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DOUBLET-TRIPLET SPLITTING IN HETEROTIC PATI-SALAM STRING MODELS

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Declaration

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Abstract

This paper seeks to be a complete introduction to String Phenomenology beginning from a well-known starting point; the standard model. Providing a short overview of how the model works in its three main sectors: interactions, matter and Higgs. From there we look at the shortcomings of the model, highlighting the issues in unification and explanation of gravity. Hence, mentioning the grand unified theories, their connection to the standard model and the problems that they aim to fix.

Noticing that the grand unified theories only solve one of the two highlighted shortcomings, we suggest a radical reshuffling of the framework to string theory. We discuss what string theory means in terms of the physics behind it, both for the bosonic and superstring cases. Then we introduce the heterotic string as a marriage between the two string cases and walk through what it means in terms of dimensionality. We look at the geometric repercussions as a result of that dimensionality in terms of the worldsheet and introduce the concept of modular invariance. We extrapolate this to the partition function to create a system using boundary conditions on the worldsheet in order to build phenomenologically useful models and then define explicit rules for these models to follow in the ABK rules.

Then it is illustrated on how to obtain phenomenological information from the models through a construction of $SO(10)$ in the form of a heterotic string model, the NAHE set. Lastly, we look at how asymmetric models are needed and how asymmetric Pati-Salam holds a relation to the doublet-triplet splitting mechanism. Finding a new constraint on how the symmetry-breaking vector can be constructed.

Acknowledgements

Before we start, I would like to extend a thank you to my supervisor and friend, Prof. Alon Faraggi, who has offered unwavering support throughout the project.

Contents

1 | Motivation

1.1 The Standard Model

Our current most robust understanding of the sub-atomic domain is described by the widely known, and respected, Standard Model of Particle Physics (SM). Using Special Relativity and Quantum Mechanics to bring together three of the four presently known fundamental forces under a singular relativistic quantum gauge theory over three generations of observed matter.

The Standard Model can be split into three distinct sectors based on the quantum number spin.

1.1.1 The Interaction Sector

The three of the four fundamental forces previously mentioned are the strong, weak and electromagnetic (note that gravity is the missing fourth). Each of these have a respective gauge group and are combined under the single representation of

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \tag{1}$$

[1].Each group in the representation corresponds to its own interaction, and hence its own spin 1 force carrying particle, a gauge boson (on occasion is referred to as a vector boson).

For $SU(3)_C$, this is the strong nuclear force, and the subscript "C" is in place to represent the colour charge. The force mediator for the strong force is the gluon, represented as G_μ^a where $a = 1, .., 8$.

$SU(2)_L$ and $U(1)_Y$ are more intimately connected through electroweak theory. $SU(2)$, corresponding to weak isospin, is mediated by the W bosons. Which are

$$W^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad W^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad W^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

in the adjoint representation of $SU(2)$ [2]. In the electroweak framework, the $SU(2)_L$ symmetry acts on left-handed fermions by arranging them into doublets. In contrast, right-handed fermions are $SU(2)_L$ singlets which do not couple to the W bosons and therefore do not interact via the weak force. Hence, the subscript "L" to show the transformation only applies to left-handed particles. It is worth noting; however, that right-handed antiparticles do participate in the weak interaction as a result of them transforming to left-handed fields. The hypercharge, $U(1)_Y$, is related to the third compenent of isospin, denoted as T_3, in the Gell-Mann-Nishijima formula:

$$Q = T_3 + \frac{1}{2}Y \tag{3}$$

where Q is electric charge [3]. Particles possessing hypercharge interact via the $U(1)_Y$ gauge boson, B, which can be likened to $U(1)_{EM}$ electromagnetic interactions but with couplings proportional to hypercharge and not electric charge.

In nature, observed interactions do not exhibit full electroweak symmetry as a result of the low energy of the interactions. Instead we observe interactions mediated by electromagnetism and the weak nuclear force, implying that at some point along the energy scale spontaneous symmetry breaking must occur. Symmetry breaking occurs through the Brout-Englert-Higgs mechanism [4], which is the mixing of the B and W^0 bosons through a rotation of the Weinberg mixing angle, θ_W.

$$\begin{pmatrix} \gamma \\ Z^0 \end{pmatrix} = \begin{pmatrix} cos(\theta_W) & sin(\theta_W) \\ -sin(\theta_W) & cos(\theta_W) \end{pmatrix} \begin{pmatrix} B \\ W^0 \end{pmatrix} \tag{4}$$

The result of the mixing, as shown above, is the photon and Z^0 boson. The mediators of the electromgnetic and weak neutral current interactions the observed interactions. The charged weak bosons, $W^{\pm}$, arise from the following

$$W^{\pm} = \frac{1}{\sqrt{2}}(W^1 \mp iW^2) \tag{5}$$

The mass relation between the Z^0 and $W^{\pm}$ bosons emerges naturally from the mechanism to be

$$m_{Z^0} = \frac{m_{W^{\pm}}}{cos(\theta_W)} \tag{6}$$

reflecting the broken symmetry structure of the electroweak interaction.

1.1.2 The Matter Sector

The matter sector is home to the fermionic matter of the standard model. Characterised by the property that they all possess half integer spin, of which all observed fermions are $spin - \frac{1}{2}$. Fermions, as previously alluded to, are chiral particles that react to the weak interaction differently depending on their left- or right-handedness. Governed by the Dirac Equation [5]:

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{7}$$

displayed in natural units where c=1. Recall that $\partial\!\!\!/ = \gamma^{\mu}\partial_{\mu}$ where γ^{μ} are the γ matrices.

$$\gamma^0 = \begin{pmatrix} I^2 & 0 \\ 0 & I^2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad where \quad i = 1,2,3 \tag{8}$$

where I is the identity matrix and σ^i are the Pauli Matrices.

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

ψ in the Dirac equation are the representative of the Fermion field expressed by a spinor.

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{10}$$

The left and right elements of the spinor can be isolated through the use of the following projection operators (projectors, for short).

$$P_L = \frac{1+\gamma^5}{2} \tag{11}$$

$$P_R = \frac{1-\gamma^5}{2} \tag{12}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ [5]. From this, left-handed quarks and fermions form $SU(2)$ doublets. Using the matter representation, we can explicitly see the constituents.

- The quark doublet:

$$Q_L = \begin{pmatrix} u_r & u_g & u_b \\ d_r & d_g & d_b \end{pmatrix} \quad in \quad (\mathbf{3}, \mathbf{2})_{\frac{1}{6}} \tag{13}$$

- Up antiquarks:

$$u_L^c = (u_r^c \quad u_g^c \quad u_b^c) \quad in \quad (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}} \tag{14}$$

- Right handed electron:

$$e_L^c \quad in \quad (\mathbf{1}, \mathbf{1})_{\mathbf{1}} \tag{15}$$

- Lepton doublet:

$$L_L = \begin{pmatrix} \nu_l \\ l \end{pmatrix} \quad in \quad (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}} \tag{16}$$

- Down antiquarks:

$$d^c = (d^c_r \quad d^c_g \quad d^c_b) \quad in \quad (\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}} \tag{17}$$

The subscripts r,g and b are the red, green and blue colour charges respectively [6].

The matter representation above provides an understanding of how fermions are organised under the aforementioned $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ gauge group of the Standard Model. This is seen clearly by letting the representation be displayed as follows:

$$(\mathbf{x}, \mathbf{y})_z \tag{18}$$

then we can attribute $\mathbf{x} = \mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}$ as being representative of $SU(3)_C$. **3** being the fundamental triplet, $\bar{\mathbf{3}}$ the conjugate triplet, and **1** the singlet of the gauge group. $\mathbf{y} = \mathbf{2}, \mathbf{1}$ as the fundamental doublet and singlet of $SU(2)_L$ respectively with the subscript **z** as the hypercharge, $U(1)_Y$ [6].

Hence, for completeness, we can express the matter sector of the Standard Model under the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ gauge group as

$$3[(\mathbf{3}, \mathbf{2})_{\frac{1}{6}} + (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}} + (\mathbf{1}, \mathbf{1})_{\mathbf{1}} + (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}} + (\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}}] \tag{19}$$

The prefactor 3 representing the three generations of chiral matter.

1.1.3 The Higgs

The facilitation of the previously mentioned Brout-Englert-Higgs mechanism is one of the primary functions of the Higgs field. Expanding on the framework of electroweak symmetry breaking, the Higgs is presented as a scalar field (meaning spin-0) that transforms as a doublet under $SU(2)_L$ with a hypercharge $Y = \frac{1}{2}$. Expressed explicitly in a spinor representation as;

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = (\mathbf{1}, \mathbf{2})_{\frac{1}{2}} \tag{20}$$

[7].Indicating to us that the field couples to both $U(1)_Y$ and $SU(2)_L$. The covariant derivative that ensures gauge invariance under these two groups is given by the formula,

$$D_\mu = \partial_\mu + igT^i W_\mu^i + i\frac{1}{2}g^i B_\mu \tag{21}$$

Noting that the second and third terms are the coupling terms to $SU(2)_L$ and $U(1)_Y$ respectively. Following this, using the above derivative, we define the Higgs Lagrangian as

$$\mathcal{L}_{Higgs} = (D^\mu)^\dagger (D_\mu) - V(\phi) \tag{22}$$

where

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \tag{23}$$

as a result of the potential being constrained by the gauge invariance and the condition that it needs to be renormalisable.λ must be greater than zero for a stable vacuum and for the spontaneous symmetry breaking to occur we require that μ^2, the mass term, is negative, Due to the requirement that there must be a non-zero vacuum expectation value [7] for the symmetry breaking to happen. The expectation value is below

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad where \quad v = \sqrt{\frac{-\mu^2}{\lambda}} \tag{24}$$

Electric charge conservation demands that a scalar field be neutral [7]. Hence, setting ϕ^0 from the spinor representation as zero ensures that electromagnetism remains unbroken in the spontaneous symmetry breaking process from $SU(2)_L \otimes U(1)_Y$ to $U(1)_{EM}$.

1.1.4 Flaws with the Standard model

Despite previously being proclaimed as the "most robust understanding" of particle physics, it is not without its problems and shortfalls. For both approaches to particle physics, those being experimental and theoretical, it is this fact that the Standard Model is incomplete that provides the most exciting of questions.

Without deviating too much from our path, there are many holes in the standard model. A few of which are outlined below:

1. Dark Matter
   There is no candidate for dark matter explicitly defined in the standard model, neutrinos are too hot and light to be acceptable as a candidate. Therefore, the standard model provides no explanation for the observations such as gravitational lensing and galactic rotation curves that require dark matter to be explained.

2. Neutrino Mass and Oscillations
   The standard model declares that neutrinos must be massless; however, experimental evidence from the likes of Super-Kamiokande show that they do have mass and oscillate [8]. This is breaking lepton flavour violation implying physics that does not agree with the standard model.

3. Matter-Antimatter Asymmetry
   The standard model does not offer an explanation for why we observe more matter than antimatter [9]. Infact, it assumes an equal amount for both. There being too little CP violation in the standard model to account for this.

4. Cosmological Problems
   The standard model assumes a fixed spacetime background, we know from observations that this is untrue [10].

There are two; however, that will be highlighted most of all in this paper. Firstly, the standard model has no explanation for gravity inside its QFT based framework.

Secondly, it is not able to provide a unification of the three forces it does describe at a single energy scale unless new physics is introduced (supersymmetry). However, this is attempted in the grand unified theory [11].

1.2 Grand Unified Theories

For us to eventually progress to heterotic string models, a short discussion on grand unified theory is in order. Abbreviated as GUT, the aim is to use what was done previously in electroweak theory, where the electromagnetic and weak forces were unified, as a precedent to extend the concept to also include the strong force. It does this by increasing the energy scale from that of electroweak theory $\approx 100 GeV$ [12] to energies typically $\approx 10^{16} GeV$ [13] .

The term is used for gauge groups that embed the standard model group inside them. Examples such as $SO(10), SU(5)$ and $SO(6) \otimes SO(4)$ will become increasingly familiar as this paper continues. These unified groups are capable of decomposing to reproduce the standard model gauge group structure through appropriate symmetry breaking, hence why they are known to be GUTs. Note; however, though GUTs solves our unification issue regarding the standard model, they do not solve the incompatibility with general relativity as these unifications still do not include gravity.

1.2.1 The $SU(5)$ Group

$SU(5)$, or qualitatively known as the Georgi-Glashow Model, was the first concrete proposal for a grand unified theory. In their paper "Unity of all Elementary-Particle Forces" Georgi and Glashow Compiled three of the four fundamental forces into a rank 4 simple Lie group, $SU(5)$ [14].

Justifying the embedding, we consider the rank and dimension of each group. Recalling some group theory knowledge, the rank of a group is defined as the number of diagonal operators, for $SU(N)$ is as follows:

$$Rank(SU(N)) = N - 1 \tag{25}$$

[15] and the dimension of the group is given by $N^2 - 1$. Conducting this for $SU(5)$:

$$SU(5): \quad R = 4, \quad D = 24 \tag{26}$$

The Standard Model gauge group is:

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \tag{27}$$

with

$$R = 4, \quad D = 12 \tag{28}$$

Hence, we can see that $SU(5)$ is smallest possible simple group that contains the standard model. As a result of the matching rank of 4 between the two groups. The differing dimensions result in an implication of 12 additional gauge bosons predicted by the $SU(5)$ model, given the name of X and Y exotic bosons [16].

Under $SU(5)$, the left handed fermionic matter of the standard model is classified under $\bar{\mathbf{5}} \oplus \mathbf{10}$.

$$\bar{\mathbf{5}} \rightarrow \begin{pmatrix} d_r^c \\ d_g^c \\ d_b^c \\ e \\ -\nu_e \end{pmatrix} \tag{29}$$

$$\mathbf{10} \rightarrow \begin{pmatrix} 0 & u_b^c & -u_g^c & u_r & d_r \\ -u_r^c & 0 & u_r^c & u_g & d_g \\ u_g^c & -u_r^c & 0 & u_b & d_b \\ -u_r & -u_g & -u_b & 0 & e^c \\ -d_r & -d_g & -d_b & -e^c & 0 \end{pmatrix} \tag{30}$$

Here, the superscript, c, represents the charge conjugate and, as mentioned previously, all fermions are left-handed [6]. Using the matter representation, we can decompose $\bar{\mathbf{5}} \oplus \mathbf{10}$ under the standard model gauge group,
$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$.

$$\begin{aligned} \bar{\mathbf{5}} &= (\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}} \\ \mathbf{10} &= (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{3}, \mathbf{2})_{\frac{1}{6}} \oplus (\mathbf{1}, \mathbf{1})_{1} \end{aligned} \tag{31}$$

For the inclusion of the right-handed equivalents, we apply the same methodology only now with the conjugate representation of $\mathbf{5} \oplus \bar{\mathbf{10}}$.

$$\begin{aligned} \mathbf{5} &= (\mathbf{3}, \mathbf{1})_{-\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{2})_{\frac{1}{2}} \\ \bar{\mathbf{10}} &= (\mathbf{3}, \mathbf{1})_{\frac{2}{3}} \oplus (\bar{\mathbf{3}}, \mathbf{2})_{-\frac{1}{6}} \oplus (\mathbf{1}, \mathbf{1})_{-1} \end{aligned} \tag{32}$$

[6]. The gauge bosons arise from the adjoint representation of the $SU(5)$ gauge group, which in matter representation decomposes as

$$\mathbf{24} = (\mathbf{8}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{3}, \mathbf{2})_{-\frac{5}{6}} \oplus (\bar{\mathbf{3}}, \mathbf{2})_{\frac{5}{6}} \tag{33}$$

The force mediators reproduce themselves here in the first three terms of the representation. Those being the gluon, W and B (hypercharge) gauge fields [6]. The remaining two are the aforementioned X and Y exotic bosons. These are predicted to facilitate the lepto-quark interactions, the implications of which will be discussed shortly.

To break $SU(5)$ to the standard model group, we have to introduce a scalar field within the adjoint Higgs sector. The vacuum expectation value of which needs to be aligned in the direction of the hypercharge generator;

$$\langle \phi \rangle \propto diag(2, 2, 2, -3, -3) \tag{34}$$

The electroweak symmetry is further broken by another Higgs field in the **5** or $\bar{\mathbf{5}}$ representation.

Challenges With $SU(5)$

Despite its elegance, $SU(5)$ clashes with observed phenomenology on a couple of occasions.

The first is that the additional bosons (X and Y) facilitate transitions between quarks and leptons, abbreviated to leptoquark interactions. These violate baryon number conservation, leading to the dreaded rapid proton decay. The X and Y bosons would mediate interactions such as

$$p \to e^+ + \pi^0 \tag{35}$$

This is highly disagreeable and stands in opposition to observed data set by experiments such as Super-Kamiokande [8]. Such experiments show the proton to be incredibly stable, with a lifetime far above what the $SU(5)$ model predicts. In fact, the proton's renowned stability (exceeding 10^{32} years [17]) is a point of bewilderment in modern particle physics at the time of writing this paper.

Secondly, $SU(5)$ does not naturally accommodate the right-handed neutrino. This is now essential in any physically viable model due to the observation of neutrino oscillations and neutrino mass. To include this, one must modify the representation by introducing a singlet requiring a larger symmetry group, such as $SO(10)$.

1.2.2 The $SO(6) \otimes SO(4)$ Group

$SO(6) \otimes SO(4)$ is known qualitatively as the Pati-Salam model [18] and is a significant feature of this paper. Introduced in 1973 by Jogesh Pati and Abdus Salam, the model is a combination of simple groups rather than a single unified group like $SU(5)$. Its semi-simple gauge structure is commonly quoted as

$$SU(4)_C \otimes SU(2)_L \otimes SU(2)_R \tag{36}$$

and allows a unification of quarks and leptons into shared multiplet despite the model not being a unified group. The structure preserves central features of the standard model while also extending it's symmetry which allows for new insights. As explored previously for $SU(5)$, the group must have a rank equal to or greater than the standard model to be considered a candidate for unification.

$$\begin{aligned} SU(3)_C \otimes SU(2)_L \otimes U(1)_Y : R = 4, \quad D = 12 \\ SU(4)_C \otimes SU(2)_L \otimes SU(2)_R : R = 5, \quad D = 14 \end{aligned} \tag{37}$$

Hence, the Pati-Salam model proves to be viable candidate for unification. A notable difference being that $SU(4)_C$ extends the colour symmetry from $SU(3)_C$ to incorporate leptons as a fourth colour charge. The left- and right-handed fermions, respectively, in the Pati-Salam model are grouped as

$$\begin{aligned} (\mathbf{4}, \mathbf{2}, \mathbf{1}) \rightarrow \begin{pmatrix} u_r & u_g & u_b & \nu \\ d_r & d_g & d_b & e \end{pmatrix}_L \\ (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \rightarrow \begin{pmatrix} u_r & u_g & u_b & \nu \\ d_r & d_g & d_b & e \end{pmatrix}_R \end{aligned} \tag{38}$$

[6] Notice the inclusion of a right-handed neutrino here, notably absent in the standard model but now required due to the observed neutrino mass, as clarified earlier [8].

Applying a group theory perspective, $SU(4) \cong SO(6)$ and $SU(2)_L \otimes SU(2)_R \cong SO(4)$. Thus, the Pati-Salam gauge structure is locally isomorphic to $SO(6) \otimes SO(4)$, first postulated by Harald Fritzsch and Peter Minkowski in 1975. This is the most useful way of expressing the Pati-Salam model in relation to the formalisms explored in this paper. This is a result of the fact that the $SO(6) \otimes SO(4)$ structure allows natural embedding in the spinorial representation of $SO(10)$.

$$\mathbf{16} = (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \tag{39}$$

[19] The Higgs sector governs the symmetry breaking, and several breaking chains exist for Pati-Salam. A typical path starts with:

$$\begin{aligned} &SU(4)_C \to SU(3)_C \otimes U(1)_{B-L} \\ &\implies SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L} \\ &\implies SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \end{aligned} \tag{40}$$

This breaking path makes it easy to see that the Pati-Salam model represents an intermediate step in the breaking of a larger gauge group to the standard model. That larger gauge group is $SO(10)$.

1.2.3 The $SO(10)$ Group

$SO(10)$ is of particular importance to us considering the previous gauge groups we have discussed. As we have just seen, the Pati-Salam model is a subgroup of $SO(10)$ and the same can be said of the Georgi-Glashow model. Thus, $SO(10)$ offers a unified framework of realistic models, and the breaking of the group becomes the backbone of the research done later in the paper.

As an orthogonal group, $SO(10)$ can organically support real representations and therefore cannot directly support complex chiral representations that the standard model demands. We resolve this by using the spinorial representation of $SO(10)$, as alluded to in the previous section, in which chiral fermions can be consistently placed.

The **16** representation of $SO(10)$ has the ability to encapsulate all standard model fermions of a single generation, shown below:

$$\mathbf{16} \rightarrow \begin{pmatrix} u_r^c & u_g^c & u_b^c & \nu_e^c & u_r & u_g & u_b & \nu_e \\ d_r^c & d_g^c & d_b^c & e^+ & d_r & d_g & d_b & e^- \end{pmatrix}_L \tag{41}$$

[20] Importantly, both the $SO(6) \otimes SO(4)$ and $SU(5)$ can be embedded into the **16** representation.

- Pati-Salam can be done directly;

$$\mathbf{16} \rightarrow (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \tag{42}$$

- $SU(5)$ requires an extension by $U(1)$ to match the rank of $SO(10)$. Hence, we embed $SU(5) \otimes U(1)$:

$$\mathbf{16} \rightarrow \mathbf{10} \oplus \bar{\mathbf{5}} \oplus 1 \tag{43}$$

  The additional singlet corresponds to the right-handed neutrino, which we pointed out was an issue with the original $SU(5)$ GUT due to the observed neutrino mass.

The gauge bosons in $SO(10)$ are included in the adjoint **45** representation.

$$\mathbf{45} \rightarrow (\mathbf{8}, \mathbf{1}, \mathbf{0}) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{0}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{0}) \oplus 33 \quad extra \tag{44}$$

As we see, upon decomposition of the **45** representation gives the three familiar standard model bosons: the gluon octet, the electroweak triplet and the hypercharge singlet. Although much like when $SU(5)$ gave us the additional X and Y gauge fields, $SO(10)$ gives us 33 additional gauge fields [6]. These are predicted to be mediators of quark and lepton transitions, leptoquarks, and are a current area of experimental searching beyond the standard model[16].

For a comprehensive discussion of the GUTs discussed here, along with their phenomenological implications, see [21] and [6].

2 | Framework of String Theory

From our previous chapter, we know that one of our key shortfalls of the standard model is its incompatibility of the model's quantum field theory framework with general relativity. Furthermore, despite attempts made by grand unified theories, it is still shown that gravity remains resistant to quantisation [22] due to Einstein's relativity being non-renormalizable at high energies. Hence, it becomes clear that a different structure is required to marry the two descriptions.

String theory offers this 'quantum gravity' structure [23]. It does this by postulating that 0-dimensional particles can be modelled as 1-dimensional strings whose vibrational modes correspond to different particles. By doing this the infinite divergences of general relativity do not appear as a result of the physics of low-energy and high-energy being astonishingly similar, and the divergences found in the short-distance singularities of point particle theories are non-existent.

To add to this string theory promotes naturalness, or physical viability, on a more powerful level than quantum field theory does. Using much stricter boundary conditions to compensate for nonphysical phenomena, e.g. tachyonic states breaking special relativity postulates by travelling faster than light. From a quantum field theory perspective, the space of quantum field theories that are considered viable is larger than those found in the low-energy quantum field theory description of string theory. This space, known as the "swampland", is in reference to quantum field theories that look consistent on their own but cannot come from any string theory, and hence cannot be embedded in a quantum gravity theory. In contrast, theories that can come from a string theory are said to be in the 'landscape.' This idea is explored much more deeply in 'The String Landscape and the Swampland' by Cumrun Vafa [24].

2.1 Physics of the String

Before we start to build models using string theory as our framework, it is important to understand the physics that describes what we are working with.

As the string is modelled to be a 1-dimensional vibrating object, thus beginning the strong connection between geometry and string theory, we can make use of the wave equation to describe its propagation as it moves through spacetime. The surface that it creates as it does this is referred to as the worldsheet, first coined in ref [25].

$$\frac{\partial^2 X^\mu}{\partial \tau^2} - \frac{\partial^2 X^\mu}{\partial \sigma^2} = 0 \tag{1}$$

x^μ represents the string, σ serves as a spatial coordinate on the worldsheet and τ serves as time-like coordinate. The partial differentiation is solved by the Fourier method, giving us the left and right movers of the string denoted by:

$$\begin{aligned} x^\mu &= x^\mu_L + x^\mu_R \\ &= x^\mu(\tau + \sigma) + x^\mu(\tau - \sigma) \end{aligned} \tag{2}$$

$x^\mu(\tau \pm \sigma)$ being left and right movers respectively.

2.1.1 The Action

The action defines the theory, telling us how the string behaves and is the starting point for deriving the equations of motion via the principle of least action. It is the symmetries of the action that lead us to physical constraints such as the requirement of extra dimensions or the presence of specific gauge symmetries.

Nambu-Goto Action

The Nambu-Goto action is the first of the two most common ways to describe the action of the string. Of the two, the Nambu-Goto action places more emphasis on the geometry of the worldsheet. It begins by the parameterisation of the string trajectory in space-time, known as the worldline [26]. As an action this is written;

$$\begin{aligned} S &= -m \int ds \\ &= -m \int d\tau \sqrt{-g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}} \\ &= \frac{1}{2} \int d\tau (\frac{1}{e} \dot{X}^2 - em^2) \end{aligned} \tag{3}$$

[27]. The first line, an integral over ds, is a summation travelling along the worldine. $\dot{X}$ is equivalent to the derivative of X with respect to the proper time, τ. The 'e' is a parameter that ensures that the parametrisation of τ has no effect on the action. Thus rendering it invariant. These parameters that serve this function are known as einbein [27].

As shown before, the string modelled in spacetime is given a proper time parameter, τ, and a spatial parameter, σ. Hence, the worldline requires a $X = X(\tau, \sigma)$ parameterisation. Resulting in

$$S = -T \int d\sigma d\tau \sqrt{-det \frac{\partial x^{\mu}}{\partial \sigma^{\alpha}} \frac{\partial x^{\nu}}{\partial \sigma^{\beta}} \eta_{\mu\nu}} \tag{4}$$

[27]. Noting the use of $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$ Minkowski metric signature, where α, β are defined as $0, 1$ respectively and the introduction of the string tension parameter, T. The entities held within the determinant inside the square root form an induced metric to which we let,

$$h_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial \sigma^{\alpha}} \frac{\partial x^{\nu}}{\partial \sigma^{\beta}} \eta_{\mu\nu} = \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \eta_{\mu\nu} \tag{5}$$

Allowing us to write the following,

$$S_{NG} = -T \int d\sigma d\tau \sqrt{-det(h_{\alpha\beta})} \tag{6}$$

recognised as the Nambu-Goto action [27]. Taking a geometric approach, the Nambu-Goto action tells us the string tries to minimise the surface area of its worldsheet which is reminiscent of how a particle's action will minimise its path length on a spacetime geodesic. Nambu-Goto action's geometric emphasis is a valuable tool as it helps to develop a visualisation of the strings motion through the worldsheet making it good for generalisations like D-brane actions (see [28]).

Polyakov Action

Despite the intuitive nature of the Nambu-Goto action there are certain calculations, namely quantisation, that demand a more malleable form of the action to work with. Enter the classically equivalent Polyakov action, achieved by plugging in the induced metric to remove the determinant present in the Nambu-Goto form and introducing an independant metric on the worldsheet

$$S_P = \frac{-T}{2} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{7}$$

[29]. Though this is classically equivalent to the Nambu-Goto action, the absence of the determinant term in the Polyakov action makes it much nicer to work with. Alongside the simplification of quantisation and path integral formulation, the Polyakov action also allows for symmetries like Weyl invariance and conformal field theories adding to the reparameterisation invariance already found in the Nambu-Goto action.

Modified Flat-Gauge Action

The conformal field theory is of particular importance both as it shows that the Polyakov action of the string can be likened to a 2D field theory and throughout. The

Polyakov action itself describes free scalar fields, $X^{\mu}(\tau, \sigma)$, on a 2D curved space. By first defining Weyl rescaling as

$$h_{\alpha\beta(\sigma,\tau)} \rightarrow e^{f(\sigma,\tau)} h_{\alpha\beta(\sigma,\tau)} \tag{8}$$

then by fixing independent components of $g_{\alpha\beta}$ it can be said that $h_{\alpha\beta(\sigma,\tau)} \rightarrow \eta_{\alpha\beta}$ [30]. Thus, it can be displayed as the following conformal gauge,

$$h_{\alpha\beta} = e^{\phi(\sigma,\tau)} \eta_{\alpha\beta} \tag{9}$$

fixing the gauge resulting in a locally flat metric. Note, this is conformally equivalent to Minkowski. Now the action is transformed to

$$S = \frac{-T}{2} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{10}$$

Which, in conformal field theory, is a theory of free scalar fields on a flat 2D space [31]. Hence the name, modified flat gauge action.

2.1.2 Quantisation

Linking back to the goals of string theory, it is essential that the theory is able to be quantised to ensure harmony with both quantum mechanics and quantum field theory. Alongside this, fitting with the concept of vibrating strings, quantisation determines the vibrational modes of the strings.

A canonical approach is a common route for the quantisation of strings and begins from the following commutation relation.

$$[X^{\mu(\sigma',\tau)}, X^{\nu(\sigma,\tau)}] = \eta^{\mu\nu} \delta(\sigma - \sigma') T^{-1} \tag{11}$$

[27]. Our aim is to obtain the 2-dimensional energy-momentum tensor, this is done by varying the action with respect to the worldsheet metric and requiring the condition

that it vanishes in the light-cone (or worldsheet) coordinates. Which are

$$X^+ = \frac{1}{\sqrt{2}}(x^0 + x')$$
$$X^- = \frac{1}{\sqrt{2}}(x^0 - x') \tag{12}$$

[29]. Hence, the previous statement is shown by

$$\frac{\partial S}{\partial g_{\alpha\beta}} \stackrel{!}{=} 0 \tag{13}$$

Where $\stackrel{!}{=}$ is synonymous with 'must equal to.' This implies

$$T_{\alpha\beta} = 0 \tag{14}$$

where $T_{\alpha\beta}$ is the energy momentum tensor [32]. The above equation gives us the consistency that we are demanding from string theory. Qualitatively, it tells us that the physics remains unchanged under any stretch or contraction of the worldsheet, a symmetry that holds both at the classical level and post quantisation.

To express components of the tensor, we require the usage of the complex worldsheet coordinates

$$z = \tau + i\sigma$$
$$\bar{z} = \tau - i\sigma \tag{15}$$

[29]. Which allow us to declare the following,

$$T_{z\bar{z}} = 0$$
$$T_{zz} = T(z) = \frac{-1}{2}\partial_z X^\mu \partial_z X_\mu \tag{16}$$
$$T_{\bar{z}z} = T(\bar{z}) = \frac{-1}{2}\partial_{\bar{z}} X^\mu \partial_{\bar{z}} X_\mu$$

Where $T_{z\bar{z}} = 0$ is indication that the energy-momentum tensor is traceless. T_{zz} and $T_{\bar{z}\bar{z}}$ are known as the two constraint operators [31].

With use of the Virasoro Algebra shown below,

$$[L_n, L_m] = (n - m)L_{m+n} \tag{17}$$

[33].T(z) can be expanded in terms of Fourier modes.

$$T(z) = \sum_n z^{(-n-2)} L_n \tag{18}$$

From the perspective of quantum mechanics, L_n and $\tilde{L_n}$ are creation and annihilation operators of the harmonic oscillator. Which is explicitly seen when looking at the form of the fourier modes below,

$$L_m = \frac{1}{2} \sum \alpha_{n-m} \cdot \alpha_n \tag{19}$$

However, in quantum theory the energy-momentum tensor, which is central to the structure of conformal field theory, is not always naturally traceless. Regardless of the fact that it is a necessary condition for conformal invariance [32]. This breakdown is a result of an anomaly present in the Virasoro algebra governing the symmetries of the worldsheet. Thus, a modification to the algebra is required to restore consistency.

$$[L_n, L_m] = (n - m)L_{m+n} + \frac{c}{12}(n(n^2 - 1))\delta_{n+m} \tag{20}$$

The central charge, c, of the virasoro algebra is synonymous with the dimensions of the target space [34]. This, in turn, is equivalent to the number of free scalar fields on the world sheet. The anomaly occurs because the Louiville mode fails to decouple under quantisation of the theory. Hence, our previous conformal invariance condition, $h_{\alpha\beta(\sigma,\tau)} \rightarrow \eta_{\alpha\beta}$ (discussed in Section 2.1.1), is now unrecoverable in the theory. Indicating that the all important conformal invariance has been broken. Thus, in order to make this anomaly vanish, we force the bosonic string to exist in 26 dimensions in order to make $c = 0$. More detail will be provided on the dimensionality of strings, both bosonic and fermionic, later on.

The previously expressed constraint equations,$T(z)$ and $T(\bar{z})$, present on the energy-momentum tensor are constructed from Fourier modes, expressed in terms of the Virasoro generators [30] . Quantum mechanically, these are interpreted as conditions that characterise physical states. To be clear, the constraint equations translate to the condition that the Virasoro operators (L_n and $\tilde{L_n}$) annihilate the physical state. Thus, with the introduction of the zero-point energy parameter a, for the right movers

$$\begin{aligned} L_n|Phys\rangle &= 0 \qquad for \quad n > 0 \\ (L_0 + a)|Phys\rangle &= 0 \\ (L_0 + \tilde{L_0})|Phys\rangle &= 0 \end{aligned} \tag{21}$$

and hence similar expression exists for the left movers

$$\begin{aligned} L_n|Phys\rangle &= 0 \qquad for \quad n > 0 \\ (L_0 - a)|Phys\rangle &= 0 \\ (L_0 - \tilde{L_0})|Phys\rangle &= 0 \end{aligned} \tag{22}$$

[27]. Though we will currently leave it undetermined, the role of the zero-point energy is crucial to the internal consistency of the theory and to address normal ordering ambiguities in the definition of the L_0 operator. Such ambiguities exist due to the nontrivial nature of commutation relations for quantum operators, placing an emphasis on the importance of the ordering of the modes. Furthermore, the zero-point energy is essential in preserving the conformal invariance at the quantum level.

The Virasoro generators have clear geometric implications for the worldsheet of the string. L_{-1} and $\tilde{L_{-1}}$ being translations with L_0 and $\tilde{L_0}$ being dilations and rotations. As these operators are responsible for infinitesimal conformal transformations, this further reinforces the fundamental role of the Virasoro algebra in conformal field theory [30].

Note that there are alternative methods of quantisation such as the Fadeev-Popov formalism which makes use of covariant path integral formalism. In this instance, we get a contribution from ghost fields. These are mathematical tools from quantum field theory that ensure consistency and unitarity of the the theory. These are used as an alternative to the c in virasoro anomaly equation, and hence for the bosonic string is

$$c_{gh} = -26 \tag{23}$$

and the total anomaly contribution is given as,

$$c_B = 1 \cdot D + (-26) \tag{24}$$

More infomation on the Fadeev-Popov method of quantisation is given in [35].

The Light-Cone Gauge

At this current stage, the string spectrum includes negative norm states. This is a direct consequence of the time-like commutation relation and the existence of residual gauge freedoms. Systematically, we can eliminate these unphysical states by considering the light-cone gauge. Defined below as

$$X_{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{(D-1)}) \tag{25}$$

[27].The choosing of the light-cone gauge aligns the gauge condition with the direction of light propagation, that is we decouple the dimensions travelling along the strings propagation and only concerning ourselves with the transverse dimensions. This fixes the remaining gauge freedoms and ensures that the resulting physical spectrum is free from negative norm states.

The use of the light-cone gauge simplifies the spectrum and, furthermore, guarantees unitarity. Despite this, it is not naturally Lorentz invariant. The invariance returns if we demand the condition that the spacetime dimension is 26 and that $a = 1$.

These conditions become essential to the consistency of the theory, as they are the result of the requirement of closure of the Lorentz algebra in quantum theory.

The gauge eliminates the unphysical degrees of freedom, isolating the transverse dimensions. This becomes particularly significant in the context of tensor structure string states, a point of importance for the closed strings we are discussing.

$$\psi^{\mu}_{\frac{1}{2}}\partial\bar{X}^{\nu}_{1}|0\rangle_{NS} \tag{26}$$

The above is an example of such a string state, the graviton to be exact [27]. Here, μ and ν correspond to components in the Lorentz group, determining the properties that govern state transformations. We can identify that it is a graviton as the tensor product implies spin-2, the key characteristic of the graviton [5]. The light-cone gauge is responsible for $\nu = 1, 2$, due to the limiting nature of the gauge decoupling from a third dimension of spacetime. Further clarity on how a string state arises is provided momentarily.

2.1.3 Supersymmetry

Although supersymmetry (SUSY) is not string theory specific, as it is not a theory that exists at a particular energy level. In the context we require, introducing SUSY here becomes worthwhile. Presenting itself as a common theme throughout, particularly on the concept of superstrings. Before that; however, we note that the bosonic string which we have been discussing demands supersymmetry. There are two reasons for this, both can be seen from the work thus far.

The first of which is that the action contains only bosonic fields, X^{μ}, shown most explicitly by the modified flat gauge action. Hence, only bosonic excitations can come from quantisation. Thus, production of space-time spin-$\frac{1}{2}$ states, fermions [5], observed in the particle content of nature is impossible.

The second is that quantising the string in the conformal gauge produces tachyons, which is unphysical. This can be seen through the use of the mass shell condition, $-M^2 = P^\mu P_\mu$ [29]. Taking our physical states, we can say that for the closed string

$$(L_0 - a)|phys\rangle = (\tilde{L_0} - a)|phys\rangle = 0 \tag{27}$$

Which we sum to be

$$(L_0 + \tilde{L_0} - 2a)|phys\rangle = 0 \tag{28}$$

Thus,

$$\begin{aligned} \frac{1}{2}\alpha_0\alpha_0 + \sum_n \alpha_{-n}\alpha_n + \frac{1}{2}\tilde{\alpha_0}\tilde{\alpha_0} + \sum_n \tilde{\alpha_{-n}}\tilde{\alpha_n} &= 2a \\ \alpha_0\alpha_0 &= -\sum_n \alpha_{-n}\alpha_n - \sum_n \tilde{\alpha_{-n}}\tilde{\alpha_n} + 2a \\ \frac{1}{T\pi}P^\mu P_\mu &= -\sum_n \alpha_{-n}\alpha_n - \sum_n \tilde{\alpha_{-n}}\tilde{\alpha_n} + 2a \end{aligned} \tag{29}$$

The summations are representative of the left and right moving excitation terms from the string vibrations, known as the level number operators, we can write them as N and $\tilde{N}$ [30].

$$\frac{1}{T\pi}M^2 = N + \tilde{N} - 2a \tag{30}$$

For a closed string we require that, $N = \tilde{N}$. Typically, to have a complete description of a string state we write it with centre of mass momentum included as

$$|0, P^\mu\rangle = ie^{-P_\mu X^\mu}|0\rangle \tag{31}$$

for a vacuum state and the following first excited state as

$$\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}1|0, P^\mu\rangle \tag{32}$$

[27]. Continuing in this way for higher excitations, but as closed string states can be said to be the tensor product of right- and left-moving states. The ground state for a

closed string, disregarding the momentum, can be shown as

$$|0_i\rangle \otimes |0_i\rangle \tag{33}$$

which is when $N = \tilde{N} = 0$. Implies that

$$\frac{1}{T\pi}M^2 = -2 \quad letting \quad a = 1 \tag{34}$$

obtaining a spin-0 tachyon [36]. This, plus the previous point, motivates the need for supersymmetry in closed bosonic string theory.

We introduce 'superpartners' for the bosons, X^μ, where each superpartner is fermionic in nature. Displayed as

$$X^\mu \to X^\mu, \psi^\mu(z, \bar{z}) \tag{35}$$

where $\psi^\mu(z, \bar{z})$ are the fermionic superpartners, accurately described as D dimensional worldsheet Majorana fermions. Expressed in its Weyl components as

$$\psi^\mu(z, \bar{z}) = \begin{pmatrix} \psi^\mu \\ \bar{\psi}^\mu \end{pmatrix} \tag{36}$$

[5]. Taking the modified flat gauge action and applying the above to include the superpartners we arrive at

$$S = \int d^2\sigma \left[\partial_z X^\mu \, \partial_{\bar{z}} X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right] \tag{37}$$

Using the gamma matrices in 2-dimensions as ρ^α. We can further this expression of the action, by use of light-cone coordinates, to

$$S = \frac{i}{\pi} \int d^2 z \, (\psi_z \partial_{\bar{z}} \psi_z + \bar{\psi}_{\bar{z}} \partial_z \bar{\psi}_{\bar{z}}) \tag{38}$$

explicitly depicting decoupled left and right degrees of freedom. We note that although this action has been adjusted for the superpartners' presence, the introduction of these fermionic fields does not instantly imply the presence of neither spacetime fermions nor spacetime supersymmetry. Only global worldsheet supersymmetry. As a result of the fields still transforming as bosons under the spacetime Lorentz group. In their current state, they are massless. Local worldsheet supersymmetry can be found from the use of the non-gauged fixed action.

The fermionic formulation also suffers from negative norm states that we seen also plagued the bosonic formulation earlier. The Fadeev-Popov formulism declares that the contribution from ghost fields in the fermionic case is

$$c_{fg} = +11 \tag{39}$$

Where D is dimension, each Majorana fermion contributes $c = \frac{1}{2}D$ [27]. Hence, the superstring gives

$$c_{total} = -26 + 11 + D + \frac{D}{2} \tag{40}$$

and again, like before, we require $c = 0$. Thus, we deduce $D = 10$, which leads to the conclusion that the superstring is consistent in 10 dimensions. Mirroring the bosonic case, we turn to the light-cone gauge to decouple the negative norm states.

Bosonic coordinates of a closed string demand the following boundary conditions,

$$X(\sigma, \tau) = X(\sigma + \pi, \tau) \tag{41}$$

[27]. In comparison, fermionic fields on the closed string obey either periodic (Ramond, R) or anti-periodic (Neveu–Schwarz, NS) boundary conditions on the basis that left and right movers require separate boundary conditions.

$$\psi_{\pm} = \pm\psi(\sigma + \pi, \tau)|^{R}_{NS} \tag{42}$$

[27]. This leads to different mode expansions for the left and right moving cases. For

the left:

$$\begin{aligned} \psi^{\mu}_{+}(\pi) &= \psi^{\mu}_{+}(0) \qquad & R: \psi^{\mu}_{+} &= \sum d^{\mu}_{n} e^{(-2in(\tau+\sigma))} \\ \psi^{\mu}_{+}(\pi) &= -\psi^{\mu}_{+}(0) \qquad & NS: \psi^{\mu}_{+} &= \sum b^{\mu}_{r} e^{(-2ir(\tau+\sigma))} \end{aligned} \tag{43}$$

and for the right:

$$\begin{aligned} \psi^{\mu}_{-}(\pi) &= \psi^{\mu}_{-}(0) \qquad & R: \psi^{\mu}_{-} &= \sum d^{\mu}_{n} e^{(-2in(\tau-\sigma))} \\ \psi^{\mu}_{-}(\pi) &= -\psi^{\mu}_{-}(0) \qquad & NS: \psi^{\mu}_{-} &= \sum b^{\mu}_{r} e^{(-2ir(\tau-\sigma))} \end{aligned} \tag{44}$$

both under the conditions that $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + 1/2$.

The equal-time anti-commutation relations for the fermionic coordinates generate the corresponding anti-commutation relations for the fermionic oscillation modes. Both for the Ramond (R) and Neveu-Schwarz (NS) sectors.

$$\begin{aligned} \{b^{\mu}_{r}, b^{\nu}_{s}\} &= \eta^{\mu\nu}\delta_{r+s} \\ \{d^{\mu}_{m}, d^{\nu}_{n}\} &= \delta_{m+n} \end{aligned} \tag{45}$$

[27] Which, if we take the NS boundary conditions, creates a unique non-degenerate Fock space. Note that the Fock space is an algebraic construction which, from a single-particle Hilbert space, creates a space of quantum states of a unknown number [37]. Unique, as the lowest energy state is

$$\begin{aligned} & b^{\mu}_{\frac{-1}{2}}|0\rangle \\ \implies & \{b^{\mu}_{\frac{-1}{2}}, b^{\nu}_{\frac{1}{2}}\} = \eta^{\mu\nu} \end{aligned} \tag{46}$$

when written as a Lorentz vector [5].

In comparison, the R boundary conditions obey Dirac algebra. Seen below,

$$\{d^{\mu}_{0}, d^{\nu}_{0}\} = \eta^{\mu\nu} \tag{47}$$

Thereby, the zero modes are proportional to Dirac gamma states. Thus, the states $d_0^\mu|0\rangle$, generate space-time fermions as a result of them being able to transform in the spinorial representation of the Lorentz group. For a single real Ramond fermion, the ground state is doubly degenerate and thus can be described by [27],

$$d_0|0\rangle \quad ; \quad |0\rangle \tag{48}$$

Worldsheet Supercurrent

When the action of the superstring is varied with respect to the 2-dimensional gravitino, the superpartner of the graviton, the resulting quantity is the supercurrent [38].

$$\begin{aligned} T_f(z) &= \psi^\mu \partial_z X_\mu(z) \\ \bar{T}_f(\bar{z}) &= \bar{\psi}^\mu \partial_{\bar{z}} X_\mu(\bar{z}) \end{aligned} \tag{49}$$

Akin to the energy-momentum tensor, the supercurrent provides insight on string behaviour in terms of its internal symmetries and spacetime with the gravitino field signifying its link to gravitational interactions. A worthy note to remember for later, the supercurrent results in constraints that play a pivotal role in the choice of γ vector when creating asymmetric basis vector models.

Modified Mass Shell Condition

The physical state condition changes under worldsheet supersymmetry from

$$(L_0 - 1)|Phys\rangle = 0 \quad to \quad (L_0 - \frac{1}{2})|Phys\rangle = 0 \tag{50}$$

reflecting the contributions of the fermionic modes to the zero-point energy [27].

2.1.4 Fermion Number

We introduce the fermion number operator, defined as the number of fermionic excitations acting on the non-degenerate NS ground state and the number of zero-mode excitations in the R sector acting on its vacua. Shown mathematically below [27],

$$(-1)^{N_{NS}} \quad and \quad (-1)^{N_R} \tag{51}$$

2.1.5 Construction of the Heterotic String

In the closed-string theory we are concerned with, the left- and right-moving modes propagate independently along the string. The decoupling of the left and right modes allows for the thought process where different symmetries can be inflicted on either sector. In particular, we can impose worldsheet supersymmetry on the left-moving sector, allowing for fermionic representation, while the right- moving sector remains bosonic. The motivation being that you can describe both fermionic and bosonic states by the same string, we call this formulation the 'heterotic string construction.'

Conformal Anomalies

As we have seen for both the bosonic string and the superstring, any consistent string theory demands that the conformal anomaly vanishes. Both sectors of the heterotic string contribute to this anomaly, and the total central charge is defined as the sum of these contributions.

$$c_{total} = c_{bg} + c_{fg} + c_{X^\mu} \cdot D + c_{\psi^\mu} \cdot D \tag{52}$$

c_{bg} and c_{fg} we have seen earlier to be the ghost fields coming from gauge fixing the path integral in Fadeev-Popov quantisation [27]. The X^μ and ψ^μ subscripts are known to be the spacetime coordinates and their fermionic superpartners where each bosonic coordinate contributes 1 and each Majorana fermion contributes $\frac{1}{2}$. Applying these and applying the vanishing condition

$$0 = -26 + 11 + D + \frac{D}{2} \tag{53}$$

Dimensionality

From this we are able to retrieve the dimensionality of the left and right sectors individually. The left superstring sector, which contains both the bosons and their fermionic superpartners, will contain both the X^μ and ψ^μ contributions.

$$C_L = -26 + 11 + D + \frac{D}{2} = 0 \implies 10 = D \tag{54}$$

Showing the 10 dimensions of the superstring, as previously found.

For the right-moving bosonic sector, we discount the fermionic superpartners, only containing the X^μ contributions.

$$C_R = -26 + D \implies 26 = D \tag{55}$$

Retrieving the 26 dimensions of the bosonic string. However, for consistency, the discrepancy between the left and right sectors must be resolved. To match both sides in 10 dimensions, we compactify the bosonic sector onto a 16 dimensional flat torus. Thus, we begin the importance of modular invariance, a common theme for us going forward to ensure consistency. The compactified dimensions result in background fields (referred to as vertex operators) that create a space-time current under $U(1)$ symmetry [31]. Furthermore, we note that there are only two gauge groups that allow this compactification without reintroducing anomalies, $SO(32)$ and $E_8 x E_8$. These gauge groups define the two consistent heterotic string theories commonly referred to as the HO and HE models, respectively [39].

Introduction to the Moduli

The phenomenological goals strongly tied to string theory demand that we go further, taking our now completely 10-dimensional torus and compactifying it by 6 additional dimensions to obtain a phenomenologically viable 4-dimensional theory. This must be done on a Calabi-Yau manifold [27] (see Figure 2.1) to preserve the $\mathcal{N} = 1$ space-time supersymmetry found in the standard model [40]. This results in effective gauge groups like E_6 in 4 dimensions, which are of particular interest to those studying GUTs.

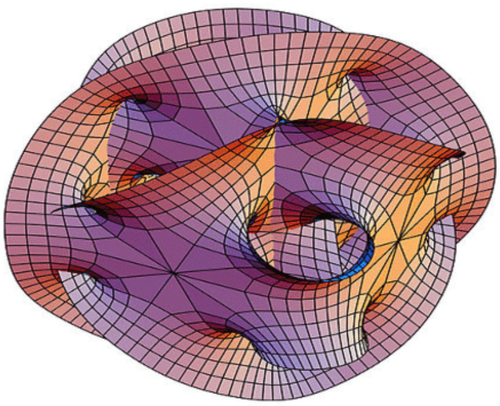

Figure 2.1: A 3D projection of a Calabi-Yau manifold. [41]

There is an alternative to this path in the Narain construction, in which all bosonic coordinates are compactified uniformly on a flat torus defined by an even, self-dual Lorentzian lattice [42]. This method allows for a wider variety of gauge groups in the phenomenologically viable 4 dimensions. However, unlike our current formulation, the Narain construction exists in $\mathcal{N} = 4$ supersymmetry. Hence, a reduction to $\mathcal{N} = 1$ supersymmetry is required to allow for chiral matter. In this case, this is carried out through orbifolding techniques (explored further in [43] and [44]) which will not be covered because they are not especially relevant to our discussions primarily concerned with the free-fermionic approach.

Even so, regardless of the geometric-based Calabi-Yau compactification of the free-fermionic approach or the lattice-based compactification of the Narain construction, both introduce the set of moduli parameters. Determining the size and shape of the six-dimensional compactified space. The moduli's presence, regardless of formulation, decrees that any realistic string model must address moduli stabilisation. A moduli that is not fixed allows for continuous changes in gauge structure, resulting in a model that is not predictive, defeating the phenomenological goals of string theory.

Free-Fermionic Construction

Continuing with our formulation, as previously mentioned, we will progress exclusively with the free-fermionic construction. In comparison to the traditional bosonic formulation, the free-fermionic approach changes the interpretation of the additional degrees of freedom required to cancel the conformal anomaly. We replace the idea in bosonic string theory, where these additional degrees of freedom are spacetime dimensions, with the interpretation that the additional degrees of freedom are free fermions propagating on the worldsheet. This new reasoning gives us the ability to directly formulate the theory in the phenomenologically relevant 4 dimensional spacetime. Pioneered by two groups independently: KLT (Kawai, Lewellen, and Tye) [45] and ABK (Antoniadis, Bachas, and Kounnas) [46].

For the heterotic string, conformal invariance imposes the following for the left- and right-moving central charges.

$$\begin{aligned} C_L &= -26 + 11 + D + \frac{D}{2} + N_{f_L} \cdot \frac{1}{2} = 0 \\ C_R &= -26 + D + N_{f_R} \cdot \frac{1}{2} = 0 \end{aligned} \tag{56}$$

where N_{f_L} and N_{f_R} are the number of left- and right-moving real worldsheet fermions, respectively, and the -26 and 11 are from the Fadeev-Popov ghost fields [27]. As previously alluded, a key characteristic of the free-fermionic formulation is that the theory is constructed in 4 dimensions. Hence, we set $D = 4$. Application of the anomaly cancellation condition yields,

$$\begin{aligned} C_L &= -26 + 11 + 4 + \frac{4}{2} + N_{f_L} \cdot \frac{1}{2} = 0 \implies N_{f_L} = 18 \\ C_R &= -26 + 4 + N_{f_R} \cdot \frac{1}{2} = 0 \implies N_{f_R} = 44 \end{aligned} \tag{57}$$

18 left-moving real fermions and 44 right-moving real fermions. We determine that the left movers must be Majorana fermions, its own antiparticle. We deduce this because the left-moving sector, if we recall, is bosonic and Majorana fermions are intrinsically massive, not observable massless fields. A condition required for the left-moving sector. Secondly, we determine the right movers to be Majorana-Weyl fermions, as they are real, chiral and massless (observable) [27] . Recalling that the right moving sector is the superstring component of the string, we require these properties of the right movers as the massless nature is required for gauge bosons and gauge symmetry in 4 dimensions.

Boson-Fermion Equivalence

We must pay attention to the reasoning that the bosonic and fermionic constructions are, in fact, equivalent [47]. Such a relation arises from the equivalence of bosons and fermions explicitly in 2 dimensions. In which bosonic operators, $e^{\pm iX}$, can be expressed in terms of fermionic fields, $y \pm i\omega$.

$$e^{\pm iX} = y \pm i\omega \tag{58}$$

Interestingly, these allow us to transform back to the bosonic formulation. Using the vertex operators, these combinations retrieve a similar conformal weight under Operator Product Expansions (OPE) [31]. This results in the conclusion that, in moduli space, they exhibit equivalence at specific points. Moving away from these fixed points in moduli space while preserving conformal invariance is a topic of current research and, therefore, is not a well-developed area. So, in general, we consider them equivalent [27].

Worldsheet Content

In the light-cone gauge, we remind ourselves that the fields are restricted to the two transverse spacetime dimensions. Displaying this by letting the index $\mu = 1, 2$. We have the following left moving components on the worldsheet,

$$\begin{aligned} &X^{\mu}_{L}(z) \quad ; \quad transverse \quad bosons \\ &\psi^{\mu}_{L}(z) \quad ; \quad fermionic \quad superpartners \\ &\chi_i y_i \omega_i \quad ; \quad 18 \quad internal \quad fermions \end{aligned} \tag{59}$$

and the corresponding right movers,

$$\begin{aligned} &X^{\mu}_{R}(\bar{z}) \quad ; \quad transverse \quad bosons \\ &\phi^{1,..,44}_{a} \quad ; \quad 44 \quad internal \quad real \quad fermions \end{aligned} \tag{60}$$

Thus, we determine the total number of real fermions being 64. This, we will see, determines the size of the basis vectors used in specifying boundary conditions and constructing models

2.2 Worldsheet Dynamics

2.2.1 Worldsheet Topology

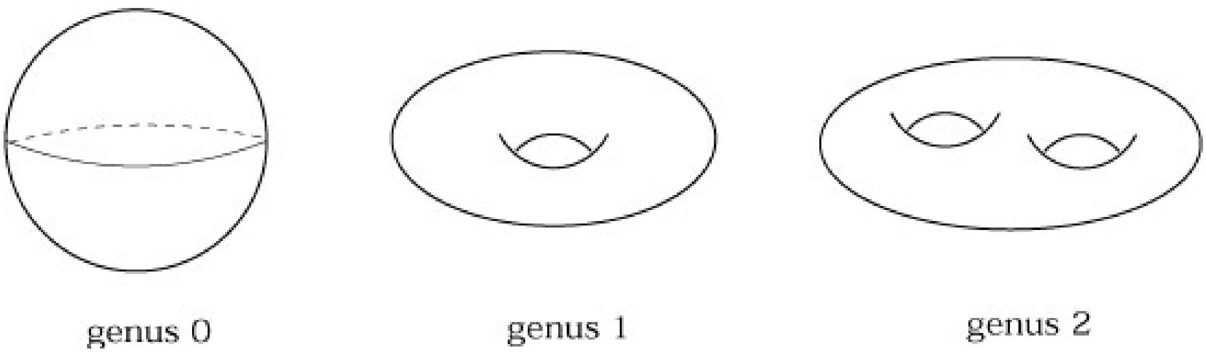

Figure 2.2: Riemann surfaces of different genus. [48]

As the free fermions propagate on the worldsheet, the surface produced is a genus-g-Riemann surface [49]. A Riemann surface is a one-complex-dimensional, two-real-dimensional, smooth manifold and the genus counts how many "holes" or "handles" are in the surface [50], see Figure 2.2. The string worldsheet, at different loop orders, corresponds to such a surface. At tree level (g = o) we have a sphere, at one loop level (g=1) we have a torus, two loops (g = 2) produce a double torus, and so on. As the genus increases, the surface becomes more complex and requires higher-order corrections in the string path integral in the Polyakov picture. The Polyakov picture describes the behaviour of a string by taking a path integral over all possible embeddings of a string worldsheet in spacetime.

$$A_n = \sum_{g=0}^{\infty} \int \mathcal{D}h\mathcal{D}X^{\mu} \int dz_1...d^2z_n V_1(z_1, \bar{z}_1)...V_n(z_n, \bar{z}_n) \tag{61}$$

Specifically, summing over the amplitude,A_n , for a process involving n external string states where each of these terms is a path integral over a Riemann surface of genus-g [27].

$$A_n = \sum_{g=0}^{\infty} A_n^{(g)} \tag{62}$$

Conformal invariance allows for the physical string states to be inserted as vertex operators. In terms of the Riemann surface, these are operators $V(z, \bar{z})$, which correspond to the creation or annihilation of string states. Thus, allowing us to map the worldsheet to the Riemann surface. The mapping makes it far easier to mathematically describe the worldsheet, as the description of a Riemann sphere/torus is better understood than trying to describe the form of the worldsheet.

When conducting the path integral, we only integrate over physically inequivalent paths, paths that are not related by coordinate changes (diffeomorphisms) or Weyl rescaling. That is, we only integrate over the full moduli space. Integrating over physically equivalent paths will lead to the overcount of identical physical states in the partition function [27]. In our scenario, the partition

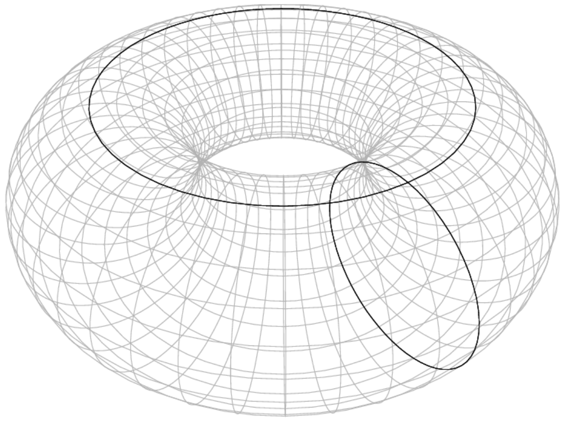

Figure 2.3: A torus with the two non-contractable loops shown in bold. [51]

function is defined to be the one-loop (torus) vacuum-to-vacuum amplitude.

On a torus, there are two non-contractable loops, as shown in Figure 2.3. As discussed before, a fermion field can be periodic (Ramond) or anti-periodic (Neveu-Schwarz). The free fermions can propagate around these loops, picking up a phase as they do so in accordance with their periodicity.

$$f \to -e^{-i\pi\alpha(f)} f \tag{63}$$

f being the fermion and $\alpha(f)$ is the phase [27]. This forms the boundary conditions placed on the worldsheet fermions. Such boundary conditions and periodicity will be explored more deeply in due course in relation to spin structures.

2.2.2 The Partition Function

At the tree level g = 0, all reparametrisations are local [49]. However, in considering higher orders, further constraints arise. These constraints, required for worldsheet consistency, can be uncovered through the torus level g=1 amplitudes in the absence of external states. An analysis not dissimilar from techniques found in quantum statistical mechanics (see [52]), the partition function is computed by considering the periodic temperature.

Taking the worldsheet time coordinate as compactified on a circle, introducing it as a complex parameter, τ.

$$\tau = i\beta - \frac{\theta}{2\pi} \tag{64}$$

The partition function, $Z(\beta,\theta)$, is found by adding all physical states, $|s\rangle$, in the Hilbert space, $\mathcal{H}$. Followed by an integration over the inequivalent tori.

$$Z(\beta,\theta) = \sum_{s\in\mathcal{H}} \langle s|e^{i\theta p}e^{-2\pi\beta H}|s\rangle = Tr_{\mathcal{H}}e^{i\theta p}e^{-2\pi\beta H} \tag{65}$$

Where H is worldsheet Hamiltonian and P is worldsheet momentum operator [27]. Note that the first exponential is spatial propagation and the second is temporal.

In this case, these operators are related to the Virasoro generators (specifically the zero modes of the energy-momentum tensor).

$$\begin{aligned} H &= L_0 + \bar{L_0} - \frac{1}{24} \\ P &= L_0 + \bar{L_0} \end{aligned} \tag{66}$$

[30]. The $\frac{1}{24}$ appears as a normal ordering constant. Hence, by substituting for the complex parameter, τ, and defining $q = e^{2\pi i\tau}$ we can express the partition function as

$$Z(\tau) = q^{\frac{-1}{48}}\bar{q}^{\frac{-1}{48}}Tr_{\mathcal{H}}(q^{L_0}\bar{q}^{\bar{L}_0}) \tag{67}$$

This form depicts the partition function being a function on the moduli space, useful as modular invariance will place constraints on the theory.

In the free-fermionic construction, the complete Hilbert space includes contributions from worldsheet fermions. Fermions whose boundary conditions must be specified. As alluded to in the previous section, fermions can be either periodic (Ramond) or anti-periodic (Neveu-Schwarz) on the two independent non-contractable loops present on the torus and as a result pick up a phase. The choice, of anti-periodic or periodic, defines the spin structure.

For any given fermion there exist four possible combinations,

- NS-NS : Anti-periodic in both directions.
- R-R : Periodic in both directions.
- R-NS : Periodic in time, anti-periodic in space.
- NS-R : Anti-periodic in time, periodic in space.

Each choice contributes differently to the partition function, where the total fermionic contribution to the partition function is written as

$$Z_f(\tau) = \prod_{i=1}^{64} Z_i \begin{bmatrix} \theta \\ \beta \end{bmatrix} (\tau) \tag{68}$$

where the 64 components come from our earlier derivations showing we need 18 left-moving and 44 right-moving, and 2 from spacetime transverse fermions for the free-fermionic model [27]. Each Z_i depends on the spin structure for that specific fermion. The components of $\begin{bmatrix} \theta \\ \beta \end{bmatrix}$ are representative of the periodicity choices, β for time and θ for spatial.

Explicitly, these spin structure contributions are written as the following,

- NS-NS :

$$Z_{NS}^{NS}(\tau) = Tr_{NS}(q^{L_0 - \frac{1}{48}}) \tag{69}$$

- R-R :

$$Z_R^R(\tau) = Tr_R((-1)^F q^{L_0 - \frac{1}{48}}) \tag{70}$$

- R-NS :

$$Z_{NS}^R(\tau) = Tr_R(q^{L_0 - \frac{1}{48}}) \tag{71}$$

- NS-R :

$$Z_R^{NS}(\tau) = Tr_{NS}((-1)^F q^{L_0 - \frac{1}{48}}) \tag{72}$$

Discussion of spin structures will continue in their own section.

2.2.3 Modular Invariance

To understand the important concept of modular invariance, we first note that in the heterotic string framework, or any perturbative string theory model for that matter, the one-loop amplitude is found by examining the string path integral over the worldsheet. As explored earlier, at the one-loop level, the worldsheet creates a Riemann surface of genus-1 (the torus).

Mapping the torus to the complex plane by cutting it along the two non-contractable loops and letting the coordinates (σ_1, σ_2) parametrize the worldsheet with periodicities λ_1, λ_2, respectively. We are able to define a complex parameter,

$$z = \sigma_1 + i\sigma_2 \tag{73}$$

allowing us to specify the torus on the complex plane, $T^2 = \mathbb{C}/\Lambda$ [53], by

$$z \Longleftrightarrow z + n_1\lambda_1 + n_2\lambda_2 \quad where \quad n_i \in z \tag{74}$$

To classify inequivalent tori we define the modular parameter, or complex structure, as

$$\tau = \frac{\lambda_2}{\lambda_1} \quad with \quad Im(\tau) > 0 \tag{75}$$

which embodies the complex parameter and the two real degrees of freedom inherent to the geometry of the torus [54]. However, in this instance, there are still tori considered conformally equivalent able to be obtained. Such an equivalence is found if their complex structures can be related through a transformation in the modular group, $SL(2,\mathbb{Z})$ [28]. Suppose we don't know the intricacies of $SL(2,\mathbb{Z})$, then the group explicitly acts as

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \quad where \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}) \tag{76}$$

Where the generators of the group are,

$$\begin{aligned} \tau &\rightarrow \tau + 1 \\ \tau &\rightarrow \frac{-1}{\tau} \end{aligned} \tag{77}$$

The transformations describe the reshaping or rotating of the torus while leaving its physical properties unchanged and hence, the physics remains the same [55]. This is what the term modular invariance describes. This invariance is essential for the consistency of string theory, ensuring the independence from the choice of coordinate parametrisation of the worldsheet for the physical amplitudes and avoiding over-counting of physically equivalent states. In reference to this, the space of inequivalent tori is given by the fundamental domain of the group. Shown as,

$$\mathcal{F} \equiv \{\tau \in \mathbb{C} : \quad |\tau| \geq 1, \quad \frac{-1}{2} \leq |Re(\tau)| < \frac{1}{2}, \quad Im(\tau) > 0\} \tag{78}$$

[30] where the contribution from all inequivalent tori is given by the integration over this domain and the requirement that the partition function is invariant under modular transformations. Which is calculated as the following,

$$Z = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{|Im(\tau)|^2} Z_B^2(\tau, \bar{\tau}) \sum_{spin-structure} C \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \prod_{f=1}^{64} Z_f \begin{bmatrix} \theta \\ \beta \end{bmatrix} (\tau) \tag{79}$$

where $Z_B^2(\tau, \bar{\tau})$ is the bosonic contribution, given as

$$Z_B(\tau) = \frac{1}{\sqrt{Im(\tau)}|\eta(\tau)|} \tag{80}$$

and $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{81}$$

where q is our previously defined, $q = e^{2\pi i \tau}$ [56]. The rest of the elements we have discussed previously, apart from $C \begin{bmatrix} \theta \\ \beta \end{bmatrix}$ which are phase coefficients chosen to preserve modular invariance. Despite its intimidating appearance, we will come to see that the partition function can be reduced to a more simple definition: the sum over all the string states, both massless and massive, that we have to take into account when propagating the string around the closed loop.

2.2.4 Spin Structures

Its important to notice that modular invariance places constraints on the fermionic sectors of the theory. Every real worldsheet fermion must be given boundary conditions on the torus. Such boundary conditions define the spin structures of the fermion. We remind ourselves of the phase picked up by the fermions when they propagate around the two non-contractable loops of the torus,

$$f \rightarrow -e^{i\pi\alpha(f)} f \tag{82}$$

where we will now define that an Ramond propagation gives $\alpha(f) = 1$ and a Neveu-Schwarz propagation gives $\alpha(f) = 0$ [27]. Hence, there are four possible combinations of these conditions in correspondence with what we have seen in the partition function section, represented by the vectors:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{83}$$

These define the transformation properties of real fermions on the torus, and are known as the spin structures. In general, two real fermions can be combined to form a complex fermion.

$$f = \frac{1}{\sqrt{2}} f_1 + i f_2 \tag{84}$$

and a corresponding representation for the $\bar{f}$ case. In this instance, the boundary conditions have the ability to be complex

$$f \rightarrow -e^{-i\pi\alpha(f)} f \quad where \quad \alpha(f) \in (-1, 1] \tag{85}$$

[27]. However, for the purposes of our analysis we shall restrict oursleves to real fermions. The full one-loop partition function for the theory is given by:

$$Z = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{|Im(\tau)|^2} Z_B^2(\tau, \bar{\tau}) \sum_{spin-structure} C \begin{bmatrix} a \\ b \end{bmatrix} Z_{long, \frac{3}{2}} \begin{bmatrix} a_\psi \\ b_\psi \end{bmatrix} \prod_{f=1}^{64} Z_f \begin{bmatrix} \theta(f) \\ \beta(f) \end{bmatrix} (\tau) \tag{86}$$

Notice this is different from our previous partition function expression in equation (79). $C \begin{bmatrix} a \\ b \end{bmatrix}$ are phase coefficients associated with the spin structure $(a.b)$ where $a, b = 0$ or 1. $Z_{long, \frac{3}{2}}$ accounts for the contribution of longitudinal gravitino modes with spin-3/2. The other elements coincide with those present in our previous expression.

The conditions are formulated from the previously expressed generators of the $SL(2, \mathbb{Z})$ group,

$$\begin{aligned} \tau &\rightarrow \tau + 1 \quad (T - Channel) \\ \tau &\rightarrow \frac{-1}{\tau} \quad (S - Channel) \end{aligned} \tag{87}$$

demanding that the partition function is now invariant under these modular transformations. We note that both the measure $\frac{d\tau d\bar{\tau}}{|Im(\tau)|^2}$ and the bosonic contribution Z_B are modular invariant by nature. This can be seen by applying the modular transformations. Each spin structure corresponds to a partition function that sums the relevant fermionic states. For example, for the NS-NS spin structure,

$$Z\begin{bmatrix}0\\0\end{bmatrix} = C\begin{bmatrix}0\\0\end{bmatrix} Tr(q^{H_{NS}}) = C\begin{bmatrix}0\\0\end{bmatrix} q^{\frac{-1}{24}} \prod_{n=1}^{\infty}(1+q^{1-\frac{1}{2}}) \tag{88}$$

[57]. Interestingly, one can liken this to the grand partition function of a one-dimensional ideal Fermi gas with $E_r = r$ energy levels.

In order to express these partition functions in terms of modular functions, we require the ϑ function. This is defined as

$$\vartheta_3(\tau) = \eta(\tau) q^{\frac{-1}{24}} \prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}})^2 \tag{89}$$

by this it can be shown that the NS-NS spin structure can be displayed in a simplified form.

$$Z\begin{bmatrix}0\\0\end{bmatrix} = \frac{\vartheta_3^{\frac{1}{2}}(\tau)}{\eta^{\frac{1}{2}}(\tau)} = Tr(e^{i\tau H_{NS}}) \tag{90}$$

For completeness sake, the same formalism can be used for the other spin structures.

$$\begin{aligned} Z\begin{bmatrix}0\\1\end{bmatrix} &= \frac{\vartheta_4^{\frac{1}{2}}(\tau)}{\eta^{\frac{1}{2}}(\tau)} = Tr((-1)^F e^{i\tau H_{NS}}) \\ Z\begin{bmatrix}1\\0\end{bmatrix} &= \frac{\vartheta_2^{\frac{1}{2}}(\tau)}{\eta^{\frac{1}{2}}(\tau)} = Tr(e^{i\tau H_R}) \\ Z\begin{bmatrix}1\\1\end{bmatrix} &= \frac{\vartheta_1^{\frac{1}{2}}(\tau)}{\eta^{\frac{1}{2}}(\tau)} = Tr((-1)^F e^{i\tau H_R}) \end{aligned} \tag{91}$$

Hence, it is shown that the fermionic partition function can be reduced to a product of the right and left moving ϑ functions, reflecting the spin structure of each sector.

2.2.5 Prelude to the ABK Rules

We will now begin to see where ABK rules [46] begin to arise, and thus their deep connection to modular invariance. These rules will become invaluable as we go on through basis vector models later on. Without the ABK rules, we would not have the basis vector model we will be evaluating.

We start with analysing how the T and S channel modular transformations affect the spin structures of the worldsheet fermions.

The S-channel, $\tau \rightarrow \frac{-1}{\tau}$, changes the torus cycles, $\lambda_1 \leftrightarrow \lambda_2$.

Using the modular transformation properties of the S- and T- channels on the spin structures, which can be found through Poisson resummation [58], we are left with the following.

$$
\begin{aligned}
&T-channel:\\
&\qquad \eta \rightarrow e^{i\pi/12}\eta\\
&\qquad \vartheta_1 \rightarrow e^{i\pi/4}\vartheta_1\\
&\qquad \vartheta_2 \rightarrow e^{i\pi/4}\vartheta_2\\
&\qquad \vartheta_3 \leftrightarrow \vartheta_4
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
&S-channel:\\
&\qquad \eta \rightarrow (-i\tau)^{1/2}\eta\\
&\qquad \frac{\vartheta_1}{\eta} \rightarrow e^{-i\pi/2}\frac{\vartheta_1}{\eta}\\
&\qquad \frac{\vartheta_2}{\eta} \leftrightarrow \frac{\vartheta_4}{\eta}\\
&\qquad \frac{\vartheta_3}{\eta} \rightarrow \frac{\vartheta_3}{\eta}
\end{aligned}
\tag{93}
$$

Again, we reinforce that the prevailing insight is that the the fermionic partition function is the product of the spin structures of all 64 fermions. Modular transformations are effectively a mapping of one spin sector (ϑ function product) to another. Therefore, since we are requiring modular invariance of the total partition function, this demands that each pair of spin structures connected through one of these transformations must provide an equal contribution to the overall partition function. Thus, the one-loop fermionic partition function is shown as the sum

$$\sum_{Spin-Structures} C \begin{bmatrix} \overrightarrow{\alpha} \\ \overrightarrow{\beta} \end{bmatrix} Z \begin{bmatrix} \overrightarrow{\alpha} \\ \overrightarrow{\beta} \end{bmatrix} \tag{94}$$

where α, β contain the boundary conditions [27]. Such boundary conditions are represented for $\overrightarrow{\alpha}$ as

$$\alpha = \{\alpha(\psi^{\mu}), ..., \alpha(\omega^{6}) | \alpha(\bar{y}^{1}, ..., \alpha(\bar{\phi}^{8})\} \tag{95}$$

Where each $\alpha(f)$ specifies the boundary condition for the corresponding fermion. This organises the spin structure according to the right- and left-moving fermions, left of the pipe representing left movers, and visa versa. This captures the behaviour around one of the non-contractable loops of the torus and a similar description exists for the β element corresponding to the second loop.

We can include an example for the permitting of spin structures in modular invariance. Deploying the T-Channel and letting $\alpha = 0$ and $\beta = 1$,

$$\vartheta_3^{\frac{1}{2}} \to \vartheta_4^{\frac{1}{2}} \implies Z \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to Z \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{96}$$

obtaining a corresponding result for the other spin structures with the different viable combinations of α and β values.

However, we note that $\vartheta_1 = 0$ by definition, as a result of the spin structure associated being naturally modular invariant under the transformations. The transformation behaviour for the right-moving sector will have similar expressions by taking the complex conjugates of the left moving sector's ϑ functions.

Under the $\alpha = 1$ and $\beta = 0$ condition where,

$$\vartheta_2^{\frac{1}{2}} \rightarrow e^{i\pi/8 \sum_f \alpha_f} \vartheta_2^{\frac{1}{2}} \implies Z \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow e^{i\pi/8} Z \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{97}$$

The phase factor in the exponentional comes from the number of fermions, l, which have the corresponding boundary conditions, α_f.

To preserve modular invariance under the full modular group, the coefficients C are required to satisfy the consistency condition

$$C \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = -exp(\frac{i\pi}{8} \sum \alpha_f) C \begin{bmatrix} \alpha \\ \alpha + \beta + 1 \end{bmatrix} \quad where \quad \sum_f \alpha_f = 0 \quad mod \quad 8 \tag{98}$$

[46]. The negative sign is accounting for the contributions from the fermion zero modes and the product over the η functions present in the partition function. Furthermore, the condition on $\sum \alpha_f$ is in relation to the cancellation of the anomalies in the theory.

Our ϑ_1 function can then be explicitly expressed as follows.

$$\vartheta_1(Z, \tau) = 2e^{\frac{i\pi\tau}{4}} sin(\pi z) \prod_{n=1}^{\infty} f(\tau, z, n) \tag{99}$$

ϑ_1 is important it provides a measure to define which spin structures contribute to the path integral and ensures that the modular sum over spin structures is well defined. In context to later on, it reflects the need for GSO projections.

When transforming the spin structures under the S-channel, such as the following;

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}^m : (\frac{\vartheta_4^{\frac{1}{2}}}{\eta^{\frac{1}{2}}})^m \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}^m : (\frac{\vartheta_2^{\frac{1}{2}}}{\eta^{\frac{1}{2}}})^m \tag{100}$$

We must place further conditions on the coefficients, C, in order to preserve the modular invariance. This time, we require,

$$C\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = exp(\frac{i\pi}{4}\sum \alpha_f \cdot \beta_f)C\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \quad where \quad \alpha_f \cdot \beta_f = 0 \quad mod \quad 4 \tag{101}$$

[46]. Here $\alpha_f \cdot \beta_f$ is denoting a Lorenzian product, which is simply as below.

$$\sum_{left} \alpha_f \cdot \beta_f - \sum_{right} \alpha_f \cdot \beta_f \tag{102}$$

Collectedly, these constraints define the constraints that ensure the modular invariance at one-loop, otherwise known as the ABK rules.

From this, the entirety of the partition function is able to be described by boundary condition basis vectors in the form of

$$b = \{\overrightarrow{b}_1, \overrightarrow{b}_2, ..., \overrightarrow{b}_n\} \tag{103}$$

where the terms contained within the partition function are of the form

$$Z\begin{pmatrix}\overrightarrow{\alpha}\\ \overrightarrow{\beta}\end{pmatrix} \tag{104}$$

where each of the components are

$$\overrightarrow{\alpha} = \alpha_1 \overrightarrow{b}_1 + \alpha_2 \overrightarrow{b}_2 + ... + \alpha_n \overrightarrow{b}_n \tag{105}$$

and similar for β. Here, the only coefficients possible are $\alpha_i, \beta_i \in \{0, 1\}$ when assuming only the NS and R sectors. The coefficients, $C\begin{bmatrix}b_i\\ b_j\end{bmatrix}$, will all be subject to consistency constraints as a result of modular invariance, constraints set by the ABK rules [59].

This formalism will be used to construct our models.

2.2.6 Constraints on the Supercurrent

Lastly, and of particular interest to us in building asymmetric Pati-Salam models, consistency conditions must also be imposed on the supercurrent for the preservation of worldsheet supersymmetry. In particular, the supercurrent T_F must be globally defined under the transformation of worldsheet fermions in accordance with the specified boundary conditions of each sector (up to a specific sign) [27].

Recall the worldsheet supercurrent, in the case of the heterotic string,

$$T_F = \psi^{\mu}\partial X_{\mu} \quad where \quad \mu = 1, ..., 8 \tag{106}$$

However, in the fermionic case, internal degrees of freedom are introduced.

$$T_F = \psi^{\mu}\partial X_{\mu} + f_{abc}\psi^{a}\psi^{b}\psi^{c} \tag{107}$$

f_{abc} are structure constants part of a semi-simple Lie algebra, corresponding to an internal symmetry. The said internal symmetry group is taken to be $SU(2)^6$ and the 18 internal fermions from the left-moving worldsheet content (equation (59)) are grouped into 6 representations in the adjoint representation of $SU(2)$ [27]. The supercurrent can then be said to take the form,

$$T_F = \psi^{\mu}\partial X_{\mu} + \sum_{i=1}^{6} \chi_i y_i w_i \tag{108}$$

where each $\chi_i y_i w_i$ product correspond to one of the 6 representations of $SU(2)$. Our requirement for the supercurrent to be well defined under modular transformations demands that each product, $\chi_i y_i w_i$, pick up the same phase as a spacetime component, $\psi^{\mu}\partial X_{\mu}$.

The transformation property of the product under the worldsheet translations is dependent on the parity of the product assigned to the NS sector. Mathematically, when $\psi^{\mu} \rightarrow -\psi^{\mu}$ then $\chi_i y_i w_i \rightarrow -\chi_i y_i w_i$. This implies that in the NS sector, the boundary conditions must contain an odd number of 0s, resulting in an overall negative contribution as a 0 is negative in the NS sector. Thus, valid boundary condition combinations for (χ, y, w) are

$$(\chi, y, w) : (1, 1, 0), (1, 0, 1), (0, 1, 1), (0, 0, 0) \tag{109}$$

[27]. In the opposite respect, when $\chi_i y_i w_i \to \chi_i y_i w_i$, we require an even number of 0 in the boundary conditions. These restrictions ensure the consistency and global well-definedness of the supercurrent on the worldsheet and will become relevant when constructing basis vectors for the basis vector models we will be discussing.

2.2.7 U(1) Charges and Fermion Number

As a concluding point before we progress onto the actual model building, we note that every complex fermion generates a worldsheet current under a U(1) charge. This produces Cartan generators of the 4 dimensional gauge group, the U(1) charge with respect to these are given by

$$Q(f) = \frac{1}{2}\alpha(f) + F(f) \tag{110}$$

where $F(f)$ is fermion number and $\alpha(f)$ is the complex fermion boundary conditions [60].

In the case where the fermion is in a non-degenerate vacuum, the fermion number assigned is $F(f) = +1$ and $F(f^*) = -1$. Comparatively, the Ramond sector with its degenerative ground state, the fermion number assignment is

$$F : |+\rangle = 0|0\rangle, \quad F : |-\rangle = -1|+\rangle \tag{111}$$

The above formulation makes a connection between the string boundary conditions directly to gauge charges, which becomes invaluable when we consider the phenomenological processes in string theory. The degenerate nature of the Ramond sector will be explored shortly and thus, this concludes our chapter on the physical foundations on which we build basis vector models.

3 | Rules of Heterotic String Models

In the previous chapter, we have worked through the physics of string theory to arrive at a point where we have a formalism to construct what are known as basis vector models. These models are built on two constituents: A set of basis vectors containing the boundary conditions following the structure in equation (95) and one-loop phases that intersect the said basis vectors. To quote my supervisor, Prof. Alon Faraggi, "these are the ingredients and the ABK rules are the recipe."

As we discussed, the basis vectors arise from the partition function as

$$b = \{\overrightarrow{b}_1, \overrightarrow{b}_2, ..., \overrightarrow{b}_n\} \tag{1}$$

where each of the elements within the basis vector are groups of boundary conditions on the worldsheet fermions. For example, naming one of the elements α (as we did in the previous chapter, equation(95))

$$\alpha = \{\alpha(\psi^\mu), ..., \alpha(\omega^6)|\alpha(\bar{y}^1, ..., \alpha(\bar{\phi}^8)\} \tag{2}$$

where $\alpha(f)$ is a boundary condition parameter associated with the worldsheet fermion taking the following values,

$$\alpha(f) = \begin{cases} 0 & \text{Anti-periodic (NS)} \\ 1 & \text{Periodic (R)} \\ \frac{1}{2} & \text{Complex} \end{cases} \tag{3}$$

[56]. We note that in non-supersymmetric models, $\alpha(f)$ is typically only R or NS hence only taking the values 1 or 0.

All of these basis vectors form the following finite additive group, given by

$$\Xi = \sum_{i=1}^{n} m_i b_i \qquad m_i = 0, ..., N_i - 1 \tag{4}$$

with $N_i b_i = 0 \quad mod \quad 2$ where N_i is the smallest possible integer. This is known as the model space and $\{b_i\}$ must be linearly independent. If a basis vector can be generated from the others from a linear combination, it must be excluded. Each sector of the model corresponds to an element of this group, and hence contributes to the partition function through its spin structure.

3.1 ABK Rules

The ABK rules are introduced to ensure the modular invariance of the partition function and the consistency of the conformal field theory that underlies the theory. First, we state the constraints that the basis vectors, along with their interactions with other basis vectors, must obey.

3.1.1 Basis Vector Rules

1. The total number of real fermions must be an even number.
2. The identity vector, that is the vector that includes boundary conditions for all worldsheet fermions, must be included.

$$\mathbb{1} \in \Xi \tag{5}$$

3. The set of basis vectors are required to satisfy,

$$\sum m_i b_i = 0 \Longleftrightarrow m_i = 0 \quad mod \quad N_i \quad \forall i \tag{6}$$

4.

$$N_{ij} b_i \cdot b_j = 0 \mod 4 \tag{7}$$

N_{ij} being the least common multiple (LCM) of b_j and b_i.

5.

$$N_i b_i \cdot b_i = 0 \mod 8 \tag{8}$$

3.1.2 Phase Coefficient Rules

To obtain phenomenologically viable models, we must define constraints on the one-loop phases to preserve modular invariance.

1. $C \begin{pmatrix} b_i \\ b_j \end{pmatrix} = \delta_{b_i} e^{\frac{2\pi i}{N_j} n} = \delta_{b_j} e^{\frac{2\pi i}{N_i} m} e^{\frac{i\pi}{2} b_i \cdot b_j}$
2. $C \begin{pmatrix} b_i \\ b_j \end{pmatrix} = e^{\frac{i\pi}{2} b_i \cdot b_j} C \begin{pmatrix} b_j \\ b_i \end{pmatrix}^*$
3. $C \begin{pmatrix} b_i \\ b_i \end{pmatrix} = -e^{\frac{i\pi}{4} b_i \cdot b_i} C \begin{pmatrix} b_i \\ \mathbb{1} \end{pmatrix}$
4. $C \begin{pmatrix} b_i \\ b_j + b_k \end{pmatrix} = \delta_{b_i} C \begin{pmatrix} b_i \\ b_j \end{pmatrix} C \begin{pmatrix} b_i \\ b_k \end{pmatrix}$

   Where the index δ_{b_i} guarantees that the spacetime statistics for the bosons and fermions are correct. Given by

$$\delta_{b_i} = e^{-i\pi b_i(\psi^\mu)} = \begin{cases} -1 & \text{If } b_i \text{ contains spacetime fermions, } \psi^\mu \\ +1 & \text{Otherwise} \end{cases} \tag{9}$$

These rules, combined with the earlier explained basis vector rules, make up the full ABK formalism [46]. Forming the theoretical basis for the construction of the heterotic string vacua in both non-supersymmetric and supersymmetric models. The rules ensure that the string model constructed is modular invariant, physically consistent and well defined across all the sectors generated by the basis vectors in a model.

Specifically, they place constraints on how the boundary conditions can be chosen. It is these choices that determine the resulting matter spectrum of the model.

3.2 Mass Formula and Level-Matching Condition

We know that for the construction of heterotic strings using the free-fermionic formulation, the, the physical spectrum is built upon the finite additive group Ξ (see previous section). Where each of its sectors are comprised of basis vectors containing boundary conditions on the worldsheet fermions.

Letting α be a member of this group, $\alpha \in \Xi$. Then each α will create its own distinct Hilbert space, $\mathcal{H}_\alpha$, assembled by taking the vacuum state, $|0\rangle_\alpha$, and acting on it with left- and right-moving fermionic oscillators. The complete nature of this Hilbert space, through its natural inner product, makes it suitable for quantum-mechanical interpretation. The physical states present in the space are required to satisfy the Virasoro matching condition to enforce consistency between the left and the right moving sectors of the heterotic string.

The mass squared of the string state is given for each sector by,

$$\begin{aligned} M_L^2 &= -\frac{1}{2} + \frac{\alpha_L \cdot \alpha_L}{8} + N_L \\ M_R^2 &= -1 + \frac{\alpha_R \cdot \alpha_R}{8} + N_L \end{aligned} \tag{10}$$

[61]. The L and R subscripts intuitively represent the left and right sectors respectively. α are the components of the vector and N is the total number of oscillators that act on the vacuum, $|0\rangle_\alpha$.

The Virasoro matching condition demands that the mass contributions, M_L^2 and M_R^2, be equal. This makes sure that only level-matched states are kept in the physical spectrum. A product of the theory being a closed string theory, hence this requirement becomes essential to the consistency [61].

The frequency of such fermionic oscillators is determined by,

$$\nu_f = \frac{1 + \alpha(f)}{2} \tag{11}$$

and a similar expression for its conjugate,

$$\nu_{f^*} = \frac{1 - \alpha(f)}{2} \tag{12}$$

[61] Here $\alpha(f)$ is determined from the aforementioned boundary condition parameter. These frequencies dictate how each fermionic oscillator contributes the total oscillator number, N.

$$N_{L,R} = \sum \nu_{L,R} = \sum_{f \in L,R} \nu_f + \sum_{f \in L,R} \nu_{f^*} \tag{13}$$

In the NS sector, where $\alpha(f) = 0$, and hence the frequencies become:

$$\nu_f = \nu_{f^*} = \frac{1}{2} \tag{14}$$

In contrast, for the R sector when $\alpha(f) = 1$. The frequencies are,

$$\nu_f = 1, \quad \nu_{f^*} = 0 \tag{15}$$

This is in relation to the presence of doubly degenerate spinorial vacua, $|+i\rangle$ and $|+i\rangle$ [62]. These are born as a result of the intrinsic spin structure of the Ramond sector fermions. The vacua are eigenstates of fermion number, differing only by relative phase; such a degeneracy plays a central role in the supersymmetric (or non-supersymmetric) nature of the model.

3.3 Generalised Gliozzi-Scherk-Olive Projections

In the free-fermionic formulation, generalised Gliozzi-Scherk-Olive (GGSO) projections play an integral role in the phenomenology of the heterotic string. GGSOs contribute to the modular invariance of the one loop partition function and, most importantly, the projections eliminate unphysical states like tachyons. Thus, GGSO projections ensure the phenomenological viability of the theory and further preserve the consistency.

We again consider our group Ξ and our sector α. For the GGSO projection, the states $|s\rangle_\alpha \in \mathcal{H}_\alpha$ must satisfy the following [63].

$$e^{i\pi b_j \cdot F_\alpha} = \delta C \begin{bmatrix} \alpha \\ b_j \end{bmatrix}^* |s\rangle_\alpha \tag{16}$$

Where

- $b_j \in \{b_i\}$
- F_α is fermion number operator for α.
- δ_α is a sign convention.
- $C \begin{bmatrix} \alpha \\ b_j \end{bmatrix}$ is the GGSO phase coefficient.

The object $b_j \cdot F_\alpha$ is defined as the difference in the fermion number between the left and right movers.

$$b_j \cdot F_\alpha = \sum_{f \in Left} b_j F_\alpha(f) - \sum_{f \in Right} b_j F_\alpha(f) \tag{17}$$

States that satisfy this projection condition are retained in the spectrum and deemed physical. In the case that a state does not satisfy the equality, it is projected out. That is, removed from the Hilbert space and therefore will not contribute to observable quantities.

The GGSO projections are the product of the requirement for modular invariance of the one-loop partition function. As we discussed earlier, the partition function can be expressed as the sum over all sectors and their interactions. In certain supersymmetric theories, take $\mathcal{N} = 4$ models for example, this sum can result in cancellations between the bosonic and fermionic contributions. These cancellations will be incorporated by the model through the GGSOs, and as such, the modular invariance of the partition function is preserved.

The uses of the GGSOs are most apparent in phenomenological processes such as analysing the string spectrum. One of the primary uses is the removal of tachyons, which are states where $M^2 < 0$ and thus signify vacuum instability. The GGSOs project out these contributions, allowing for consistent constructions that are free of unphysical tachyons.

3.4 Basis Vector Projections in Practice

The scalar product, or "projecting" two basis vectors, is defined to take into account the contribution from both left- and right-moving fermions as follows,

$$b_i \cdot b_j = \{ \sum_{\substack{Complex\\Left}} + \frac{1}{2} \sum_{\substack{Real\\Left}} - (\sum_{\substack{Complex\\Right}} + \frac{1}{2} \sum_{\substack{Real\\Right}}) \} b_i(f) b_j(f) \tag{18}$$

[64] where each complex fermion is counted as two real fermions, accounting for the complexification.

The process can be seen more explicitly when defining some basis vectors to work with,

$$\begin{aligned} b_1 &= \{\psi_{1,2}\chi_{1,2}y_{5,6}|\bar{y}_{3,4}\bar{y}_{5,6}\bar{\psi}_{1,..,5}\bar{\eta}_1\} \\ b_2 &= \{\psi_{1,2}\chi_{3,4}y_{1,2}\omega_{5,6}|\bar{y}_{1,2}\bar{\omega}_{5,6}\bar{\psi}_{1,..,5}\bar{\eta}_2\} \end{aligned} \tag{19}$$

Recalling from earlier that left of the pipe denotes the left-moving sector and right of the pipe denotes right-moving sector. We can adjust our phase for the intersection

$$e^{i\pi b_2 \cdot F_{b_1}} = \delta_{b_1} C \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{20}$$

.

To project a state, we analyse the overlap of the basis vectors, this can be thought of as carrying out $b_1 \cap b_2$ for the case of b_1 on b_2.

So to start, we write our vectors again and identify through inspection which elements survive an intersection and which don't.

$$\begin{aligned} b_1 &: \{\psi_{1,2}\cancel{\chi_{1,2}}\cancel{y_{5,6}}|\cancel{y_{\bar{3},4}}\cancel{y_{\bar{5},6}}\psi_{1,..,5}^{\bar{}}\cancel{\bar{\eta}_1}\} \\ b_2 &: \{\psi_{1,2}\cancel{\chi_{3,4}}\cancel{y_{1,2}}\cancel{\omega_{5,6}}|\cancel{y_{\bar{1},2}}\cancel{\omega_{\bar{5},6}}\psi_{1,..,5}^{\bar{}}\cancel{\bar{\eta}_2}\} \end{aligned} \tag{21}$$

We can see the only overlap is in $\psi_{1,2}$ and $\psi_{1,..,5}^{\bar{}}$. This will be denoted as the following.

$$\begin{gathered} b_1 : \{\psi_{1,2}\chi_{1,2}y_{5,6}|\bar{y}_{3,4}\bar{y}_{5,6}\bar{\psi}_{1,..,5}\bar{\eta}_1\} \\ b_2 : \{\psi_{1,2}\chi_{3,4}y_{1,2}\omega_{5,6}|\bar{y}_{1,2}\bar{\omega}_{5,6}\bar{\psi}_{1,..,5}\bar{\eta}_2\} \\ b_1 \cap b_2 : 1,0,0,0,0,0,1,0 \end{gathered} \tag{22}$$

1 denoting an overlap, 0 denoting no overlap. We note that the first two terms in $b_1 \cap b_2$ we pay special attention to because of degeneracy and that the $\bar{\psi}_{1,..,5}$ term (the seventh term in $b_1 \cap b_2$) contains 5 worldsheet fermions due to the subscript on $\bar{\psi}$.

3.5 Projections and Chirality

Here we can start to consider a phenomenological outlook with respect to how chirality appears in basis vector models. Reminding ourselves of combinatorics where

$$\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n}{m!(n-m)!} \tag{23}$$

[65]. Where it can be thought of as m negative states in n positive states.

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = |-\rangle|-\rangle|+\rangle|+\rangle \tag{24}$$

So returning to our projection, ψ_1 and ψ_2 can be displayed as

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix}_+ \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix}_- \tag{25}$$

where the subscript representing the chirality and $n = 2$ representing the $\psi_{1,2}$. Using the same formalism for the $\bar{\psi}_{1,..,5}$ we have,

$$\left[\begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix}\right]_+ \quad \left[\begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \end{pmatrix}\right]_- \tag{26}$$

However, although there was no overlap in $y, \bar{y}, \omega, \bar{\omega}$ and $\bar{\eta}$ they still need to be considered in the overall projection. As there are 5 terms not overlapped we can consider their combinatorial representation the same as above.

Combining all of our terms together,we can express the projection fully in terms of combinatorics and hence, through chirality.

$$\begin{matrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}_+ \left[\begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix}\right]_+ \left[\begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix}\right]_+ \\ \begin{bmatrix} 2 \\ 2 \end{bmatrix}_- \left[\begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \end{pmatrix}\right]_- \left[\begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix}\right]_+ \end{matrix} \tag{27}$$

where the first term comes from χ and ψ and is said to be the chiral operator, the reason we have two lines for left and right. The second term is $\bar{\psi}$ and the third term is from the non-overlapping terms. This expression has a net positive chirality ([+],[+],[+] and [-],[-],[+]).

Alternatively, we can express the same projection in a net negative chirality by flipping the the second term.

$$\begin{aligned}&\begin{bmatrix}2\\0\end{bmatrix}_+\left[\begin{pmatrix}5\\1\end{pmatrix}+\begin{pmatrix}5\\3\end{pmatrix}+\begin{pmatrix}5\\5\end{pmatrix}\right]_-\left[\begin{pmatrix}5\\0\end{pmatrix}+\begin{pmatrix}5\\2\end{pmatrix}+\begin{pmatrix}5\\4\end{pmatrix}\right]_+\\&\begin{bmatrix}2\\2\end{bmatrix}_-\left[\begin{pmatrix}5\\0\end{pmatrix}+\begin{pmatrix}5\\2\end{pmatrix}+\begin{pmatrix}5\\4\end{pmatrix}\right]_+\left[\begin{pmatrix}5\\0\end{pmatrix}+\begin{pmatrix}5\\2\end{pmatrix}+\begin{pmatrix}5\\4\end{pmatrix}\right]_+\end{aligned} \tag{28}$$

When conducting a projection, we choose one chiral representation and ignore the other. This is projecting out, from the chiral perspective.

Returning to the phase,

$$e^{i\pi b_2\cdot F_{b_1}} = \delta_{b_1} C \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{29}$$

the choice governs whether $C\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is positive or negative. For consistency, we require that this be kept the same throughout the formalism.

4 | Constructing $SO(10)$ in String Models

It is now time to introduce some models, heading towards a reproduction of the observable $SO(10)$ of the standard model which acts as a foundation for not only creating a Pati-Salam model, but also the of breaking $SO(10)$ to other models such as flipped $SU(5)$.

4.1 The Single Basis Vector Model

As a starting point, we examine the model defined by a single basis vector. As the simplest possible case, this model will be an illustrative example of how boundary conditions, GGSO projections, Virasoro conditions and modular invariance work together.

As per the ABK rules, when constructing a single basis vector model, there is only one choice available for the set of boundary conditions to have the identity element in the Ξ group.

$$B = \{\mathbb{1}\} \tag{1}$$

Where $\mathbb{1}$ is the identity vector assigning boundary conditions to all worldsheet,

$$\mathbb{1} : \{\psi^\mu, \chi^{1,\ldots,6}, y^{1,\ldots,6}, \omega^{1,\ldots,6} | \bar{y}^{1,\ldots,6}, \bar{\omega}^{1,\ldots,6}, \bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\ldots,8}\} \tag{2}$$

[27]. Where we use the notation that,

- Left Movers:
  - Complex Fermions: $\psi^{\mu}, \chi^{1,\ldots,6}$
  - Real Fermions: $y^{1,\ldots,6}, \omega^{1,\ldots,6}$
- Right Movers:
  - Complex Fermions: $\bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,\ldots,8}$
  - Real Fermions: $\bar{y}^{1,\ldots,6}, \bar{\omega}^{1,\ldots,6}$

This minimal model has only two elements in Ξ.

$$\Xi = \{0, \mathbb{1}\} \tag{3}$$

Where the 0 is representative of the anti-periodic (NS) boundary conditions for all of the fermions. These two sectors determine the full space of possible states that this model creates. Sector by sector, we can express this as,

$$\{\overrightarrow{1}, 2 \cdot \overrightarrow{1} = \overrightarrow{0}\} \tag{4}$$

The even number of real fermions, combined with the single basis vector, determines that the ABK rules are satisfied. When the fermions are paired together, periodicity is preserved, having the same boundary conditions across the sectors.

For phenomenological applications, we are interested in the massless states as these states produce physically relevant degrees of freedom at sufficiently low energy. That is, the massless states avoid the Planck energy scale.

We also remind ourselves that physical consistency demands the absence of tachyonic states ($M^2 < 0$).

We progress with our two sectors searching for the phenomenologically viable massless states.

Ramond Sector

Beginning with $\overrightarrow{1}$, where all fermions are periodic. We apply the Virasoro matching condition,

$$M_L^2 = -\frac{1}{2} + \frac{\alpha_L \cdot \alpha_L}{8} + N_L = -1 + \frac{\alpha_R \cdot \alpha_R}{8} + N_L = M_R^2 \tag{5}$$

and we compute the products $\alpha_{L,R} \cdot \alpha_{L,R}$,

$$\begin{aligned} \alpha_L \cdot \alpha_L &= \frac{1}{2} \times 20 = 10 \\ \alpha_R \cdot \alpha_R &= \frac{1}{2} \times 44 = 22 \end{aligned} \tag{6}$$

Thus, the mass formula becomes

$$\begin{aligned} M_L^2 &= -\frac{1}{2} + \frac{\alpha_L \cdot \alpha_L}{8} + N_L = \frac{3}{4} + N_L > 0 \\ M_R^2 &= -1 + \frac{\alpha_R \cdot \alpha_R}{8} + N_L = \frac{9}{2} + N_R > 0 \end{aligned} \tag{7}$$

and since both M_L^2 and M_R^2 are greater that zero, there are no massless states in this sector [27].

Neveu-Schwarz Sector

In the $2 \cdot \overrightarrow{1} = \overrightarrow{0}$ sector, all fermions are anti-periodic. As a result,

$$\alpha_{L,R} \cdot \alpha_{L,R} = 0 \tag{8}$$

Inferring that all the worldsheet fermions are included in the $\mathbb{1}$ sector, which we already knew. Implying that the mass formulae is,

$$\begin{aligned} M_L^2 &= -\frac{1}{2} + \frac{\alpha_L \cdot \alpha_L}{8} + N_L = -\frac{1}{2} + N_L > 0 \\ M_R^2 &= -1 + \frac{\alpha_R \cdot \alpha_R}{8} + N_L = -1 + N_R > 0 \end{aligned} \tag{9}$$

Thus to obtain massless states from this we require that,

$$N_L = \frac{1}{2}$$
$$N_R = 1 \tag{10}$$

We take our fermionic oscillator frequency equations and apply the information we have deduced to find the allowed oscillator modes for the NS fermions,

$$\nu_{f,f^*} = \frac{1 \pm \alpha(f)}{2} = \frac{1 \pm 0}{2} = \frac{1}{2} \tag{11}$$

[27]. Hence, to build the phenomenological massless states we require the following ingredients:

- A left-moving oscillator with $N_L = \frac{1}{2}$
- One of the following:
  - Two right-moving fermionic oscillators to make $N_R = 2 \times \frac{1}{2} = 1$
  - One right moving bosonic oscillator to make $N_R = 1$

Alternatively, we can force a tachyon appear if we consider a single right-moving fermionic oscillator as

$$N_R = \frac{1}{2} \implies M_R^2 = -\frac{1}{2} \tag{12}$$

[27]. Thus we are showing that the Virasoro constraints indicate to us which states are allowed on the basis of mass and physicality.

4.1.1 Massless States

As we have just discussed, all the massless states for this model lie in the NS sector. After using the Virasoro constraints to find the allowed massless states, we interpret the quantum excitations in a space-time language. The form of these oscillator excitations tells us the gauge properties of a string state; these properties are now matched to particle types. We will now demonstrate this phenomenology for the NS sector.

As stated, the requirements to build the massless states are either a left moving fermionic oscillator with a right moving bosonic oscillator or a left moving fermionic oscillator with two right moving fermionic oscillators. Such requirements leave us with the following string states.

- A familiar state which we can begin here is the Graviton.

$$\psi^{\mu}_{\frac{1}{2}}\partial\bar{X}^{\nu}_{1}|0\rangle_{NS} \tag{13}$$

  Here, $\partial\bar{X}^{\nu}_{1}$ is the right-moving boson. Leaving $\psi^{\mu}_{\frac{1}{2}}$ as the left-moving fermionic oscillator. We note here that bosonic states in general are analogous to the graviton, antisymmetric tensor and the dilaton [27].

- $$\{\chi^{i}_{\frac{1}{2}}y^{i}_{\frac{1}{2}}\omega^{i}_{\frac{1}{2}}\}\partial\bar{X}^{\nu}_{1}|0\rangle_{NS} \qquad \{i = 1, ..., 6\} \tag{14}$$

  For this case, $\{\chi^{i}_{\frac{1}{2}}y^{i}_{\frac{1}{2}}\omega^{i}_{\frac{1}{2}}\}$ is the left-moving fermionic oscillator. These states are the $SU(2)^6$ adjoint representations of gauge bosons.

- $$\psi^{\mu}_{\frac{1}{2}}\bar{\phi}^{a}_{\frac{1}{2}}\bar{\phi}^{b}_{\frac{1}{2}}|0\rangle_{NS} \qquad \{a, b\} = 1, ..., 44 \tag{15}$$

  Now we have $\bar{\phi}^{a}_{\frac{1}{2}}\bar{\phi}^{b}_{\frac{1}{2}}$ as two right-moving fermionic oscillators analogous to the $SO(44)$ gauge group representation of gauge bosons.

- $$\{\chi^{i}_{\frac{1}{2}}y^{i}_{\frac{1}{2}}\omega^{i}_{\frac{1}{2}}\}\bar{\phi}^{a}_{\frac{1}{2}}\bar{\phi}^{b}_{\frac{1}{2}}|0\rangle_{NS} \tag{16}$$

  In particular, this state aligns with the scalars in the $SU(2)^6 \otimes SO(44)$ gauge group [27].

As an exercise, we will include the tachyonic state despite its unphysicality, which as previously stated is found by considering a state with a single right moving fermionic oscillator, $\bar{\phi}^a_{\frac{1}{2}}$.

$$\bar{\phi}^a_{\frac{1}{2}}|0\rangle_{NS} \tag{17}$$

4.1.2 GGSO Projections

Continuing with our formalism, we now use the GGSO projections to leave us with a completely physical spectrum. We recall the condition for the projection as;

$$e^{i\pi b_j \cdot F_\alpha} = \delta C \begin{bmatrix} \alpha \\ b_j \end{bmatrix}^* |s\rangle_\alpha \tag{18}$$

We apply it to this model by taking the NS as α and b_j as the $\mathbb{1}$. Thus, we compute that

$$\begin{aligned} e^{i\pi \mathbb{1} \cdot F_{NS}}|s\rangle_{NS} &= \delta_{NS} C \begin{pmatrix} NS \\ \mathbb{1} \end{pmatrix}^* |s\rangle_{NS} \\ &= \delta_{NS}\delta_{\mathbb{1}}|s\rangle_{NS} \\ &= \delta_{\mathbb{1}}|s\rangle_{NS} \\ &= -1|s\rangle_{NS} \end{aligned} \tag{19}$$

by the setting of $\delta_{NS} = 1$ and $\delta_{\mathbb{1}} = -1$ from the modular invariance rules and we using that

$$\begin{aligned} C \begin{pmatrix} b_i \\ b_j \end{pmatrix} &= \delta_{b_i} e^{\frac{2\pi i}{N_j} n} = \delta_{b_j} e^{\frac{2\pi i}{N_i} m} e^{\frac{i\pi}{2} b_i \cdot b_j} \\ \implies C \begin{pmatrix} NS \\ NS \end{pmatrix} &= \delta_{NS} e^{\frac{2\pi i (0)}{0}} = \delta_{NS} \end{aligned} \tag{20}$$

and

$$C\begin{pmatrix} NS \\ b_j \end{pmatrix} = \delta_{b_j} e^{\frac{2\pi i m_{NS}}{N_{NS}}} e^{\frac{i\pi}{2} b_j \cdot NS} = \delta_{b_j} \tag{21}$$

as $e^{\frac{i2\pi m_{NS}}{NS}} = 1$, $e^{\frac{i\pi b_j NS}{2}} = 1$ and $b_j \cdot NS = b_j \cdot 0 = 0$ [27].

Now that we have a general expression for a surviving state, we substitute $|s\rangle$ for one of the proposed states to see if that state is in the model.

Taking the Graviton first,

$$e^{i\pi \mathbb{1}\cdot F_{NS}}(\psi^{\mu}_{\frac{1}{2}} \partial \bar{X}^{\nu}_{1} |0\rangle_{NS}) = \delta_{\mathbb{1}}(\psi^{\mu}_{\frac{1}{2}} \partial \bar{X}^{\nu}_{1} |0\rangle_{NS}) = -\psi^{\mu}_{\frac{1}{2}} \partial \bar{X}^{\nu}_{1} |0\rangle_{NS} \tag{22}$$

we can see that it survives as the result is of $-|s\rangle_{NS}$ form. Here we absolutely must note that, according to the identity proven in Equation (21), the graviton state will be universally present in all consistent string models owing to this formulation. This instils a description of gravity in our models, solving one of the issues we outlined with the standard model.

We continue with the other states we proposed,

$$\begin{aligned}
&e^{i\pi \mathbb{1}\cdot F_{NS}} \psi^{\mu}_{\frac{1}{2}} \bar{\phi}^{a}_{\frac{1}{2}} \bar{\phi}^{b}_{\frac{1}{2}} |0\rangle_{NS} = \delta_{\mathbb{1}} \psi^{\mu}_{\frac{1}{2}} \bar{\phi}^{a}_{\frac{1}{2}} \bar{\phi}^{b}_{\frac{1}{2}} |0\rangle_{NS} = -\psi^{\mu}_{\frac{1}{2}} \bar{\phi}^{a}_{\frac{1}{2}} \bar{\phi}^{b}_{\frac{1}{2}} |0\rangle_{NS} \\
&e^{i\pi \mathbb{1}\cdot F_{NS}} \{\chi^{i}_{\frac{1}{2}} y^{i}_{\frac{1}{2}} \omega^{i}_{\frac{1}{2}}\} \partial \bar{X}^{\nu}_{1} |0\rangle_{NS} = \delta_{\mathbb{1}} \{\chi^{i}_{\frac{1}{2}} y^{i}_{\frac{1}{2}} \omega^{i}_{\frac{1}{2}}\} \partial \bar{X}^{\nu}_{1} |0\rangle_{NS} = -\{\chi^{i}_{\frac{1}{2}} y^{i}_{\frac{1}{2}} \omega^{i}_{\frac{1}{2}}\} \partial \bar{X}^{\nu}_{1} |0\rangle_{NS} \\
&e^{i\pi \mathbb{1}\cdot F_{NS}} \{\chi^{i}_{\frac{1}{2}} y^{i}_{\frac{1}{2}} \omega^{i}_{\frac{1}{2}}\} \bar{\phi}^{a}_{\frac{1}{2}} \bar{\phi}^{b}_{\frac{1}{2}} |0\rangle_{NS} = \delta_{\mathbb{1}} \{\chi^{i}_{\frac{1}{2}} y^{i}_{\frac{1}{2}} \omega^{i}_{\frac{1}{2}}\} \bar{\phi}^{a}_{\frac{1}{2}} \bar{\phi}^{b}_{\frac{1}{2}} |0\rangle_{NS} = -\{\chi^{i}_{\frac{1}{2}} y^{i}_{\frac{1}{2}} \omega^{i}_{\frac{1}{2}}\} \bar{\phi}^{a}_{\frac{1}{2}} \bar{\phi}^{b}_{\frac{1}{2}} |0\rangle_{NS}
\end{aligned} \tag{23}$$

Thus showing that the other states also survive the projection. However, when we perform this for the tachyon we proposed,

$$e^{i\pi \mathbb{1}\cdot F_{NS}} \bar{\phi}^{a}_{\frac{1}{2}} |0\rangle_{NS} = \delta_{\mathbb{1}} \bar{\phi}^{a}_{\frac{1}{2}} |0\rangle_{NS} = -\bar{\phi}^{a}_{\frac{1}{2}} |0\rangle_{NS} \tag{24}$$

we see that this also survives. Hence, we call this particular model tachyonic and unphysical [27].

4.1.3 Proving Modular Invariance

It can be shown here that modular invariance is preserved here through the partition function, though this is decidedly trivial, as we know that the ABK rules with which we formulated the model ensure this. Nevertheless, in an effort for completeness, the partition function can receive three contributions from spin structures.

$$Z = C \begin{bmatrix} 0 \\ 0 \end{bmatrix} Z \begin{bmatrix} 0 \\ 0 \end{bmatrix} + C \begin{bmatrix} 0 \\ 1 \end{bmatrix} Z \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C \begin{bmatrix} 1 \\ 0 \end{bmatrix} Z \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{25}$$

Using the GGSO coefficient relations and one-loop phases conditions,

$$\begin{aligned} C \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -C \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= +1 \end{aligned} \tag{26}$$

Thus, we can say

$$Z = Z \begin{bmatrix} 0 \\ 0 \end{bmatrix} - Z \begin{bmatrix} 1 \\ 0 \end{bmatrix} - Z \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{27}$$

Then evaluating each component of the partition function,

$$\begin{aligned} Z \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \frac{\vartheta_3^{10}}{\eta^{10}} \cdot \frac{\bar{\vartheta}_3^{22}}{\bar{\eta}^{22}} \\ Z \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{\vartheta_4^{10}}{\eta^{10}} \cdot \frac{\bar{\vartheta}_4^{22}}{\bar{\eta}^{22}} \\ Z \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \frac{\vartheta_2^{10}}{\eta^{10}} \cdot \frac{\bar{\vartheta}_2^{22}}{\bar{\eta}^{22}} \end{aligned} \tag{28}$$

Expressing this all together,

$$Z = \frac{1}{\eta^{10}\bar{\eta}^{22}}(\vartheta_3^{10}\bar{\vartheta}_3^{22} - \vartheta_4^{10}\bar{\vartheta}_4^{22} - \vartheta_2^{10}\bar{\vartheta}_2^{22}) \tag{29}$$

Which is a invariant under the modular transformations, $\tau \to \tau + 1$ and $\tau \to -1/\tau$ [27]. Which is expected as we have worked within the confines of modular invariance constraints.

This check can be performed on later, and more elaborate models, but we shall omit the partition function formalism of showing modular invariance as it becomes increasingly lengthy for a result we expect due to the constraints we work within.

Thus, we have completed an analysis of the simplest basis vector model, $\{\mathbb{1}\}$, we will now build upon this.

4.2 The Addition of the S Vector

Although our single basis vector model satisfied the modular consistency requirements. The tachyonic mode present in it makes it physically unstable. In an effort to increase the physicality of the model, we add basis vectors. We will now introduce the second basis vector, S. Resulting in our total model being,

$$\{\mathbb{1}, \vec{S}\} \tag{30}$$

[27]. Unlike the identity vector, $\mathbb{1}$, this is where we start to add conditions on specific worldsheet fermions. S is defined by the following,

$$\vec{S} : \{\psi^{\mu}, \chi^{1,\ldots,6}\} \tag{31}$$

corresponding to Ramond conditions for the stated fermions and all others are NS [27]. The 6 internal χ fermions are in fact analogous to the Majorana-Weyl superpartners of the 6 compactified internal dimensions, hence their inclusion. This is also why this vector may be referred to as the supersymmetric vector.

4.2.1 ABK Rules on S

As with any model, the construction must satisfy the ABK rules for modular invariance. Calculating the basis vector projections using,

$$b_i \cdot b_j = \{ \sum_{\substack{Complex \\ Left}} + \frac{1}{2} \sum_{\substack{Real \\ Left}} - (\sum_{\substack{Complex \\ Right}} + \frac{1}{2} \sum_{\substack{Real \\ Right}}) \} b_i(f) b_j(f) \tag{32}$$

We have

$$\begin{aligned} S \cdot \mathbb{1} &= 4 \\ S \cdot S &= \frac{8}{2} - 0 \end{aligned} \tag{33}$$

These values allow us to show modular invariance constraints are satisfied through the ABK rules.

$$\begin{aligned} Rule: N_{ij} b_i b_j &= 0 \quad mod \quad 4 \\ \implies N_{\vec{S},\mathbb{1}} &= N_{\vec{S},\mathbb{1}} \cdot \vec{S} \cdot \vec{\mathbb{1}} \\ &= 1 \cdot 4 \\ &= 4 \\ &= 0 \quad mod \quad 4 \end{aligned} \tag{34}$$

$$\begin{aligned} Rule: N_i b_i b_j &= 0 \quad mod \quad 8 \\ \implies N_{\vec{S}} \cdot b_{\vec{S}} \cdot b_{\vec{S}} &= N_S \cdot \vec{S} \cdot \vec{S} \\ &= 2 \cdot 4 \\ &= 8 \\ &= 0 \quad mod \quad 8 \end{aligned} \tag{35}$$

Hence, the extended construction satisfies the modular invariance constraints.

The additive group Ξ for this model, generated by the linear combinations, gives 4 sectors,

$$\Xi = \{\mathbb{1}, \vec{S}, \mathbb{1} + \vec{S}, NS\} \tag{36}$$

The NS sector, where all fermions are anti-periodic, does not provide a new basis vector when combined with $\vec{S}$. Hence, $S + NS$ is not included in Ξ [27].

4.2.2 Massive States

Of the sectors in Ξ, only the $\vec{S}$ and NS sectors contain observable massless states. $\mathbb{1}$ and $\mathbb{1} + \vec{S}$ produce massive states. Shown to be true by the Virasoro condition;

$$\begin{aligned} \alpha_L \cdot \alpha_L = 6 &\implies M_L^2 = -\frac{1}{2} + \frac{6}{8} = \frac{1}{4} > 0 \\ \alpha_R \cdot \alpha_R = 22 &\implies M_R^2 = -1 + \frac{22}{8} > 0 \end{aligned} \tag{37}$$

We note the absence of N_L and N_R in these calculations, these are omitted as these sectors are Ramond in nature.

GGSO Projections

As we know, the GGSO projections are imposed to preserve modular invariance and eliminate unphysical states. As such, this is also required for this model.

Using the same process as in Equation (19) in the preceding section, we impose the same condition as before on a GGSO projection. That is, acting a GGSO on a state must give a negative state for the state to survive, which in this case is now,

$$e^{i\pi\vec{S}\cdot F_{NS}}|s\rangle_{NS} = -1|s\rangle_{NS} \tag{38}$$

[27]. Now we apply this projection condition to the states in the model, beginning

with the graviton.

$$\begin{aligned} e^{i\pi(1-0)}\{\psi^{\mu}_{\frac{1}{2}}\partial X^{\nu}_{1}|0\rangle_{NS}\} &= \delta_S\{\psi^{\mu}_{\frac{1}{2}}\partial_{+1}X^{\nu}_{1}|0\rangle_{NS}\} \\ &= -\{\psi^{\mu}_{\frac{1}{2}}\partial_{+1}X^{\nu}_{1}|0\rangle_{NS}\} \end{aligned} \tag{39}$$

As expected, we obtain a negative state. Thus, the state survives which is an expected result as the graviton state always survives.

Next in line is,

$$\begin{aligned} & e^{i\pi S\cdot F_{NS}}\left(\{\chi_{\frac{1}{2}},y_{\frac{1}{2}},\omega_{\frac{1}{2}}\}\partial\bar{X}^{\mu}_{+1}\mid 0\rangle_{NS}\right) \\ &= e^{i\pi(S(\chi)\cdot F_{NS}+S(y)\cdot F_{NS}+S(\omega)\cdot F_{NS}+S(\partial\bar{X})\cdot F_{NS})}\left(\{\chi_{\frac{1}{2}},y_{\frac{1}{2}},\omega_{\frac{1}{2}}\}\partial\overline{X}^{\mu}_{+1}\mid 0\rangle_{NS}\right) \\ &= e^{i\pi(S(\chi)+S(y)+S(\omega))}(\{\chi_{\frac{1}{2}};y_{\frac{1}{2}};\omega_{\frac{1}{2}}\}\partial\bar{X}^{\mu}_{+1}|0\rangle_{NS}) \end{aligned} \tag{40}$$

Where we have used Fermion number to simplify as $F_{NS} = 1$ for fermions and 0 for bosons. Recalling the composition of $\vec{S}$ and we can simplify further to see we do indeed have a negative state.

$$\begin{aligned} &= e^{i\pi(1+0+0)}(\{\chi_{\frac{1}{2}};y_{\frac{1}{2}};\omega_{\frac{1}{2}}\}\partial\bar{X}^{\mu}_{+1}|0\rangle_{NS}) \\ &= -\{\chi_{\frac{1}{2}}\}\partial\bar{X}^{\mu}_{+1}|0\rangle_{NS} \end{aligned} \tag{41}$$

Thus the state survives. We note that since y, w have a positive contribution, $y_{\frac{1}{2}}\partial\bar{X}^{\mu}_{+1}|0\rangle_{NS}$ and $\omega_{\frac{1}{2}}\partial\bar{X}^{\mu}_{+1}|0\rangle_{NS}$ are projected out. Recalling that this state is representative of the $SU(2)^6$ bosons, and as we are now only left with the χ from $\{\chi, y, \omega\}$ we have broken the gauge group by adding $\vec{S}$ to the model. By the same method, we see that only $\chi_{\frac{1}{2}}$ survives from $\{\chi_{\frac{1}{2}};y_{\frac{1}{2}};\omega_{\frac{1}{2}}\}\bar{\phi}^{a}_{\frac{1}{2}}\bar{\phi}^{b}_{\frac{1}{2}}|0\rangle_{NS}$.

For the adjoint scalar representations of $SO(44)$ we use their fermion numbers to deduce that,

$$\begin{aligned} & e^{i\pi(S(\psi^{\mu})\cdot F_{NS}(\psi^{\mu})+S(\bar{\phi}^{a})\cdot F(\phi^{a})\cdot S(\bar{\phi}^{b})\cdot F(\phi^{b})} = e^{i\pi} \\ & \implies \psi^{\mu}_{\frac{1}{2}}\bar{\phi}^{a}_{\frac{1}{2}}\bar{\phi}^{b}_{\frac{1}{2}}|0\rangle_{NS} = -1 \end{aligned} \tag{42}$$

and hence, it is a surviving state.

And finally, and perhaps most importantly, the tachyon.

$$e^{i\pi(0\cdot 1)}\{\bar{\phi}^{a}_{\frac{1}{2}}|o\rangle_{NS}\} = +\{\{\bar{\phi}^{a}_{\frac{1}{2}}|o\rangle_{NS}\} = Positive \quad Result \tag{43}$$

Which does not satisfy the survivability showing that the unphysical tachyon is now projected out. A step in the right direction as our model is now more physically viable than it once was [27].

4.2.3 Massless States

Now that we have considered the NS and the $1 + NS$ sectors, we move onto the $\vec{S}$ sector where the massless states exist.

Here, by the Virasoro condition, the left moving sector is purely periodic.

$$\begin{aligned} \alpha_L &= S_L \\ S_L \cdot S_L &= 4 \\ \implies M_L^2 &= -\frac{1}{2} + \frac{4}{8} = 0 \end{aligned} \tag{44}$$

The right moving sector remains in the NS vacua.

$$\begin{aligned} S_R \cdot S_R &= 0 \\ \implies M_R^2 &= -1 + \frac{0}{8} = -1 \end{aligned} \tag{45}$$

As always, we seek to avoid tachyonic states so we suggest that either one right moving bosonic oscillator or two right moving fermionic oscillators must be excited.

Here, we define complex fermions,

$$\psi^{\mu}_{1,2} = \frac{1}{\sqrt{2}}(\psi^{\mu}_1 + i\psi^{\mu}_2), \quad \chi_{i,j} = \frac{1}{\sqrt{2}}(\chi_i + i\chi_j) \tag{46}$$

where the indices i, j correspond to $\chi_{1,2}, \chi_{3,4}, \chi_{5,6}$ internal degrees of freedom. Together with their conjugates. Using this, we define the $S-$vacuum as

$$\begin{aligned} |S\rangle_L = &|\pm\rangle \quad |\pm\rangle \quad |\pm\rangle \quad |\pm\rangle \quad |0\rangle_L \\ &\psi^{\mu}_{1,2} \quad \chi_{1,2} \quad \chi_{3,4} \quad \chi_{5,6} \end{aligned} \tag{47}$$

[27]. Thus, we obtain $2^4 = 16$ state choices. Which we can represent in combinatorics as

$$\left[\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}\right] \tag{48}$$

By our requirement that we need either one right moving bosonic oscillator or two right moving fermionic oscillators to avoid a tachyon. We have two possible massless states,

$$\begin{aligned} Gravitino &: |s\rangle_L \partial \bar{X}^{\mu}_{+1} |0\rangle_R \\ Gaugino &: |s\rangle_L \bar{\phi}^{a}_{\frac{1}{2}} \bar{\phi}^{b}_{\frac{1}{2}} |0\rangle_R \end{aligned} \tag{49}$$

[27]. We note the gravitino has spin-$\frac{3}{2}$ and the gaugino spin-$\frac{1}{2}$.

GGSO Projections

We have to perform GGSOs for both $\mathbb{1}$ and $\vec{S}$. Recall our fermion number designations, where $F(|+\rangle) = 0$ and $F(|-\rangle) = 1$. Using the ABK rules, it can be shown algebraically from

$$e^{i\pi \mathbb{1} \cdot F_S} |s\rangle_S = \delta_S C \begin{pmatrix} S \\ \mathbb{1} \end{pmatrix}^* |s\rangle_S \tag{50}$$

that we can form matrices of relative contributions from the phases

$$C\begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix}, \quad C\begin{pmatrix} \mathbb{1} \\ S \end{pmatrix}, \quad C\begin{pmatrix} S \\ \mathbb{1} \end{pmatrix}, \quad C\begin{pmatrix} S \\ S \end{pmatrix} \tag{51}$$

Which are,

$$\begin{array}{cc} & \begin{array}{cc} \mathbb{1} & S \end{array} \\ \begin{array}{c} \mathbb{1} \\ S \end{array} & \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \end{array} \tag{52}$$

or

$$\begin{array}{cc} & \begin{array}{cc} \mathbb{1} & S \end{array} \\ \begin{array}{c} \mathbb{1} \\ S \end{array} & \begin{bmatrix} -1 & +1 \\ +1 & +1 \end{bmatrix} \end{array} \tag{53}$$

By modular invariance, we can fix the phases $C\begin{pmatrix} S \\ S \end{pmatrix}$ and $C\begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix}$, leaving the other two as independent. Doing this greatly simplifies the process for larger models [27]. This thought process allows the GGSO projection for $\mathbb{1}$ to be written with respect to the number of negative states.

$$e^{i\pi\mathbb{1}\cdot F}|s\rangle = (-1)^{\epsilon}|s\rangle = +|s\rangle \tag{54}$$

Where ϵ is representing the number of negative states. Thus, we require an even number of negative states to obtain the negative condition for the GGSO projection. Hence, we turn to combinatorial notation as a intuitive representation the surviving states;

$$\begin{aligned} Gaugino: &\left[\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix}\right] \bar{\phi}^a_{\frac{1}{2}}\bar{\phi}^b_{\frac{1}{2}}|0\rangle \\ Gravitino: &\left[\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix}\right] \partial\bar{X}^{\mu}_{+1}|0\rangle \end{aligned} \tag{55}$$

Decomposing the states allows us to determine the gravitino count.

$$\left[\begin{pmatrix}1\\0\end{pmatrix}\left(\begin{pmatrix}3\\0\end{pmatrix}+\begin{pmatrix}3\\2\end{pmatrix}\right)+\begin{pmatrix}1\\1\end{pmatrix}\left(\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}3\\3\end{pmatrix}\right)\right]=4 \tag{56}$$

Which implies the presence of the $\mathcal{N}=4$ supersymmetry [27].

The $\vec{S}$ projections,

$$e^{i\pi S\cdot F_S}|s\rangle_S=\delta_s\begin{pmatrix}S\\S\end{pmatrix}^{*}|s\rangle_S=+|s\rangle_S \tag{57}$$

which acts the same as $\mathbb{1}$ projections because of $S_R=0$. Thus, we can infer that the spectrum survives through the projections of $\vec{S}$.

As a concluding statement, the GGSOs of $\{\mathbb{1},\vec{S}\}$ leave us with $\mathcal{N}=4$ supersymmetry and $SO(44)$ gauge group as a result of the states that have been shown to survive.

4.3 The b_1 Vector

Taking stock of what we have with $\{\mathbb{1},\vec{S}\}$,

- No tachyons.
- No matter.
- $N=4$ supersymmetry.
- $SO(44)$ gauge group.

Though the result of no tachyons is promising, our aim to reproduce the $SO(10)$ of the standard model in order to get a realistic model requires matter, $\mathcal{N}=1$ supersymmetry and of course, a smaller gauge group.

The first step in this direction, and in the direction of what will come to know as the NAHE set, is to add the vector b_1 to the model,

$$b_1 = \{\psi^\mu_{1,2}, \chi_{1,2}, y_{3,4}, y_{5,6} | \bar{y}_{3,4}, \bar{y}_{5,6}, \bar{\psi}^\mu_{1,...,5}, \bar{\eta}_1\} \tag{58}$$

[27]. Extending our model to $\{\mathbb{1}, S, b_1\}$ and giving us the sectors

$$\Xi = \{NS, \mathbb{1} + S, \mathbb{1} + b_1, S + b_1, \mathbb{1} + S + b_1, \mathbb{1}, S, b_1\} \tag{59}$$

It is here where we start to see how the sectors start to pile up after the addition of so many basis vectors.

4.3.1 ABK Rules

In the order of proceedings for string model building, we of course start with the ABK rules. As we have done this for two previous models and thus gained an understanding of the process of building a model, we will begin to speed up our walk-throughs.

As we know, we do this in order to preserve the model's modular invariance, as we have added a new basis vector. So we confirm the following,

$$\begin{aligned} N_{\mathbb{1}b_1} \cdot \mathbb{1} \cdot b_1 &= 2(-8+4) = 0 \mod 4 \\ N_{Sb_1} \cdot S \cdot b_1 &= 2(2-0) = 0 \mod 4 \\ N_{b_1} \cdot b_1 \cdot b_1 &= 0 \mod 8 \end{aligned} \tag{60}$$

to show that the b_1 vector is permissible.

4.3.2 Mass Spectrum

Deploying the Virasoro condition,

$$\begin{aligned} b_{1_R} \cdot b_{1_R} &= 8 \\ b_{1_L} \cdot b_{1_L} &= 4 \\ \implies M_L^2 = -\frac{1}{2} + \frac{4}{8} + 0 &= -1 + \frac{8}{8} + 0 = M_R^2 \end{aligned} \tag{61}$$

Which is showing that we have $N_R = N_L = 0$, no oscillators are present, meaning a Ramond vacua. Hence, b_1 contributes to the massless states but no tachyonic states to the spectrum.

4.3.3 GGSO Projections

For the GGSO projections, we expand our relative contribution matrix by b_1

$$\begin{array}{c} \\ \mathbb{1} \\ S \\ b_1 \end{array} \begin{array}{c} \begin{array}{ccc} \mathbb{1} & S & b_1 \end{array} \\ \begin{bmatrix} -1 & -1 & . \\ -1 & -1 & . \\ . & . & . \end{bmatrix} \end{array} \tag{62}$$

From the ABK rules we can determine the projection conditions for the massless states in various sectors as,

$$\begin{aligned} C\begin{bmatrix} \mathbb{1} \\ b_1 \end{bmatrix} &= e^{\frac{i\pi}{2}\mathbb{1}\cdot b_1} = -1 \\ C\begin{bmatrix} S \\ b_1 \end{bmatrix} &= e^{\frac{i\pi}{2}S\cdot b_1} = +1 \end{aligned} \tag{63}$$

Now we analyse the typical massless states of the NS sector, $\chi_{1,..,6}\, \bar{\phi}^a_{\frac{1}{2}} \bar{\phi}^b_{\frac{1}{2}} |0\rangle_{NS}$,

$\{\chi^i_{\frac{1}{2}}, y^i_{\frac{1}{2}}\omega^i_{\frac{1}{2}}\}\partial\bar{X}^\mu_1|0\rangle_{NS})$, and $\psi^\mu_{\frac{1}{2}}\bar{\phi}^a_{\frac{1}{2}}\bar{\phi}^b_{\frac{1}{2}}|0\rangle_{NS}$.

Using the fermion number contributions we determine,

$$
\begin{aligned}
e^{i\pi b_1 F_{NS}}\chi^{1,2}_{\frac{1}{2}}|0\rangle_{NS} &= -1 \\
e^{i\pi b_1 F_{NS}}\chi^{3,4}_{\frac{1}{2}}|0\rangle_{NS} &= +1 \\
e^{i\pi b_1 F_{NS}}\chi^{5,6}_{\frac{1}{2}}|0\rangle_{NS} &= +1
\end{aligned} \tag{64}
$$

[27]. Leaving us only with the $\chi^{1,2}_{\frac{1}{2}}$ contribution, expected as this is contained with the b_1 basis vector.

For the next states which are composite states,

$$
\begin{aligned}
\chi_{1,..,6}\,\bar{\phi}^a_{\frac{1}{2}}\bar{\phi}^b_{\frac{1}{2}}|0\rangle_{NS} \\
\psi^\mu_{\frac{1}{2}}\bar{\phi}^a_{\frac{1}{2}}\bar{\phi}^b_{\frac{1}{2}}|0\rangle_{NS}
\end{aligned} \tag{65}
$$

The $\bar{\phi}^a_{\frac{1}{2}}\bar{\phi}^b_{\frac{1}{2}}$ have contributions represented by $\{0,1\}, \{1,0\}, \{1,1\}$ and $\{0,0\}$ corresponding to the following fermion numbers.

$$
\{0,1\} = (-1,+1), \quad \{1,1\} = (+1,+1), \quad \{0,0\} = (-1,-1), \quad \{1,0\} = (+1,-1), \tag{66}
$$

Thus, using these contributions we find that the mixed states are projected out, only leaving the $\bar{\phi}\bar{\phi} \in \{1,1\}, \{0,0\}$ combinations. As a result of this, the left over complex fermions in the two brackets total to 8 and 14 thus the once $SO(44)$ gauge group is now broken down to $SO(16) \otimes SO(28)$.

Following this, we conduct the projections for b_1 on the S sector.

$$
\begin{aligned}
S = &\psi^\mu_{1,2} \quad \chi_{1,2} \quad \chi_{3,4} \quad \chi_{5,6} = +1 \\
&|\pm\rangle \quad |\pm\rangle \quad |\pm\rangle \quad |\pm\rangle
\end{aligned} \tag{67}
$$

Recalling that the S basis vector does not contain $\chi_{3,..,6}$, we consider the gravitino for the supersymmetry analysis

$$\left[\begin{pmatrix}4\\0\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}4\\4\end{pmatrix}\right]\partial\bar{X}^{\nu}_{\frac{1}{2}}|0\rangle \tag{68}$$

The composition of b_1 enforces that we have two Ramond and two Neveau-Schwarz boundary conditions. Hence, we only need to consider the $\begin{pmatrix}4\\2\end{pmatrix}$ contribution from above. After the GGSO, we are able to split this representation as we did in the previous model to determine the count of the gravitino.

$$\begin{pmatrix}4\\2\end{pmatrix}\rightarrow\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\Longleftrightarrow\{\psi^{\mu},\chi_{12},\chi_{34},\chi_{56}\}\rightarrow\{\psi^{\mu},\chi_{12}\}\{\chi_{34},\chi_{56}\} \tag{69}$$

[27]. Showing the breaking of supersymmetry in both combinatorics and in terms of the boundary conditions. The conclusion is therefore drawn that the addition of b_1 breaks the supersymmetry of $\mathcal{N}=4$ to $\mathcal{N}=2$.

For the next state we have,

$$\left[\begin{pmatrix}4\\0\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}4\\4\end{pmatrix}\right]\bar{\phi}^{a}_{\frac{1}{2}}\bar{\phi}^{b}_{\frac{1}{2}}|0\rangle_R \tag{70}$$

For $e^{i\pi b_1\cdot F_s}$, breaking this down as we did before leaves us with two chiral representations, think back to the chirality discussion earlier in the paper, as

$$\left[\begin{pmatrix}2\\0\end{pmatrix}+\begin{pmatrix}2\\2\end{pmatrix}\right]_{+}\quad\text{or}\quad\left[\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}\right]_{-} \tag{71}$$

For projections of $\bar{\phi}^a_{\frac{1}{2}}\bar{\phi}^b_{\frac{1}{2}}$, we find that

$$A : [\{\bar{\psi}^{1,..,5}, \bar{\eta}^1, \bar{y}^{3,..,6}\}\{\bar{\psi}^{1,..,5}, \bar{\eta}^1, \bar{y}^{3,..,6}\}]_+ \quad and \quad [\{\bar{w}^{1,..,6}, \bar{y}^{1,2}, \bar{\eta}^{2,3}, \bar{\phi}^{1,..,8}\}\{\bar{w}^{1,..,6}, \bar{y}^{1,2}, \bar{\eta}^{2,3}, \bar{\phi}^{1,..,8}\}]_+$$
$$B : [\{\bar{\psi}^{1,..,5}, \bar{\eta}^1, \bar{y}^{3,..,6}\}_-\{\bar{w}^{1,..,6}, \bar{y}^{1,2}, \bar{\eta}^{2,3}, \bar{\phi}^{1,..,8}\}_+]_- \tag{72}$$

Matching the results that cancel the positive or negative chirality to give a neutral chiral state leave us with the following invariant states

$$\left[\binom{2}{0} + \binom{2}{2}\right]_+ \otimes B \quad and \quad \left[\binom{2}{1} + \binom{2}{1}\right]_- \otimes A \tag{73}$$

These products of the surviving chiral representations complete the $\mathcal{N} = 2$ supermultiplets. The states complete $\mathcal{N} = 2$ supersymmetry.

The number of supersymmetries are equal to the amount of independent massless gravitinos, which originate from the S sector;

$$|s\rangle_S \otimes \partial X^\mu|0\rangle_R \tag{74}$$

Further projections deduce whether $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry is preserved. In this case, 2 gravitinos survive, and as such, is consistent with $\mathcal{N} = 2$.

4.3.4 New States from b_1

The inclusion of b_1 imlpies that there must be new states in the model. From the definition of the b_1 vector we can deduce that;

$$b_1 : \psi_{1,2} \quad \chi_{1,2} \quad y_{3,4} \quad y_{5,6} \quad \bar{y}_{3,4} \quad \bar{y}_{5,6} \quad \bar{\psi}_{1,..,5} \quad \bar{\eta}_1 \tag{75}$$

where each element can be $|+\rangle$ or $|-\rangle$. Thus, we once again turn to combinatorics to represent the vector using $\left[\begin{pmatrix} 12 \\ i \end{pmatrix}\right]$ where the index $i = 0, 1, .., 12$. Performing GGSOs with $\mathbb{1}$ will tell us which of these states survive,

$$\begin{aligned} Choose: \quad & \delta_{b_i} C \begin{pmatrix} b_i \\ \mathbb{1} \end{pmatrix} = +1 \\ \implies e^{i\pi \mathbb{1}\cdot F_{NS}} |S\rangle_{b_1} &= \delta_{b_i} C \begin{pmatrix} b_i \\ \mathbb{1} \end{pmatrix} |s\rangle_{b_1} = +1 |s\rangle_{b_1} \end{aligned} \tag{76}$$

The projection leaves only the even i's as an even number of negative states is the only way to survive under this projection, thus reducing the i index to now run from $0, 2, 4, .., 12$ [27].

Doing the same for the S sector, under the choice of $\delta_{b_1} C \begin{pmatrix} b_1 \\ S \end{pmatrix} = +1$ yeilds the following;

$$\begin{aligned} S: \quad 1 \qquad 1 \quad 10 \times 0's \\ [\begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}] \begin{pmatrix} 10 \\ even \end{pmatrix} \end{aligned} \tag{77}$$

And for b_1 sector under the choice, $\delta_{b_1} C \begin{pmatrix} b_1 \\ b_1 \end{pmatrix} = +1$;

$$\begin{aligned} S: \quad 1 \qquad 1 \quad 10 \times 1's \\ [\begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}] \begin{pmatrix} 10 \\ even \end{pmatrix} \end{aligned} \tag{78}$$

These are the states left over after the projections have been completed, with $S + b_1$

sector providing the superpartners below

$$S + b_1 : \quad \chi_{3,4} \quad \chi_{5,6} \quad y_{3,4} \quad y_{5,6} \quad \bar{y}_{3,4} \quad \bar{y}_{5,6} \quad \bar{\psi}_{1,\dots,5} \quad \bar{\eta}_1 \tag{79}$$

This completes our addition of the b_1 vector, and furthermore, our chapters on introductory model building. We will now move to bigger, better, and more phenomenologically useful models.

4.4 Reproduction of $SO(10)$

We now introduce the foundational model on which $SO(10)$ is encoded. The NAHE set. The NAHE model builds on the models we have been working with and is expressed fully from five vectors

$$\{\mathbb{1}, S, b_1, b_2, b_3\} \tag{80}$$

[27]. The constituents, like before, define boundary conditions on the torus for worldsheet fermions and their mutual consistency is guaranteed by the modular invariance constraints and the GGSO projections. So despite the models large phenomenological repercussions, there is nothing new here, the model is simply just more expansive.

Our new boundary conditions added to the model are defined by

$$\begin{aligned} b_2 &= \{\psi^\mu, \chi_{3,4}, y_{1,2}, \omega_{5,6} | \bar{y}_{1,2}, \bar{\omega}_{5,6}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^2\} \\ b_3 &= \{\psi^\mu, \chi_{5,6}, \omega_{1,2}, \omega_{3,4} | \bar{\omega}_{1,2}, \bar{\omega}_{3,4}, \bar{\psi}^{1,\dots,5}, \bar{\eta}^3\} \end{aligned} \tag{81}$$

[27]. These vectors, along with the previous $\mathbb{1}, S, b_1$ construct a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold-like structure with a hidden permutation symmetry among b_1, b_2 and b_3.

4.4.1 ABK Rules of NAHE

As states, there is nothing new to the procedures here, hence we know that the first step is to show ABK rule compliance.

$$
\begin{aligned}
& b_1 \cdot \mathbb{1} = -4 = 0 \mod 4 \\
& b_2 \cdot b_1 = -4 = 0 \mod 4 \\
& N_{b_2} b_2 \cdot b_2 = -8 = 0 \mod 8 \\
& N_{b_2} b_2 \cdot S = 4 = 0 \mod 4 \\
& b_3 \cdot S = 2 \\
& b_3 \cdot b_2 = -4 = 0 \mod 4 \\
& b_3 \cdot b_3 = -8 = 0 \mod 4 \\
& b_3 \cdot b_1 = -8 = 0 \mod 4 \\
& b_3 \cdot \mathbb{1} = -4 = 0 \mod 4
\end{aligned} \tag{82}
$$

Thus confirming the addition of b_2 and b_3 is allowed by ABK rules and their is torus worldsheet modular invariance.

4.4.2 GGSOs of NAHE

Due to the larger nature of the NAHE set is easier to display the GGSO phases in the matrix as we have done previously,

$$
\begin{array}{c} \\ \mathbb{1} \\ S \\ b_1 \\ b_2 \\ b_3 \end{array}
\begin{array}{c} \begin{array}{ccccc} \mathbb{1} & S & b_1 & b_2 & b_3 \end{array} \\
\left[\begin{array}{ccccc}
-1 & +1 & -1 & \pm 1 & \pm 1 \\
+1 & +1 & +1 & \pm 1 & \pm 1 \\
-1 & +1 & -1 & \pm 1 & \pm 1 \\
\pm 1 & \mp 1 & \pm 1 & \pm 1 & \pm 1 \\
\pm 1 & \mp 1 & \pm 1 & \pm 1 & \pm 1
\end{array}\right] \end{array} \tag{83}
$$

They're are six new GGSO phases that have arisen because of the new basis vectors.

4.4.3 Massless Spectrum

NS Sector

The vector bosons from $\psi^{\mu}\bar{\phi}^{a}\bar{\phi}^{b}|0>_{NS}$ are now broken by the GGSO projections into,

$$\begin{array}{cccccc} \psi^{\mu}_{\frac{1}{2}} & \{\overline{\psi}_{1,\ldots,5}\overline{\psi}_{1,\ldots,5}\} & \{\overline{\eta}_1\overline{y}_{3,\ldots,6}\} & \{\overline{\eta}_2\overline{y}_{1,2}\overline{\omega}_{5,6}\} & \{\overline{\eta}_3\overline{\omega}_{1,\ldots,4}\} & \{\overline{\phi}_{1,\ldots,8}\} \\ & SO(10) & SO(6)_1 & SO(6)_2 & SO(6)_3 & SO(16) \end{array} \tag{84}$$

The fermionic states that were in $(\chi_1, y_i, \omega_i)\partial\bar{X}^{\mu}|0\rangle_{NS}$ are now projected out.

Recalling the states we had from the b_1 GGSO projections,

$$\begin{array}{l} \chi_{1,2}\left\{\overline{w}_{1,\ldots,6}, \overline{y}_{1,2}, \overline{\eta}_2, \overline{\eta}_3, \overline{\phi}_{1,\ldots,8}\right\}\left\{\overline{w}_{1,\ldots,6}, \overline{y}_{1,2}, \overline{\eta}_2, \overline{\eta}_3, \overline{\phi}_{1,\ldots,8}\right\} \\ \chi_{1,2}\left\{\overline{\psi}_{1,\ldots,5}, \overline{\eta}_1, \overline{y}_{3,4}, \overline{y}_{5,6}\right\}\left\{\overline{\psi}_{1,\ldots,5}, \overline{\eta}_1, \overline{y}_{3,4}, \overline{y}_{5,6}\right\} \\ \chi_{34,56}\left\{\overline{\psi}_{1,\ldots,5}, \overline{\eta}_1, \overline{y}_{3,4}, \overline{y}_{5,6}\right\}\left\{\overline{w}_{1,\ldots,6}, \overline{y}_{1,2}, \overline{\eta}_2, \overline{\eta}_3, \overline{\phi}_{1,\ldots,8}\right\} \end{array} \tag{85}$$

After the b_2 and b_3 projections we are left with,

$$\begin{array}{l} \chi_{1,2}\left\{\overline{\eta}_2, \overline{w}_{5,6}, \overline{y}_{1,2}\right\}\left\{\overline{\eta}_3, \overline{w}_{1,\ldots,4}, \overline{\phi}_{1,\ldots,8}\right\} \\ \chi_{1,2}\left\{\psi_{1,\ldots,5}\right\}\left\{\overline{\eta}_1, \overline{y}_{3,\ldots,6}\right\} \\ \\ \chi_{3,4}\left\{\psi_{1,\ldots,5}\right\}\left\{w_{5,6}, y_{1,2}, \overline{\eta}_2\right\} \\ \chi_{3,4}\left\{\overline{\eta}_1, \overline{y}_{3,\ldots,6}\right\}\left\{w_{1,\ldots,4}, \overline{\eta}_3, \overline{\phi}_{1,\ldots,8}\right\} \\ \\ \chi_{5,6}\left\{\psi_{1,\ldots,5}\right\}\left\{w_{1,\ldots,4}, \overline{\eta}_3\right\} \\ \chi_{5,6}\left\{\overline{\eta}_1, y_{3,\ldots,6}\right\}\left\{w_{5,6}, y_{1,2}, \overline{\eta}_2\right\} \end{array} \tag{86}$$

This is the NS sector completed [27].

S Sector

Reminding ourselves that the S sector is important for the gravitino count, we start from the same point as we did with b_1.

$$\begin{array}{cccccc} S: & \{\phi^{\mu}_{\frac{1}{2}} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6}\} & \partial\bar{X}^{\mu} \\ & |\pm\rangle & |\pm\rangle & |\pm\rangle & |\pm\rangle & \end{array} \tag{87}$$

In the $\mathbb{1}$ sector we have the content;

$$\begin{array}{cccccc} \mathbb{1}: & \{\phi^{\mu}_{\frac{1}{2}} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6}\} & \partial\bar{X}^{\mu} \\ & 1 & 1 & 1 & 1 & \end{array} \tag{88}$$

Thus in combinatorics, it follows we must have;

$$S: \left[\binom{4}{1} + \binom{4}{3}\right] \tag{89}$$

Thus S sector doesn't change. Now we elaborate for b_1;

$$\begin{array}{cccccc} & \psi^{\mu}_{\frac{1}{2}} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6} & \partial\bar{X}^{\mu} \\ b_1 & 1 & 1 & 0 & 0 & \end{array} \tag{90}$$

which gives

$$\binom{2}{1}\left[\binom{2}{0} + \binom{2}{2}\right] \tag{91}$$

Now for b_2 and b_3;

$$\begin{array}{cccccc} & \psi^\mu_{\frac{1}{2}} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6} & \partial\bar{X}^\mu \\ b_2 & 1 & 0 & 1 & 0 & \\ b_3 & 1 & 0 & 0 & 1 & \end{array} \tag{92}$$

Which both give;

$$b_2 : \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{93}$$

$$b_3 : \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{94}$$

The combinatorics on b_2 show us that we have one gravitino remaining, implying the $\mathcal{N} = 1$ supersymmetry we desire. Though the choice on phase in the b_3 projections makes a decision on whether this is gravitino is kept or not. Setting $C \begin{pmatrix} S \\ b_3 \end{pmatrix} = +1$ keeps the gravitino, but setting it to -1 will project it out.

However, as our goal with this is purely phenomenological, we want to keep the gravitino to preserve the $\mathcal{N} = 1$ supersymmetry to match the standard model. That is not to say $\mathcal{N} = 0$ aren't explored and constructed, just like the extended supersymmetry models ($\mathcal{N} > 2$), its just that their physicality is debated as the cosmological constant is not 0 [56].

b_1 Sector

The b_1 sector is where we see the spinorial 16 of the standard model arise. Using our combinatorial representations we have the states, $\mathbb{1}$:

$$\begin{array}{ll} b_1 : & \{\psi^\mu_{1,2}, \quad \chi_{12}, y_{3,4}, y_{5,6} | \bar{y}_{3,4}, \bar{y}_{5,6}, \bar{\psi}^\mu_{1,\ldots,5}, \bar{\eta}_1\} \\ \mathbb{1} : & \begin{pmatrix} 12 \\ even \end{pmatrix} \end{array} \tag{95}$$

S:

$$\begin{aligned} b_1 : &\quad \{\psi^{\mu}_{1,2}, \quad \chi_{12}, y_{3,4}, y_{5,6} | \bar{y}_{3,4}, \bar{y}_{5,6}, \bar{\psi}^{\mu}_{1,...,5}, \bar{\eta}_1\} \\ S : &\quad \begin{pmatrix} 12 \\ even \end{pmatrix} \quad [\begin{pmatrix} 10 \\ even \end{pmatrix}] \end{aligned} \tag{96}$$

b_2

$$\begin{aligned} b_1 : &\quad \{\psi^{\mu}_{1,2}, \quad \chi_{12}, y_{3,4}, y_{5,6} | \bar{y}_{34}, \bar{y}_{5,6}, \bar{\psi}^{\mu}_{1,...,5}, \bar{\eta}_1\} \\ b_2 : &\quad \quad 1 \quad\quad\quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad\quad 1 \quad\quad 0 \end{aligned} \tag{97}$$

Which we can express as;

$$\begin{aligned} &\begin{pmatrix} 2 \\ 0 \end{pmatrix} [\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix}][\begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix}] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 2 \\ 0 \end{pmatrix} [\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}][\begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix}]... \end{aligned} \tag{98}$$

The ellipse representing other components coming from CPT invariance, ensuring that we are complying with what is seen in the standard model.

These states have physical interpretations that draw relations to the $SO(10)$ representation of the standard model.

$$[\begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix}] \implies 16 \quad \text{generations of SO(10)} \tag{99}$$

and by deduction that makes the odd states the $\bar{16}$ representation.

$$[\begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \end{pmatrix}] \implies \bar{16} \quad \text{generations of SO(10)} \tag{100}$$

However, as we choose one chiral representation for the spectrum, we only have the

16 representation present.

For b_3, we get an identical result as for b_2. Furthermore, like above, the projection or retention of generations from b_1 is dependent on the choice of phase from $C\begin{pmatrix} b_1 \\ b_3 \end{pmatrix}$ and $C\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. In the event that we choose them to be equal, then we have 16 generations from b_1.

For both b_2 and b_3, we get another 16 generations from either, as their results are congruent to b_1. Thus, totalling 48 generations across the model.

4.4.4 Permutation Symmetry

b_1, b_2 and b_3 exhibit a permutation symmetry.

$$b_1 \rightarrow b_2 \rightarrow b_3 \rightarrow b_1 \tag{101}$$

This symmetry cyclically permutes the internal coordinates of the vectors and thus plays a role in the geometrical representation of orbifold points. Though this is not what we are interested in, it is worth a mention. More can be found in [66].

4.4.5 Remaining Sectors

As we know, all sectors of our Ξ group need to be considered. Hence, the sectors

$$\begin{gathered} Scalars \begin{cases} S + b_1 \\ S + b_2 \\ S + b_3 \end{cases} \\ Periodic \left\{ \mathbb{1} + b_1 + b_2 + b_3, \{\phi_{1,\ldots,8}\} \right. \end{gathered} \tag{102}$$

also need to be considered.

We can simplify the spin structure calculations by defining $\xi = \mathbb{1} + b_1 + b_2 + b_3$. Hence, we have the following

$$C\begin{pmatrix}\xi\\ \mathbb{1}\end{pmatrix}, C\begin{pmatrix}\xi\\ S\end{pmatrix}, C\begin{pmatrix}\xi\\ b_1\end{pmatrix}, C\begin{pmatrix}\xi\\ b_2\end{pmatrix}, C\begin{pmatrix}\xi\\ b_3\end{pmatrix} \tag{103}$$

Thus for the first phase,

$$\begin{aligned} C\begin{pmatrix}\xi\\ \mathbb{1}\end{pmatrix} &= C\begin{pmatrix}\mathbb{1} + b_1 + b_2 + b_3\\ \mathbb{1}\end{pmatrix} \\ &= e^{\frac{i\pi}{2}\mathbb{1}\cdot\xi} C\begin{pmatrix}\mathbb{1}\\ \xi\end{pmatrix}^* \end{aligned} \tag{104}$$

Then using $\mathbb{1}\cdot\xi = -8$, it is found that

$$\begin{aligned} C\begin{pmatrix}\mathbb{1}\\ \xi\end{pmatrix} &= \delta_{\mathbb{1}} C\begin{pmatrix}\mathbb{1}\\ \mathbb{1}\end{pmatrix} C\begin{pmatrix}\mathbb{1}\\ b_1 + b_2 + b_3\end{pmatrix} \\ &= \delta_{\mathbb{1}}^2 C\begin{pmatrix}\mathbb{1}\\ \mathbb{1}\end{pmatrix} C\begin{pmatrix}\mathbb{1}\\ b_1\end{pmatrix} C\begin{pmatrix}\mathbb{1}\\ b_2\end{pmatrix} C\begin{pmatrix}\mathbb{1}\\ b_3\end{pmatrix} \\ &\implies \delta_{\mathbb{1}}^3 C\begin{pmatrix}\mathbb{1}\\ \mathbb{1}\end{pmatrix} C\begin{pmatrix}\mathbb{1}\\ b_1\end{pmatrix} C\begin{pmatrix}\mathbb{1}\\ b_2\end{pmatrix} C\begin{pmatrix}\mathbb{1}\\ b_3\end{pmatrix} \end{aligned} \tag{105}$$

[27]. Now we apply the same technique as we have done before, looking at each of their sign contributions individually to decipher if we have a overall negative or positive contribution. As we have a negative contribution from each term and five terms then,

$$C\begin{pmatrix}\mathbb{1}\\ \xi\end{pmatrix} = -1 \tag{106}$$

Thus, allowing us to perform the GGSOs for the sector.

- S:

$$C\begin{pmatrix}\xi\\ S\end{pmatrix} = e^{\frac{i\pi}{2}S\cdot\xi}C\begin{pmatrix}S\\ \xi\end{pmatrix}$$
$$= \delta_{\mathbb{1}}^{3}C\begin{pmatrix}S\\ 1\end{pmatrix}C\begin{pmatrix}S\\ b_1\end{pmatrix}C\begin{pmatrix}S\\ b_2\end{pmatrix}C\begin{pmatrix}S\\ b_3\end{pmatrix} = -1 \tag{107}$$

- b_1

$$C\begin{pmatrix}\xi\\ S\end{pmatrix} = e^{\frac{i\pi}{2}\xi\cdot b_1}C\begin{pmatrix}b_1\\ \xi\end{pmatrix}$$
$$= \delta_{\mathbb{1}}^{3}C\begin{pmatrix}b_1\\ 1\end{pmatrix}C\begin{pmatrix}b_1\\ b_1\end{pmatrix}C\begin{pmatrix}b_1\\ b_2\end{pmatrix}C\begin{pmatrix}b_1\\ b_3\end{pmatrix} = -1 \tag{108}$$

and this implies that similarly $C\begin{pmatrix}\xi\\ b_2\end{pmatrix} = -1$ and $C\begin{pmatrix}\xi\\ b_3\end{pmatrix} = -1$.

Using these results, we find that for the $\mathbb{1}$ sector we have,

$$\begin{array}{ccc} \xi & & \bar{\phi}_1\ldots\bar{\phi}_8 \\ & \psi^\mu & |\pm\rangle\ldots|\pm\rangle \\ \mathbb{1} & \mathbb{1} & 1\ldots\quad 1 \end{array} \tag{109}$$

Deploying combinatorics for the $\phi_{1,...,8}$.

$$
\begin{array}{ccc}
 & \psi^{\mu} & [\binom{8}{0} + \binom{8}{2} + \ldots \binom{8}{8}] \\
\mathbb{1} & 1 & \vec{0} \\
S & 1 & \vec{0} \\
b_1 & 1 & \vec{0} \\
b_2 & 1 & \vec{0} \\
b_3 & 1 & \vec{0}
\end{array}
\tag{110}
$$

It can be said then, that these states transform as $2^7 = 128$ in the gauge group $SO(16) \times E_8$ [27].

For sheer completeness, we quickly state the scalar sectors,

$$
\begin{array}{ccc}
 & \{\chi, y, \omega\} & |\pm\rangle \ldots |\pm\rangle \\
\mathbb{1} & \{\} & [\binom{8}{0} + \binom{8}{2} + \ldots \binom{8}{8}] \\
S & & \\
b_1 & \chi_{1,..,6} & (\qquad) \\
b_2 & \chi_{1,2} & (\qquad) \\
b_3 & projected & out \\
 & projected & out
\end{array}
\tag{111}
$$

4.4.6 Result

As a closing statement to this section, we have shown that NAHE set provides a phenomenologically rich starting point for string model building. Correctly reproducing the observable $SO(10)$ gauge group of the standard model with 48 generations. Along, with $\mathcal{N} = 1$ supersymmetry due to the presence of a single gravitino with the correct choice of phase. As explained in the first chapter, breaking down the $SO(10)$ is how we obtain all kinds of physical models.

5 Asymmetric S-Model Pati-Salam

We have shown and demonstrated that models have strong phenomenological implications; however, analysis of string states is not the only thing the free fermionic models can be used for. Providing a string perspective on certain mechanisms is also a use of these models. In particular interest to us, we will look at an asymmetric model and its implications on a specific mechanism called doublet-triplet splitting.

5.1 Motivation for Asymmetry

Unfortunately, none of the supersymmetric partners we have talked about have been observed in any experiment currently, and despite the theoretical nature of this subject [56], we do have to pay attention to what happens in experiments for string phenomenology. As more data is released, it is continuously more likely that supersymmetry is not the solution to the phenomenological problems it aims to solve. Hence, with this in mind, the construction of non-supersymmetric models becomes more attractive. In this sense referring to a non-supersymmetric model, we mean non-supersymmetric in spacetime and that each left-mover does not necessarily have a right- moving counter part in the definition of a vector. We can see numerous examples of this in Figure 5.1.

Considering the symmetry of the model has consequences when considering the doublet-triplet splitting in relation to Pati-Salam models, which we will see shortly.

	$y^3\bar{y}^3$	$y^4\bar{y}^4$	$y^5\bar{y}^5$	$y^6\bar{y}^6$	$y^1\bar{y}^1$	$y^2\bar{y}^2$	$\omega^5\bar{\omega}^5$	$\omega^6\bar{\omega}^6$	$\omega^2\omega^3$	$\omega^1\bar{\omega}^1$	$\omega^4\bar{\omega}^4$	$\bar{\omega}^2\bar{\omega}^3$
α	1	0	0	1	0	0	1	0	0	0	1	1
β	0	0	0	1	0	1	1	0	0	1	0	1
γ	1	1	0	0	1	1	0	0	0	0	0	1

Figure 5.1: Symmetric conditions are those which have a left mover with a corresponding right mover, e.g $y_3\bar{y}_3$, where as an asymmetric condition is not matched together. [67]

5.2 Doublet-Triplet Splitting

Doublet-triplet splitting is a central challenge deeply rooted in the GUTs that the models represent. The standard model is strongly supported by experimental data at the electroweak scale, and thus the possibility of it existing at higher energy scales is explored through the GUTs. Hence, it is explored through string theory, too.

However, it is a well-known criticism of the GUTs that proton decay is predicted. For instance in the case of $SU(5) \otimes U(1)$,

$$\begin{aligned} &SO(10) \rightarrow SU(5) \otimes U(1) \rightarrow SU(3) \otimes SU(2) \otimes U(1) \otimes U(1) \\ &\mathbf{10} \rightarrow \mathbf{5}_{-1} + \bar{\mathbf{5}}_{+1} \rightarrow (\mathbf{3}, -\mathbf{1}, \mathbf{1}, \mathbf{0}) + (\mathbf{1}, \mathbf{0}, \mathbf{2}, -\mathbf{1}) + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{0}) + (\mathbf{1}, \mathbf{0}, \mathbf{2}, \mathbf{1}) \end{aligned} \tag{1}$$

where we see a triplet mediating proton decay and the Higgs doublet. We note that the $\bar{\mathbf{5}} + \mathbf{5}$ representation corresponds to the following,

$$\bar{\mathbf{5}}_{+1} + \mathbf{5}_{-\mathbf{1}} \rightarrow \bar{\psi}^{1,..,5}|0\rangle_{NS} + \bar{\psi}^{*1,..,5}|0\rangle_{NS} \tag{2}$$

Proton decay, as we alluded to in the first section, is something to be avoided as we do not see it in nature. Thus, in the GUT scale the problem is approached by requiring the triplet to be heavy, existing at the GUT scale $10^{16}GeV$, and making the doublet light, existing at the electroweak scale. In strings, we look to remove the triplet entirely. For more on doublet-splitting in relation to heterotic string models see [68] and [67].

5.3 From NAHE to Pati-Salam

Our starting point, as for many modern realistic models, is the previously crafted NAHE set.

$$\begin{aligned}
\mathbb{1} &= \{\psi^{\mu}, \chi^{1,..,6}, y^{1,..,6}, \omega^{1,..,6} | \bar{y}^{1,..,6} \bar{\omega}^{1,..,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,..,5}, \bar{\phi}^{1,..,8}\} \\
S &= \{\psi^{\mu}, \chi^{1,..,6}\} \\
b_1 &= \{\psi^{\mu}, \chi^{12}, y^{3,4}, y^{5,6} | \bar{y}^{3,4}, \bar{y}^{5,6}, \bar{\psi}^{1,...,5}, \bar{\eta}^{1}\} \\
b_2 &= \{\psi^{\mu}, \chi^{34}, y^{12}, \omega^{56} | \bar{y}^{1,2}, \bar{\omega}^{56}, \bar{\psi}^{1,...,5}, \bar{\eta}^{2}\} \\
b_3 &= \{\psi^{\mu}, \chi^{56}, \omega^{12}, \omega^{34} | \bar{\omega}^{1,2}, \bar{\omega}^{34}, \bar{\psi}^{1,...,5}, \bar{\eta}^{3}\}
\end{aligned} \tag{3}$$

There are two prominent techniques for creating an asymmetric model in modern free-fermionic formulation. The first of which being replacing the supersymmetric inducing basis vector S with the following;

$$\tilde{S} = \{\psi^{\mu}, \chi^{1,..,6} | \phi^{3,4,5,6}\} \tag{4}$$

$\tilde{S}$ models are explored extensively in [56] and in [69]. The second and the technique we are exploring in this paper is introducing further vectors. In our case, we will explore the addition of the γ vector to the model.

5.3.1 The γ Vector

The technique of using this basis vector for the building of asymmetric models was introduced for a flipped $SU(5)$ case in [70] and allows you to create a antisymmetric model while retaining the S basis vector. The vector is comprised of three sets of conditions;

1. Conditions on internal fermions.
2. Conditions on hidden fermions.
3. Conditions to cause SO(10) symmetry breaking.

5.3.2 General form of γ for $SO(6)$ x $SO(4)$

The reference [70], displays the γ vector in a general form by taking the first and second set of conditions then calling them A and B respectively. The third condition, is simply stated in the representation.

Similarly, we can create two general forms for a γ vector using this representation for the $SO(6)$ x $SO(4)$ case.

$$\begin{aligned}\gamma &= A + \{\bar{\psi}^{1,2,3} = 0, \bar{\psi}^{4,5} = 1\} + B \\ \gamma &= A + \{\bar{\psi}^{1,2,3} = 1, \bar{\psi}^{4,5} = 0\} + B\end{aligned} \tag{5}$$

The conditions imposed on by A and B need to be complaint with the ABK rules, like any other basis vector, in order to preserve modular invariance. Hence, the choice of γ vector must satisfy the following

$$\begin{aligned}N_\gamma \gamma \cdot \gamma &= 0 \quad mod \quad 8 \\ N_{z_1\gamma} z_1 \cdot \gamma &= 0 \quad mod \quad 4\end{aligned} \tag{6}$$

where N_γ is the smallest possible integer for $N_\gamma \gamma = 0$ with $N_{z_1\gamma}$ being the lowest common multiple of N_γ and N_{z_1}.

In addition to the modular invariance rules, A needs to also comply with supercurrent constraints. Which we discussed earlier and split between bosonic and fermionic cases. The former of which is as follows;

$$A = \{A(y^{1,..,6}), A(\omega^{1,..,6}) | A(\bar{y}^{1,..,6}), A(\bar{\omega}^{1,..,6})\} \tag{7}$$

which from the worldsheet supercurrent,

$$T_f(z) = i\psi^\mu \partial X^\mu(z) + i \sum_{I=1}^{6} \chi^I y^I \omega^I \tag{8}$$

imposes the constraint that if you choose a y^I you must intern choose a matching ω^I.

The condition can be expressed as,

$$(y^I, \omega^I) = (0,0) \quad or \quad (1,1), \\ for \quad I = 1,..,6 \tag{9}$$

[70]. In the second sense, when γ is fermionic, we include ψ^μ and a choice of $\chi^{1,2}$,$\chi^{3,4}$ or $\chi^{5,6}$ in the conditions imposed by A.

$$A = \{A(\psi^\mu), A(\chi^{1,2}), A(y^{1,..,6}), A(\omega^{1,..,6}) | A(\bar{y}^{1,..,6}), A(\bar{\omega}^{1,..,6})\} \tag{10}$$

Following this the mandatory consistency with the supercurrent for the fermionic case imposes,

$$(y^I, \omega^I) = \begin{cases} (1,1) \quad or \quad (0,0) \quad for \quad I = 1,2 \\ (0,1) \quad or \quad (1,0) \quad for \quad I = 3,..,6 \end{cases} \tag{11}$$

Alongside these conditions, we must note our choices for the central term, $\{\bar{\psi}^{1,2,3} = 0, \bar{\psi}^{4,5} = 1\}$ and $\{\bar{\psi}^{1,2,3} = 1, \bar{\psi}^{4,5} = 0\}$. Although the choices are different, their main purpose is the same, to break $SO(10)$ to $SO(6)$ x $SO(4)$.

5.3.3 Integrating the γ Vector into the Model

As previously discussed, the γ vector is based on a series of choices of boundary conditions. Hence, we need to make those choices in such a way that satisfies the modular invariance rules. Firstly, we state our proposed model that we will add γ

to.

$$
\begin{aligned}
\mathbb{1} &= \{\psi^\mu, \chi^{1,..,6}, y^{1,..,6}, \omega^{1,..,6} | \bar{y}^{1,..,6} \bar{\omega}^{1,..,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,..,5}, \bar{\phi}^{1,..,8}\} \\
S &= \{\psi^\mu, \chi^{1,..,6}\} \\
b_1 &= \{\psi^\mu, \chi^{12}, y^{3,4}, y^{5,6} | \bar{y}^{3,4}, \bar{y}^{5,6}, \bar{\psi}^{1,...,5}, \bar{\eta}^1\} \\
b_2 &= \{\psi^\mu, \chi^{34}, y^{12}, \omega^{56} | \bar{y}^{1,2}, \bar{\omega}^{56}, \bar{\psi}^{1,...,5}, \bar{\eta}^2\} \\
b_3 &= \{\psi^\mu, \chi^{56}, \omega^{12}, \omega^{34} | \bar{\omega}^{1,2}, \bar{\omega}^{34}, \bar{\psi}^{1,...,5}, \bar{\eta}^3\} \\
z_1 &= \{\phi^{1,..4}\} \\
e_i &= \{y^i, \omega^i | \bar{y}^i, \bar{\omega}^i\} \quad i = 1,..,6
\end{aligned}
\tag{12}
$$

Note the similarity to the NAHE set and the further similarity to the $\tilde{S}$-models in [71] and [69]. The difference being is this is an S-model and our asymmetry will be induced by the addition of the aforementioned γ vector.

The inclusion of z_1 and e_i is done for the main purpose of enlarging the space for which the model is scanned. Alongside this, the z_1 breaks the hidden gauge group and the e_i are fermionised internal coordinates on the 6-dimensional torus allowing for symmetric shifts of the six internal circles. However, the inclusion of each e_i is not mandatory and is decided by the choice of γ due to the ABK rules. We also note the existence of a z_2 basis vector,

$$
z_2 = \mathbb{1} + \sum_{i=1}^{6} e_i + \sum_{k=1}^{3} b_k + z_1 = \{\phi^{5,..,8}\} \tag{13}
$$

[71] which is included in alternate Pati-Salam models instead of b_3.

Now we will demonstrate a γ vector model, along with its moduli projection.

A γ Vector Model Example

$$
\begin{aligned}
\mathbb{1} &= \{\psi^{\mu}, \chi^{1,..,6}, y^{1,..,6}, \omega^{1,..,6} | \bar{y}^{1,..,6} \bar{\omega}^{1,..,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,..,5}, \bar{\phi}^{1,..,8}\} \\
S &= \{\psi^{\mu}, \chi^{1,..,6}\} \\
b_1 &= \{\psi^{\mu}, \chi^{12}, y^{3,4}, y^{5,6} | \bar{y}^{3,4}, \bar{y}^{5,6}, \bar{\psi}^{1,...,5}, \bar{\eta}^{1}\} \\
b_2 &= \{\psi^{\mu}, \chi^{34}, y^{12}, \omega^{56} | \bar{y}^{1,2}, \bar{\omega}^{56}, \bar{\psi}^{1,...,5}, \bar{\eta}^{2}\} \\
b_3 &= \{\psi^{\mu}, \chi^{56}, \omega^{12}, \omega^{34} | \bar{\omega}^{1,2}, \bar{\omega}^{34}, \bar{\psi}^{1,...,5}, \bar{\eta}^{3}\} \\
z_1 &= \{\phi^{1,..4}\} \\
e_i &= \{y^{i}, \omega^{i} | \bar{y}^{i}, \bar{\omega}^{i}\} \quad i = 1,..,6 \\
\gamma &= \{y^{1,..,6}, \omega^{1,...6} | \bar{\psi}^{4,5}\}
\end{aligned}
\tag{14}
$$

Firstly, we notice that $A = \{y^{1,..,6}, \omega^{1,..,6}\}$, satisfying the supercurrent constraint and there is no constraints coming from B in this case. Now we check the modular invariance rules by projecting each basis vector on γ using the basis vector techniques earlier described.

$$
\begin{aligned}
&\mathbb{1}\cdot\gamma\\
&\quad \mathbb{1}:\{\cancel{\psi^{\mu}},\cancel{\chi^{1,..,6}},y^{1,..,6},\omega^{1,..,6}|\cancel{\bar{y}^{1,..,6}}\cancel{\bar{\omega}^{1,..,6}},\cancel{\bar{\eta}^{1,2,3}},\bar{\psi}^{1,..,5},\cancel{\bar{\phi}^{1,..,8}}\}\\
&\quad \gamma:\{y^{1,..,6},\omega^{1,...6}|\bar{\psi}^{4,5}\}\\
&\qquad =0 \quad mod \quad 2\\
&S\cdot\gamma\\
&\quad S:\{\cancel{\psi^{\mu}},\cancel{\chi^{1,..,6}}\}\\
&\quad \gamma:\{\cancel{y^{1,..,6}},\cancel{\omega^{1,...6}}|\cancel{\bar{\psi}^{4,5}}\}\\
&\qquad =0 \quad mod \quad 2\\
&b_1\cdot\gamma\\
&\quad b_1:\{\cancel{\psi^{\mu}},\cancel{\chi^{12}},y^{3,4},y^{5,6}|\cancel{\bar{y}^{3,4}},\cancel{\bar{y}^{5,6}},\bar{\psi}^{1,...,5},\cancel{\bar{\eta}^{1}}\}\\
&\quad \gamma:\{y^{1,..,6},\cancel{\omega^{1,...6}}|\bar{\psi}^{4,5}\}\\
&\qquad =0 \quad mod \quad 2\\
&b_2\cdot\gamma\\
&\quad b_2:\{\cancel{\psi^{\mu}},\cancel{\chi^{34}},y^{12},\omega^{56}|\cancel{\bar{y}^{1,2}},\cancel{\bar{\omega}^{56}},\bar{\psi}^{1,...,5},\cancel{\bar{\eta}^{2}}\}\\
&\quad \gamma:\{y^{1,..,6},\omega^{1,...6}|\bar{\psi}^{4,5}\}\\
&\qquad =0 \quad mod \quad 2\\
&b_3\cdot\gamma\\
&\quad b_3:\{\cancel{\psi^{\mu}},\cancel{\chi^{56}},\omega^{12},\omega^{34}|\cancel{\bar{\omega}^{1,2}},\cancel{\bar{\omega}^{34}},\bar{\psi}^{1,...,5},\cancel{\bar{\eta}^{3}}\}\\
&\quad \gamma:\{\cancel{y^{1,..,6}},\omega^{1,...6}|\bar{\psi}^{4,5}\}\\
&\qquad =0 \quad mod \quad 2\\
&z_1\cdot\gamma\\
&\quad z_1:\{\cancel{\phi^{1,..4}}\}\\
&\quad \gamma:\{\cancel{y^{1,..,6}},\cancel{\omega^{1,...6}}|\cancel{\bar{\psi}^{4,5}}\}\\
&\qquad =0 \quad mod \quad 2
\end{aligned}
\tag{15}
$$

After these projections, which ensure γ compatibility with the model, we project the e_i vectors to observe which of the them will also be present in the model.

$$\begin{aligned} & e_i \cdot \gamma \\ & \quad e_i : \{y^i, \omega^i | \not{\bar{y}}^i, \not{\bar{\omega}}^i\} \\ & \quad \gamma : \{y^{1,..,6}, \omega^{1,...6} | \not{\bar{\psi}}^{4,5}\} \\ & \quad \neq 0 \quad mod \quad 2 \quad for \quad any \quad i \end{aligned} \tag{16}$$

As displayed, none of the symmetric shift vectors are compatible in the case. Hence, none will be present in this model. For finality this model is therefore;

$$\begin{aligned} \mathbb{1} &= \{\psi^\mu, \chi^{1,..,6}, y^{1,..,6}, \omega^{1,..,6} | \bar{y}^{1,..,6} \bar{\omega}^{1,..,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,..,5}, \bar{\phi}^{1,..,8}\} \\ S &= \{\psi^\mu, \chi^{1,..,6}\} \\ b_1 &= \{\psi^\mu, \chi^{12}, y^{3,4}, y^{5,6} | \bar{y}^{3,4}, \bar{y}^{5,6}, \bar{\psi}^{1,...,5}, \bar{\eta}^1\} \\ b_2 &= \{\psi^\mu, \chi^{34}, y^{12}, \omega^{56} | \bar{y}^{1,2}, \bar{\omega}^{56}, \bar{\psi}^{1,...,5}, \bar{\eta}^2\} \\ b_3 &= \{\psi^\mu, \chi^{56}, \omega^{12}, \omega^{34} | \bar{\omega}^{1,2}, \bar{\omega}^{34}, \bar{\psi}^{1,...,5}, \bar{\eta}^3\} \\ z_1 &= \{\phi^{1,..4}\} \\ \gamma &= \{y^{1,..,6}, \omega^{1,...6} | \bar{\psi}^{4,5}\} \end{aligned} \tag{17}$$

Following this we conduct an analysis of the model with respect to its retained and projected moduli. A retained moduli is defined by a projection with $\{y^i\omega^i|\bar{y}^j\bar{\omega}^j\}$ with the following result,

$$\{y^i\omega^i|\bar{y}^j\bar{\omega}^j\} = 0 \quad mod \quad 2 \tag{18}$$

A projected moduli is similarly defined as;

$$\{y^i\omega^i|\bar{y}^j\bar{\omega}^j\} = 1 \quad mod \quad 2 \tag{19}$$

[70] and by the same process as earlier we see that all moduli are retained.

As usual with any model, the Virasoro condition and GGSOs can be applied here to analyse the phenomenological content, but as we have done this thrice before and are familiar with the process, we will avoid doing it again as the group Ξ will be very lengthy. It is not the focus of why we have introduced this model, rather to learn more about doublet-triplet splitting.

5.4 Doublet-Triplet Splitting Mechanism on the Model

In the Pati-Salam case, the doublet-triplet splitting takes a special place in the model. According to [68], a way to ensure that proton decay will be avoided in an asymmetric model is derived from the GGSO projection condition.

$$\left(e^{i\pi b_i \cdot F_\alpha} - \delta_\alpha C \begin{pmatrix} \alpha \\ b_i \end{pmatrix}\right) |s\rangle = 0 \tag{20}$$

Selection rules for the basis vector that breaks $SO(10) \to SO(6) \otimes SO(4)$ is found. If $|\alpha_L(b_j) - \alpha_R(b_j)| = 0$ then the electroweak doublets are projected out, leaving the colour triplets in the spectrum, which holds for symmetric models. If $|\alpha_L(b_j) - \alpha_R(b_j)| = 1$ the triplets are projected, and the electroweak doublets remain in the spectrum. This is a desirable result to avoid proton decay.

However, for the constructed model, we have shown that this condition is broken and needs to be amended. Providing a counter-example and thus defining a new set of conditions for asymmetric Pati-Salam string models to project the proton decaying triplet state. Hence, we correct the doublet-selection rules for the SO(10) to Pati-Salam breaking vector;

$$|\alpha_L(b_j) - \alpha_R(b_j)| = \begin{cases} 0_S \implies Triplet \quad Retained = Proton \quad Decay \\ 1_A \implies Triplet \quad Projected \neq Proton \quad Decay \\ 2_A \implies Triplet \quad Projected \neq Proton \quad Decay \end{cases} \tag{21}$$

where the subscripts A and S represent antisymmetric and symmetric respectively.

Thus, we prove that the exploration of free fermionic string models can reveal more about physical mechanisms. Here, we have proven the existence of a third selection rule for Pati-Salam models for doublet-triplet splitting. Such rules are vital because they increase the physical viability of string theory, bridging the gap between strings and observed physics. Without the rules, the models would be cluttered with extra particles, incorrect symmetry breaking, and, most dangerously, rapid proton decay.

The new third selection rule for asymmetric Pati-Salam models opens the door for further exploration in asymmetric model building.

6 Conclusion

To close, we started with the robust standard model and assessed how it fits into a larger gauge group $SO(10)$. Then showing how this larger gauge group can be broken into other smaller subgroups with the grand unified theories, while acknowledging that neither the GUTs or the standard model are without their flaws.

Hence, we targeted two of these flaws; unification and gravity, proposing a radical solution via string theory. We examined and explained how such a solution works physically through concepts like the Nambu-Goto action and quantisation. Following this, we took an in-depth look at where these proposed mathematical objects exist in the worldsheet. Looking at how the compactification of dimensions impacted the topology of the worldsheet and introduced the important concept of modular invariance. Then we galvanised the partition function to a point where we established a system in which we could build string models.

In this system, we introduced a framework to ensure physical consistency in the models through the ABK rules and provided a way to extract physical information through the Virasoro condition and the GGSO projections.

Starting from the single identity basis vector, $\mathbb{1}$, we built a model that reproduced the $SO(10)$ group we established at the beginning. Using our string theory formalism, we add the supersymmetry vector S and $b_{1,2,3}$ that contain the 16 generations of chiral matter.

Lastly, we talk about the rising need in an asymmetric model, due to the lack of experimental evidence, and the doublet-triplet splitting mechanism. Highlighting the problems it poses to both the GUTs and our string models through rapid proton decay. We then built an asymmetric $SO(6) \otimes SO(4)$ model to demonstrate such an asymmetric model by adding a γ vector. Omitting the massless spectrum as it would be tedious and not beneficial to our conversation. Despite the model not projecting out any moduli, which would of been preferable to increase the physicality at the string energy scale and simplify the geometry, we found a new condition for an $SO(10) \rightarrow SO(6) \otimes SO(4)$ breaking vectors in the string models to project out the proton decay causing triplet. That is, $|\alpha_L(b_j) - \alpha_R(b_j)| = 2$.

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