$

This set contains the matrices βI and $i\alpha_j\sigma_j$ for $j = 1, 2, 3$ ($\alpha, \beta \in \mathbb{R}$), where I is the identity matrix and σ_j are the three Pauli Spin Matrices, given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{120}$$

Writing the matrices of $SU(2)$ as $U = e^{\beta I + i\vec{\alpha}\cdot\vec{\sigma}} = e^A$, we can calculate the determinant as

$$\det U = \det(I + A + \frac{A^2}{2} + \ldots).$$

We have expanded e^A in a Taylor expansion. We now diagonalise the matrix A through multiplaction with a matrix P, that commutes with A, and it's inverse,

$$\begin{aligned} \det U &= \det(I + PAP^{-1} + \frac{1}{2}(PAP^{-1}PAP^{-1}) + \ldots) \\ &= \det(I + A_D + \frac{A_D^2}{2} + \ldots) \end{aligned}$$

where A_D is the diagonalised matrix. We now write this in exponential form considering the matrix is diagonalised as

$$\begin{aligned} \det U &= \det \begin{pmatrix} e^{\gamma_1} & 0 \\ 0 & e^{\gamma_2} \end{pmatrix} \\ &= e^{\gamma_1 + \gamma_2} = e^{\text{tr}A}. \end{aligned}$$

Since we stated earlier that for $SU(2)$, we can write generally $U = e^{\beta I + i\vec{\alpha}\cdot\vec{\sigma}}$, then we have now obtained the result that for the matrices to be *special* unitary matrices (i.e. $\det U = 1$), we require $\text{tr} = \beta I + i\vec{\alpha}\cdot\vec{\sigma} = 0$. Since the trace of each of the Pauli spin matrices is zero, and the trace of I is non-zero, we require $\beta = 0$. The most general form of the $SU(2)$ matrices is

$$U = e^{i\vec{\alpha}\cdot\vec{\sigma}}, \qquad \alpha_i \in \mathbb{R} \tag{121}$$

with the generators σ_j given in (120). It is also common to include a factor of 1/2 so that the generators follow the algebra of $SU(2)$, which is

$$[\tau_i, \tau_j] = i\epsilon_{ijk}\tau_j \tag{122}$$

where ϵ_{ijk} is the alternating tensor and $\tau_i = \frac{1}{2}\sigma_i$.

D.3.2 Spin ±1/2

We have constructed our **2** representation of $SU(2)$. We must now construct the space on which it acts. To do this we use the eigenvectors of this representation, using the diagonal generator to form the basis. Let us denote these eigenvectors $|j, m\rangle$, where j is the maximum eigenvalue of τ_3 and m is the eigenvalue of that particular eigenvector. For our **2** representation we can write these eigenvectors as

$$|j, m_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |j, m_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{123}$$

Then we can work out the values of j, m_1, and m_2 by acting upon them with τ_3.

$$\tau_3|j,m_1\rangle = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}|j,m_1\rangle \to m_1 = \frac{1}{2}$$
$$\tau_3|j,m_2\rangle = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}|j,m_2\rangle \to m_2 = -\frac{1}{2} \tag{124}$$

and hence $j = 1/2$ since j is simply the maximum eigenvalue. We may rewrite (123) as

$$|\frac{1}{2},\frac{1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |\frac{1}{2},-\frac{1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \tag{125}$$

These states represent spin $\pm 1/2$ particles, namely the fermions of the standard model. Finally we wish to construct raising and lowering operators to relate these states. We make an assumption that these operators are related to the remaining two generators τ_1 and τ_2. The solution is a combination of the two,

$$\tau^{\pm} = \frac{1}{\sqrt{2}}(\tau_1 \pm i\tau_2) \tag{126}$$

so we obtain

$$\tau^+ = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \qquad \tau^- = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{127}$$

It is a simple calculation to show that

$$\tau^+|\frac{1}{2},\frac{1}{2}\rangle = 0 \qquad \tau^-|\frac{1}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}|\frac{1}{2},-\frac{1}{2}\rangle$$
$$\tau^+|\frac{1}{2},-\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}|\frac{1}{2},\frac{1}{2}\rangle \qquad \tau^-|\frac{1}{2},-\frac{1}{2}\rangle = 0 \tag{128}$$

It is clear that τ^+ is the raising operator, as it raises the spin of a state by one until the maximum spin, at which point it annihilates the state, and τ^- is a lowering operator, as it lowers the spin of a state by one until the minimum spin, at which point it also annihilates the state. The **2** representation of $SU(2)$ corresponds to fermions with spin $\pm 1/2$.

D.3.3 Spin ±1 and 0

$SU(2)$ also represents spin 1 and 0 bosons. However for this we require the **3** representation of $SU(2)$. This is the adjoint representation. It is generated by the matrices

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \qquad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}, \qquad T_3 = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}. \tag{129}$$

Note we still have only one diagonal matrix, and so we follow the same procedure as before. We firstly define our states as

$$|j,m_1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad |j,m_2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad |j,m_3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{130}$$

As before, we act on these states with the diagonal matrix T_3.

$$T_3|j, m_1\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow m_1 = 1 \tag{131}$$

$$T_3|j, m_2\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \rightarrow m_2 = 0 \tag{132}$$

$$T_3|j, m_3\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow m_3 = -1 \tag{133}$$

so we rewrite (130) as

$$|1, 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |1, 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |1, -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{134}$$

Our raising and lowering operators are constructed in the same way as before.

$$T^{\pm} = \frac{1}{\sqrt{2}}(T_1 \pm iT_2). \tag{135}$$

Calculating the form of $T^{\pm}$ explicitly allows us to verify the following results.

$$\begin{array}{ccc} T^{+}|1, 1\rangle = 0 & T^{+}|1, 0\rangle = |1, 1\rangle & T^{+}|1, -1\rangle = |1, 0\rangle \\ T^{-}|1, 1\rangle = |1, 0\rangle & T^{-}|1, 0\rangle = |1, -1\rangle & T^{-}|1, -1\rangle = 0 \end{array} \tag{136}$$

We have shown it is possible to represent spin $-1, 0, 1$ bosonic states by the adjoint representation of $SU(2)$.

D.3.4 A Generalisation of $SU(2)$ for higher values of Spin

We now wish to consider $SU(2)$ for any value of j. Since j is the maximum eigenvalue of the diagonal $SU(2)$ generator T_3, this is the higest spin in a given representation, so writing our $d \times d$ matrix representations of $SU(2)$ (for $d \geq 2$) for an arbitrary value of j is desirable. For example, we would require $j = 2$ to include the graviton, or $j = 3/2$ for some baryons.

For any representation of $SU(2)$, we require three generators. We shall denote these $T_{1,2,3}$, where T_3 is the one diagonal generator. These matrices follow the $SU(2)$ algebra (122). A useful tool for each representation given above was the raising and lowering operators, defined by (135). Let us use equations (122, 135) to obtain the following commutation relations.

$$\begin{aligned} [T^{+}, T^{-}] &= T^{+}T^{-} - T^{-}T^{+} \\ &= \frac{1}{2}\big((T_1 + iT_2)(T_2 - iT^2) - (T_1 - iT_2)(T_1 + iT_2)\big) \\ &= -i[T_1, T_2] = -i \cdot i\epsilon_{123}T^3 \\ &= T^3 \end{aligned} \tag{137}$$

$$\begin{aligned}
[T_3, T^{\pm}] &= T_3 T^{\pm} - T^{\pm} T_3 \\
&= \frac{1}{\sqrt{2}}\big(T_3(T_1 \pm iT_2) - (T_1 \pm iT_2)T_3\big) \\
&= \frac{1}{\sqrt{2}}([T_3, T_1] \pm i[T_3, T_2]) \\
&= \frac{1}{\sqrt{2}}(i\epsilon_{312} T_2 \pm i \cdot i\epsilon_{321} T_1) \\
&= \pm\frac{1}{\sqrt{2}}(T_1 \pm iT_2) = \pm T^{\pm}
\end{aligned} \tag{138}$$

We are now in a position to understand the meaning of these operators. We know that by definition, $T_3|j,m\rangle = m|j,m\rangle$. What effect do the raising and lowering operators have? We know from experience of our **2** and **3** representations that they raise and lower the value of m by one. Let us now verify that this is true for all values of j. We cannot directly evaluate $T^{\pm}|j,m\rangle$, but we can evaluate it using the commutation relations derived above.

$$T_3 T^{\pm}|j,m\rangle = (T^{\pm}T_3 \pm T^{\pm})|j,m\rangle = (m \pm 1)T^{\pm}|j,m\rangle \tag{139}$$

Essentially we have shown here that if we define a state $|a\rangle = T^{\pm}|j,m\rangle$, and evaluate the spin of this state, we obtain $T_3|a\rangle = (m \pm 1)|a\rangle$. Then the state $T^{\pm}|j,m\rangle$ is proportional to a state $|j, m \pm 1\rangle$, or equivalently

$$T^{-}|j,m\rangle = N_m|j, m-1\rangle \tag{140}$$

where N_j is simply a constant of proportionality. We can calculate the value of N_j by

$$\begin{aligned}
|N_j|^2 = \langle j, j-1|N_j N_j|j, j-1\rangle &= \langle j,j|T^{+}T^{-}|j,j\rangle \\
&= \langle j,j|[T^{+}, T^{-}]|j,j\rangle \\
&= \langle j,j|T^{3}|j,j\rangle = j
\end{aligned} \tag{141}$$

We have assumed here the states are normalised. Then $N_j = \sqrt{j}$. More generally[7] we can write

$$N_m = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)} \tag{142}$$

We can now calculate the minimum spin state for a given maximum spin j. We see from (140) and from the definition of a lowering operator as annihilating the minimum spin state that the minimum is at $N_m = 0$. From (142) we see that $N_m = 0$ at either $m = -j$ or $m = j+1$. Since $m \leq j$, we know $m = -j$ is the minimum.

Equation (142) shows that the possible values of spin in a given representation are between j and $-j$, and combining this with (139) shows that the different spin states are $\{-j, -j+1, \ldots, j-1, j\}$. This results in a $d = 2j+1$ representation of $SU(2)$, represented by $d \times d$ matrices. The basis vectors $|j,m\rangle$ are therefore column vectors of order d, which can be written as:

$$|j,j\rangle = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}, \quad |j,j-1\rangle = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \quad |j,j-2\rangle = \begin{pmatrix}0\\0\\1\\\vdots\\0\end{pmatrix}, \quad \ldots \quad |j,-j\rangle = \begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix} \tag{143}$$

The structures for $j = 1/2$ and $j = 1$ are described in detail above.

[7]This can be calculated using the same technique as in (141) for each value of $m = j - k$ and calculating the series.

D.4 The Special Unitary Group $SU(3)$ and Colour

We shall now construct the generators of $SU(3)$ using a similar logic to the one we used to construct $SU(2)$. Using this method, we can construct the generators of $SU(n)$ for any n, however since the standard model gauge group only involves $SU(2)$ and $SU(3)$, we shall limit ourselves to these cases.

The fundamental representation of $SU(2)$ was the set of 2×2 special unitary matrices. Hence the fundamental representation of $SU(3)$ is the set of 3×3 special unitary matrices; that is the group of matrices U with $\det U = 1$ and $UU^\dagger = I$. There are eight generators of $SU(3)$, since a general hermitian matrix H has 9 degrees of freedom, which is reduced to eight when we impose the constraint $\mathrm{tr}H = 0$, as the component $u_{33} = -(u_{11} + u_{22})$.

To construct our first three generators, we take advantage of the knowledge that $SU(3) \supset SU(2) \times U(1)$. We know that the generators of $SU(2)$ are included in the generators of $SU(3)$, so the first three generators of $SU(3)$ take the form

$$\left(\begin{array}{c:c} SU(2) & \\ \hdashline & \end{array}\right)$$

We write the first three generators of $SU(3)$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{144}$$

We can also extend the symmetric matrices $\lambda_{1,2}$ to four more symmetric matrices by

$$\begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & b & 0 \end{pmatrix}$$

Our remaining symmetric generators of $SU(3)$ are

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{145}$$

In total we required eight generators of $SU(3)$, so finally we must add another traceless generator. This is

$$\lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{146}$$

λ_j ($j = 1 \ldots 8$) are the Gell-Man matrices, and are the generators of $SU(3)$. For normalisation we may once again write a factor of $1/2$ before the matrices $\lambda_{1\ldots7}$ and a factor $1/2\sqrt{3}$ before the matrix λ_8. The most general form of $SU(3)$ is

$$U = e^{\vec{\alpha}\cdot\vec{\lambda}}, \qquad \alpha_i \in \mathbb{R}. \tag{147}$$

The algebra of $SU(3)$ is

$$[\lambda_i, \lambda_j] = i \sum_k f_{ijk} \lambda_k \tag{148a}$$

$$\{\lambda_i, \lambda_j\} = \frac{1}{3}\delta_{ij} + \sum_k d_{ijk} \lambda_k \tag{148b}$$

where f_{ijk} and d_{ijk} are structure constants that can be computed through use of the commutators and anticommutators respectively. Note here we have used the renormalisation process described above.

D.4.1 Colour

The primary role of the $SU(3)_C$ group of the standard model is to give quarks colour charge. Note that any state that is not a singlet of $SU(3)_C$ in Appendix A corresponds to a quark. Quarks come in one of three colours, and a hadron is formed by three quarks of different colour.

The fundamental representation of $SU(3)$ is given in (144-146). These are 3×3 matrices, so the basis vectors shall be column vectors consisting of three elements. Each basis vector corresponds to a different quark colour charge, so each quark state is an $SU(3)$ triplet. No other standard model particles have a colour charge, so all other particles are $SU(3)$ singlets.

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