Final Year Project

Report About the Equivalence Postulate in Quantum Mechanics

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1 Abstract

Quantum Mechanics and General Relativity are incompatible. It is reflected at least in the discontinues of energy and the probability description in Quantum Mechanics. There are two thoughts to connect the microcosmic with the macroscopic: one is to reconstruct the Quantum Mechanics using the structure of General Relativity, the other one is to reconstruct the General Relativity from the structure of Quantum Mechanics. Equivalence Postulate in Quantum Mechanics uses the first thought. This report will focus on how the Quantum Mechanics is reconstructed from the structure of Classical Mechanics. The main structure of the report is based on the papers Qauntum Mechanics from an Equivalence Principle (Faraggi & Matone, 1997), and Equivalence Postulate in Quantum Mechanics (Faraggi & Matone, 1999)

2 Introduction

The thought of reconstructing Quantum Mechanics comes from imitating the geometric structure of Classical Mechanics. The geometric structure is reflected in the formats of the formulae and also displays the relation between physics quantities. Therefore considering that Heisenberg Uncertainty separates momentum and position, the duality from the Hamilton's formula and Legendre Transformation is chosen as a start.

Then, need to check whether such reconstruction works, which means it agrees with the solid conclusions in Classical Mechanics and Quantum Mechanics. So try to obtain a way of changing the coordinates. This way should agree with the Equivalence Principle.

Equivalence Principle is generated from the General relativity. It originally is relevant to that for the same physics states, the descriptions based on different reference systems are equivalent to each other. The most famous example is that the force causing the acceleration in an inertia system is equivalent to the non-inertia force in a non-inertia system (the reference system with acceleration). Equivalence Principle in Quantum Mechanics tries to make the changes of the reference systems in Quantum Mechanics have the same effect as that in the General Relativity. In another words, because the wave function describes the possible states and their possibilities, Equivalence Principle in Quantum Mechanics tries to expand the reference system changes including the possibility descriptions.

After discovering that Möbius transformation is the way of changing coordinates which satisfies the requirements of the reconstruction, the problem that such transformation failed at transforming “0 kinetic energy” state to other states makes modifying the Hamilton-Jacobi Equation in need. Thus, by adding two inhomogeneous terms into the Hamilton-Jacobi Equation, the modification makes the transformation valid on each two states and thus satisfies the Equivalence Principle.

Such transformation and modification generate the new wave function. Because it is on the extended complex plane, the wave function can be the linear combination of the two solutions of the Schrödinger Equation (a second-order differential equation). However, the continuity and smoothness conditions also need to be satisfied at the infinity point. These conditions provides the different solutions from the classical solutions for the potential well problems.

While it is imitating the Classical Mechanics, the meanings of the imitating terms sometimes differ from the physical meanings in Classical Mechanics. For S_0, it differs from the reduced action in the $W = 0$ state; For trajectory, although there is an arguments that the second term in the generating function of the Canonical Transformation corresponds to the path density, there are even different opinions on whether the trajectory in Quantum Mechanics exists. However, whether the idea of de Broglie wave can help to explain the contradiction point ”velocity level” may worth further discussion.

3 Duality

3.1 Duality on the Format

The differences between the microcosmic world and the macroscopic world are generated by the different origins of Quantum Mechanics and General Relativity. Quantum Mechanics differs greatly from the Classical Mechanics and its limits describe Classical Mechanics whereas the General Relativity is the expansion of the Classical Mechanics. Because the formulae formats express their geometric structure, Equivalence Postulate in Quantum Mechanics tries a new perspective to reconstruct the Quantum Mechanics: Keep the formulae’s formats similar to the ones in the Classical Mechanics.

In the Classical Mechanics, for the monogenic system without the time parameter, there is a duality between momentum p and velocity $\frac{dq}{dt}$ from Lagrange Equation and Hamilton's formula.

$$p = \frac{\partial L}{\partial \frac{dq}{dt}} \tag{1}$$

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \tag{2}$$

so that the Legendre Transformation indicates that

$$L = p\frac{dq}{dt} - H \tag{3}$$

where L and H can exchange, and the equation still remains in the same form.

In the Equivalence Postulate in Quantum Mechanics, rather than using the entire theory of trajectory in the Classic Mechanics, at the beginning it only employs the form of the duality between p-$\frac{dq}{dt}$ and L -H. Under the assumption that there is a similar duality in Quantum Mechanics, then to derive what property such duality will result in, and also to compare the results with the Classical Quantum Mechanics. In this duality, p and q can be separate, which also corresponds to the Heisenberg Uncertainty: momentum and position cannot be measured simultaneously.

To specify the imitating from Classical Mechanics, take T_0 to imitate Hamiltonian H_0, and S_0 to imitate the Lagrangian L_0, p does not change, q to imitate the $\frac{dq}{dt}$in the Canonical Equation of Hamilton. The results are:

Position Description	Momentum Description
$T_0 = q\frac{\partial S_0}{\partial q} - S_0$	$S_0 = p\frac{\partial T_0}{\partial P} - T_0$
$p = \frac{\partial S_0}{\partial q}$	$q = \frac{\partial T_0}{\partial p}$
$p\frac{dq}{dS_0} = 1$	$q\frac{dp}{dT_0} = 1$
Take a derivative with respect to S_0	Take the derivative with respect to T_0
$\frac{dp}{dS_0}\frac{dq}{dS_0} + p\frac{d^2q}{dS_0^2} = 0$	$\frac{dq}{dT_0}\frac{dp}{dT_0} + q\frac{d^2p}{dT_0^2} = 0$
$\frac{1}{q\sqrt{p}}\frac{d^2q\sqrt{p}}{dS_0^2} = \frac{1}{\sqrt{p}}\frac{d^2\sqrt{p}}{dS_0^2}$	$\frac{1}{p\sqrt{q}}\frac{d^2p\sqrt{q}}{dT_0^2} = \frac{1}{\sqrt{q}}\frac{d^2\sqrt{q}}{dT_0^2}$
$= -\frac{1}{4p^2}(\frac{dp}{dS_0})^2 + \frac{1}{2p}\frac{d^2p}{dS_0^2}$	$= -\frac{1}{4q^2}(\frac{dq}{dT_0})^2 + \frac{1}{2q}\frac{d^2q}{dT_0^2}$
Define $U(S_0) := \frac{1}{4p^2}(\frac{dp}{dS_0})^2 - \frac{1}{2p}\frac{d^2p}{dS_0^2}$	Define $V(T_0) := \frac{1}{4q^2}(\frac{dq}{dT_0})^2 - \frac{1}{2q}\frac{d^2q}{dT_0^2}$
$= \frac{\{q,S_0\}}{2}$	$= \frac{\{p,T_0\}}{2}$
where $\{h(x),x\} = \frac{h'''}{h'} - \frac{3}{2}(\frac{h''}{h'})^2$	
So that $(\frac{d^2}{dS_0^2} + U(S_0))q\sqrt{p} = 0$	So that $(\frac{d^2}{dT_0^2} + V(T_0))p\sqrt{q} = 0$
and $(\frac{d^2}{dS_0^2} + U(S_0))\sqrt{p} = 0$	and $(\frac{d^2}{dT_0^2} + V(T_0))\sqrt{q} = 0$

3.2 Duality in the Transformation

In Classical Mechanics, considering the stationary (independent from the time parameter) system, Hamilton's Principle requires the line integral, which is also named as reduced action, $I = \int_{t1}^{t2} Ldt$ should have a stationary value for the actual path of motion (Goldstein et al. 2000). Hence, it need L_0 to be a constant in no matter what reference system. For the reconstruction of Quantum Mechanics not relevant to time, because the physical meaning of p and q in Quantum Mechanics are similar as they are in the Classical Mechanics, to imitate the Classical Mechanics, S_0 is required to be invariant under the coordinate transformation.

Having manifest p-q and $S_0 - T_0$ duality, then try to see what may happen when transforming coordinates. In the Classical Mechanics, Canonical Transformation is aimed at finding the free system with total energy in Hamilton-Jacobi Equation equals to zero. So in the Quantum Mechanics try to find a invertible transformation from $V(x) - E = 0$ in Schördinger Equation to other states. Therefore, define the invertible v-transformation v(x): $x \mapsto x^v$, such that $S_0(q^v) = S_0(q)$. $V(x) - E$ corresponds to the Canonical potential $U(S_0)$. Because S_0 required to be invariant under the transformation, next step try to find a concrete transformation that keeps $U(S_0)$ invariant.

Möbius transformation satisfies the requirement that $U(S_0)$ is invariant. Recall that from Section 2.1

$$\frac{1}{q\sqrt{p}}\frac{d^2q\sqrt{p}}{dS_0^2} = \frac{1}{\sqrt{p}}\frac{d^2\sqrt{p}}{dS_0^2} \tag{4}$$

so that $q\sqrt{p}$ and $\sqrt{p}$ are the two linear independent solutions of

$$(\frac{d^2}{dS_0^2} + U(S_0))y = 0 \tag{5}$$

Therefore, $q\sqrt{p}$ and $\sqrt{p}$ span the solution space $\{y|y := Aq\sqrt{p}+B\sqrt{p}\}$. Note that

$$U(S_0) = \frac{\{q, S_0\}}{2} = \frac{\{\frac{q\sqrt{p}}{\sqrt{p}}, S_0\}}{2} \tag{6}$$

such relation among the two linear independent solutions and $U(x)$ in the equation (6) is a property of Schwarzian Derivative. To abstract mathematically, assume $f_1(x)$ and $f_2(x)$ are the linear independent solutions of the differential equation

$$(\frac{\partial^2}{\partial x^2} + V(x))y(x) = 0 \tag{7}$$

For $A \times B \neq 0$, transpose the terms to obtain

$$V(x) = -\frac{(Af_1^{''} + Bf_2^{''})}{Af_1 + Bf_2} \tag{8}$$

Due to the fact that the Möbius transformation M does not change the result of Schwarzian Derivative, which means $\{M(f(x)), x\} = \{f(x), x\}$, M is a possible transformation as

$$U(S_0) = \frac{\{q, S_0\}}{2} = \frac{\{M(q), S_0\}}{2} \tag{9}$$

Additionally, Möbius transformation is the only transformation satisfying that $U(S_0)$ is invariant. For $\{f(x), x\} = \{x, x\} = 0$, because

$$\{f(x), x\} = (ln[f(x)'])'' - \frac{1}{2}(ln[f(x)'])'^2 = 0 \tag{10}$$

To solve the differential equation, take $y := ln[f(x)']'$,

$$y' - \frac{1}{2}y^2 = 0; -\frac{1}{y} = \frac{1}{2}x; y = ln[f(x)']' = -\frac{2}{x + c_1} \tag{11}$$

Then,

$$ln[f(x)'] = \int \frac{-2}{x+c_1} = -2ln(x+c_1) + c_2; \tag{12}$$

$$f(x)' = \frac{e^{c_2}}{(x+c_1)^2} \tag{13}$$

So,

$$f(x) = \frac{c_3}{x+c_1} \tag{14}$$

$f(x)$ is a Mӧbius Transformation. Furthermore, because the chain rule on Schwarzian Equation indicates that

$$\{g(f(x)), f(x)\} = (\frac{\partial x}{\partial f(x)})^2(\{g(f(x)), x\} - \{f(x), x\}) \tag{15}$$

It implies that only if g is Mӧbius Transformation, then $\{g(x), x\} = \{f(x), x\}$, because

$$\{g(f(x)), f(x)\} = (\frac{\partial x}{\partial f(x)})^2(\{g(f(x)), x\} - \{f(x), x\}) = (\frac{\partial x}{\partial f(x)})^2 \times 0 = 0 \tag{16}$$

With the v-transformation v(x) so that $S_0(q^v) = S_0(q)$, then after the Mӧbius transformation on either p or q, the new coordinate and momentum $\tilde{p}$ and $\tilde{q}$ are:

$$\begin{array}{ll}
\tilde{q} = q^v = \frac{Aq+B}{Cq+D} & \tilde{p} = p^v = \frac{Ap+B}{Cp+D} \\
\tilde{p} = \frac{dS_0}{dq^v} & \tilde{q} = \frac{dT_0}{dp^v} \\
= \frac{dS_0(q)}{dq}\frac{dq}{dq^v} & = \frac{dT_0(p)}{dp}\frac{dp}{dp^v} \\
= p\frac{(Cq+D)^2}{AD-BC} & = q\frac{(Cp+D)^2}{AD-BC} \\
\tilde{T_0} := T_0(\tilde{p}) = \tilde{q}\tilde{p} - S_0(\tilde{q}) & \tilde{S_0} := S_0(\tilde{q}) = \tilde{q}\tilde{p} - T_0(\tilde{p}) \\
= \frac{Aq+B}{Cq+D}\frac{(Cq+D)^2}{AD-BC}p - pq + T_0(p) & = \frac{Ap+B}{Cp+D}\frac{(Cp+D)^2}{AD-BC}q - pq + S_0(p) \\
= T_0(p) + \frac{ACpq^2+BDp+2BCpq}{AD-BC} & = S_0(p) + \frac{ACqp^2+BDq+2BCpq}{AD-BC}
\end{array}$$

3.3 Properties of the Duality

Mӧbius transformation is valid for the entire space because it does not lose the generality: firstly, any Mӧbius transformation is bijective, and secondly it covers the entire extended complex plane $\bar{\mathbb{C}} := \mathbb{C} \cup \{\pm\infty\}$ (although for coordinates, extended real line is enough).

The definition of the Möbius transformation is that a map M: $\bar{\mathbb{C}} \mapsto \bar{\mathbb{C}}$, which can be represented in the form $w = \frac{az+b}{cz+d}$, for a,b, c, $d \in \bar{\mathbb{C}}$ and $ac \times bd \neq 0$. Hence, for $c = 0$, $w = az + b$, if $a \neq 0$, then solve the equation to obtain that $z = \frac{w}{a} - \frac{b}{a}$, which means it is invertible. Else if $c \neq 0$, it can be solved that $z = \frac{-dw+b}{cw-a}$. Finally, $\infty \mapsto \frac{d}{c}$ and $\frac{a}{c} \mapsto \infty$. Summarize above, M is bijective and its inverse transformation is also a Möbius transformation. Additionally, infinity is not removable from the space.

Then, to see under what condition, S_0 and T_0 (position description and momentum description) are overlap, which means they can interchange. Assume there is the simplest relation between p and q:

$q^v = \alpha p$	$p^v = \beta q$.
So,	
$q^v = \frac{T_0(p^v)}{\partial p^v} = \alpha \frac{\partial S_0(q)}{\partial q}$	$p^v = \frac{S_0(q^v)}{\partial q^v} = \beta \frac{\partial T_0(p)}{\partial p}$
substitute $\frac{\partial p^v}{\partial q} = \beta$	Substitute $\frac{\partial q^v}{\partial p} = \alpha$
Then,	
$\frac{T_0(p^v)}{\partial q} = \frac{T_0(p^v)}{\partial p^v} \frac{\partial p^v}{\partial q} = \alpha\beta \frac{\partial S_0(q)}{\partial q}$	$\frac{S_0(q^v)}{\partial p} = \frac{S_0(q^v)}{\partial q^v} \frac{\partial q^v}{\partial p} = \alpha\beta \frac{\partial T_0(p)}{\partial p}$
Integral over q,	Integral over p,
$T_0(p^v) = \alpha\beta S_0(q) + c_1$	$S_0(q^v) = \alpha\beta T_0(p) + c_2$
$T_0((p^v)^v) = \alpha\beta S_0(q^v) + c_1$	$S_0((q^v)^v) = \alpha\beta T_0(p^v) + c_2$
$= (\alpha\beta)^2 T_0(p) + c_1 + \alpha\beta c_2$	$= (\alpha\beta)^2 S_0(q) + \alpha\beta c_1 + c_2$

First thought: Hoping to simplify the process, assume $S_0((q^v)^v) = S_0(q)$ and $T_0((p^v)^v) = T_0(p)$. That is $(\alpha\beta)^2 = 1$. For the stationary case, imitate the reduced action $S(q,t) = S_0(q) - Et$ and $T(p,t) = T_0 + Et$. And as required invariance: $S_0(q^v) = S_0(q)$, as well as $T_0(p^v) = T_0(p)$. So $S(q,t) = \pm T(p,t) + \gamma$. Because

$$dS = d(pq - T) = pdq + qdp - qdp - \frac{\partial T}{\partial t} dt \tag{17}$$

the sign before $T(p,t)$ is negative: $S(q,t) = -T(p,t)+\gamma$. So $\alpha\beta = -1$. Additionally, from the imitation of Legendre transformation $S(q,t) = pq - T(p,t)$, so $pq = \gamma$. Hence, the condition is that $pq = \gamma$.

Second thought: Apart from that the invariance of $S(q)$ and $T(p)$ on the transforming p and q, from the perspective of Möbius transformation

$$q^v = \frac{A_1 q + B_1}{C_1 q + D_1} = \frac{\partial T(p^v)}{\partial p^v} \frac{\partial p^v}{\partial p}, \tag{18}$$

$$p^v = \frac{A_2 p + B_2}{C_2 p + D_2} = \frac{\partial S(q^v)}{\partial q^v}\frac{\partial q^v}{\partial q} \tag{19}$$

Recall the duality in 3.2, there will be the requirement that

$$\frac{(A_1C_1q^2 + 2B_1C_1q + B_1D_1)p}{A_1D_1 - B_1C_1} = \frac{(A_2C_2p^2 + 2B_2C_2p + B_2D_2)q}{A_2D_2 - B_2C_2} \tag{20}$$

4 Modify the Classical Hamilton-Jacobi Equation

4.1 Changing the Coordinates

Shödinger Equation, which is used to derive the wave function and thus very important in Quantum Mechanics, comes from substituting the defined quantum momentum $p := -i\hbar\frac{\partial}{\partial x}$ into Hamilton-Jacobi Equation. The Equivalence Principle suggests that the descriptions about the same physics states under arbitrary reference system are equivalence with each other. However, when changing the reference system from the $W = 0$ state, the result is a fixed point. Thus, a modification on the classical Hamilton-Jacobi Equation is required to guarantee the equivalent descriptions on physics states on the different reference systems.

The classical Hamilton-Jacobi Equation is

$$H(q_1, ..., q_n, \frac{\partial F}{\partial q_1}, ..., \frac{\partial F}{\partial q_n}, t) + \frac{\partial F}{\partial t} = 0 \tag{21}$$

where F is the generating function of the Canonical Transformation. The generating function is tightly connected with the reduced action, because from the perspective of dimensions, Hamiltonian H and Lagrangian L correspond to energy and thus the generating function has the same dimension as the reduced action I. Substitute the expression of Hamiltonian H: $-\frac{p^2}{2m} + V(q) = E$, recall that $p = \frac{dS}{dq}$, it is $-\frac{1}{2m}(\frac{dS}{dq})^2 + V(q) - E = 0$, where E is the total energy and V(q) is the potential.

Take position description. Then try to change the reference system on the classical Hamilton-Jacobi Equation. As v-transformation required in 2.2, $S_0(q)$ remains invariant under the changing of the reference system, so the v-transformation here must be Möbius transformation. The only concern is that $V(x)$ may varies. Consider $U(S_0)$ defined in 2.1, the format of $(\frac{\partial^2}{\partial S_0^2} + U(S_0) = 0)$ is similar to the one in the Shödinger Equation $(-\frac{\hbar}{2m}\frac{\partial^2}{\partial q^2} + V(x) - E = 0)$. The canonical potential $U(S_0)$ is invariant under

Möbius transformation, therefore, take $V(x) - E = U(S_0)$.

Take $W := V(x) - E$. For the Equivalence Principle in Quantum Mechanics, for a coordinate q, there exists a transformation into $\tilde{q}$, such that $W(q) = W(\tilde{q})$.It is known that

$$\tilde{p} = \frac{dS_0}{dq^v} = p\frac{(Cq+D)^2}{AD-BC} = \frac{dS}{dq}\frac{(Cq+D)^2}{AD-BC} \tag{22}$$

So, it is hoped that

$$\begin{aligned} W(\tilde{q}) &= -\frac{1}{2m}(\frac{\partial S(\tilde{q})}{\partial \tilde{q}})^2 = -\frac{1}{2m}\frac{(Cq+D)^4}{(AD-BC)^2}(\frac{\partial S(q)}{\partial q})^2 \\ &= \frac{(Cq+D)^4}{(AD-BC)^2}W(q) \end{aligned} \tag{23}$$

so called Co-cycle Condition, for the modified Hamilton-Jacobi Equation.

4.2 Modify the Format of Hamilton-Jacobi Equation

The Equivalence Principle in Classical Mechanics failed at the reference system where at least one particle is at rest. For an instance, consider the case of two free particles A and B, where the mass of A is m_A, the mass of B is m_B, If take the reference system relatively rest to A, then The reduced action $I(q_A) = constant$, $I(q_B) = m_B v q_B$. However, because no non-constant transformation can force the variable $q_B \equiv constant$, A, B are not equivalent as it cannot be always true that $I(f(q_B)) = I(q_A)$. The non-equivalence for the reference system with at least one particle at rest against the goal of the Equivalence Principle: there is a coordinate transformation in which each two reference systems are equivalent.

The similar problem exists when analyse from the Classical Hamilton-Jacobi Equation, then for $W(q) = 0$, $W(q^v) = (\frac{\partial q^v}{\partial q})^{-2} W(q) = 0$, Therefore, $(\frac{\partial q^v}{\partial q})^{-2} = 0$, which means q is a constant. In another words, only on a fixed point that a state with $W(q^v) \neq 0$ can transformed to the state with $W(q) = 0$. However, for reference system changing, it is hoped that for an arbitrary coordinate, there is a transformation changes the $W(q^v) \neq 0$ to $W(q) = 0$ state. To reach the hoped goal, modifying the Classical Hamilton-Jacobi Equation is in need.The method is to provide $W(q)$ a inhomogeneous term $Q(q)$.

Define the Co-cyclic Condition as $(a;b) = (\frac{\partial c}{\partial b})^2[(a;c) - (b;c)]$. In order to have

$$\frac{1}{2m}(\frac{\partial S_0(q)}{\partial q})^2 + W(q) + Q(q) = \frac{1}{2m}(\frac{\partial S_0(q^v)}{\partial q^v})^2 + W(q^v) + Q(q^v) = 0 \quad (24)$$

noted that from 2.2,

$$(\frac{\partial S_0(q^v)}{\partial q^v})^2 = (\frac{\partial q}{\partial q^v})^2(\frac{\partial S_0(q)}{\partial q})^2 \quad (25)$$

so the remaining part $W(q^v) + Q(q^v) = (\frac{\partial q}{q^v})^2(W(q) + Q(q))$, to simplify the work of modification, using the same transformation of $S(q)$ on $W(q)+Q(q)$ because they satisfy co-cycle condition.

After writing the Quantum Hamilton-Jacobi Equation in the form

$$\frac{1}{2m}(\frac{\partial S_0}{\partial q})^2 + W(q) + Q(q) = 0 \quad (26)$$

where in Classical Mechanics the limit of $Q(q) \to 0$. Then need to find the suitable $Q(q)$. The format of $\{e^{i\alpha h}, x\}$ is a possible choice. However directly calculate $\{e^{i\alpha h}, x\}$ is a bit tedious, so using the method of substitution. Chain Rule of Schwarzian Derivative indicates that

$$\{a(b), b(c)\} = (\frac{dc}{db(c)})^2(\{a(b), c\} - \{b(c), c\}) \quad (27)$$

If take function $c := a$ that is

$$\begin{aligned}\{a(b), b(a)\} &= (\frac{da}{db(a)})^2(\{a(b), a\} - \{b(a), a\}) \\ &= (\frac{da}{db(a)})^2(0 - \{b(a), a\}) \\ &= -(\frac{da}{db(a)})^2\{b(a), a\}\end{aligned} \quad (28)$$

Then, take $a := e^h$, $b := x$, $\{e^h, x\} = -(\frac{de^h}{dx})^2\{x, e^h\}$. Meanwhile, if $a := x$, $b := e^h$, $c := h$, then

$$\begin{aligned}\{x, e^h\} &= (\frac{dh}{de^h})^2(\{x, h\} - \{e^h, h\}) \\ &= e^{-2h}(\{x, h\} + \frac{d^3}{dh^3}[ln(\frac{de^h}{dh})] - \frac{1}{2}[\frac{d}{dh}[ln(\frac{de^h}{dh})]]^2) \\ &= e^{-2h}(\{x, h\} - \frac{1}{2}).\end{aligned} \quad (29)$$

Thus,

$$\begin{aligned}\{e^h, x\} &= -(\frac{de^h}{db(x)})^2\{x, e^h\} \\ &= -(\frac{de^h}{dx})^2(\frac{dh}{de^h})^2\{x, h\} - \frac{1}{2}) \\ &= -(\frac{dh}{dx})^2 - \frac{1}{2}(\frac{dh}{dx})^2.\end{aligned} \tag{30}$$

So

$$\{e^{i\alpha h}, x\} = \{h, x\} + \frac{\alpha^2}{2}(\frac{\partial h}{\partial x})^2 \tag{31}$$

$$\{e^{i\alpha h}, x\} = \{h, x\} + \frac{\alpha^2}{2}(\frac{\partial h}{\partial x})^2 \tag{32}$$

If $\frac{\alpha^2}{2}(\frac{\partial h}{\partial x})^2$ corresponds to $-(\frac{\partial S_0}{\partial q})^2$, it implies that

$$W(q) + Q(q) = -(\frac{\partial S_0}{\partial q})^2 = \frac{\beta^2}{2}(-\{e^{\frac{2iS_0}{\beta}}, q\} + \{S_0, q\}) \tag{33}$$

Take $Q(q) = \frac{\beta^2}{2}\{S_0, q\}$, then $W(q) = -\frac{\beta^2}{2}\{e^{\frac{2iS_0}{\beta}}, q\}$. Observing the result, although $e^{i\alpha h}$ is a complex number, the output is real number.

However, the problem of non-equivalence for $W = 0$ still exists as taking the free system with $W(q) = 0$, That is $(\frac{\partial S_0}{\partial q})^2 = -2mQ$. That S_0 is invariant under the transformation implies $Q \equiv 0$. This result indicates that $W = 0$ is still away from connecting to other $W \neq 0$ state via a non-constant transformation.

4.3 Modification on the Existed Terms

Now, try to reach from the state $W = 0$ to an arbitrary state where $W \neq 0$. Because that the singularity of the state $W(q) = 0$ when connecting to other $W \neq 0$ states indicates that current W does not satisfy the Co-cyclic condition, try to modify the term $W(q)$. Denoting $(q; q^v)$ as the inhomogeneous term depending on q and q^v to be determined, which satisfies the cocyclic condition. So, $W(q^v) = (\frac{\partial q}{\partial q^v})^2 W(q) + (q; q^v)$. It is known that

$$W(q^v) + Q(q^v) = (\frac{\partial q}{\partial q^v})^2(W(q) + Q(q)) \tag{34}$$

Substitute newly defined $W(q^v)$ with additional inhomogeneous term, which gives

$$(\frac{\partial q}{\partial q^v})^2 W(q) + (q; q^v) + Q(q^v) = (\frac{\partial q}{\partial q^v})^2 (W(q) + Q(q)). \tag{35}$$

Firstly, to see the calculation property of the co-cylic condition

$$(a; b) = (\frac{\partial c}{\partial b})^2 [(a; c) - (b; c)] \tag{36}$$

Assume A, B are constants and $f(x)$, and $h(x)$ are functions.

$$(a; a) = (\frac{\partial a}{\partial a})^2 [(a; a) - (a; a)] = 0 \tag{37}$$

$$(Aa; a) = (\frac{\partial Aa}{\partial a})^2 [(Aa; a) - (a; a)] = A^2 (Aa; a) \Rightarrow (Aa; a) = 0 \tag{38}$$

$$(a; Ab) = (\frac{\partial b}{\partial Ab})^2 [(a; b) - (Ab; b)] = A^{-2} (a; b) \tag{39}$$

Because $T_0 = p\frac{dS_0}{dq} + S_0$ will not change if adding a constant to S_0, the general function of S_0 only depends on its first derivative and higher order derivatives: $F(s_0', s_0'', s_0''', ...) = 0$. Because Hamilton-Jacobi Equation is the classical limit of the function F, the modification term $(a; b)$ should only depends on the first and higher order derivative of a. So,

$$(a + B; b) = (a; b) = (a; b + B) \tag{40}$$

Then to show that the sufficient and necessary condition for the co-cyclic condition $(a; b)$ only depends on the first derivative or higher derivative of a is that in the case $a = b + \varepsilon f(b)$, it always satisfies that $(a; b) = c_1 \varepsilon f^{(3)}(b) + O(\varepsilon^2)$ with $c \neq 0$. For $(q^v; q)$ assume q^v infinitesimally differs from q, that is $q^v := q + \varepsilon f(q)$. It is required that $(a; b)$ only depends on the first and higher order derivatives of a, so

$$(q + \varepsilon f(q); q) = c_1 \varepsilon f^{(k)}(q) + O(\varepsilon^2) \tag{41}$$

To determine the k. From the calculation properties above, it can be derived that

$$\begin{aligned} (A(q + \varepsilon f(q)); Aq) &= (q + \varepsilon f(q); Aq) = A^{-2} (q + \varepsilon f(q); q) \\ &= A^{-2} c_1 \varepsilon f^{(k)}(q) + O(\varepsilon^2) \end{aligned} \tag{42}$$

If $F(Aq) := Af(q)$, then use chain rule

$$(Aq + \varepsilon F(Aq); q) = c_1 \varepsilon F^{(k)}(Aq) + O(\varepsilon^2) = c_1 \varepsilon A^{1-k} f^{(k)}(q) + O(\varepsilon^2) \quad (43)$$

Therefore k=3, and

$$(q + \varepsilon f(q); q) = c_1 \varepsilon f^{(3)}(q) + O(\varepsilon^2) \quad (44)$$

Then is to show that $(q; q^v) = -\frac{\beta^2}{4m}\{q, q^v\}$. Firstly to show the condition where the solution for $(q; q^v)$ exists. Expand the $O(\varepsilon^2)$, for ε^n, the term is

$$c_{i_1 \dots i_n} \varepsilon F^{(i_1)}(Aq) \dots F^{(i_n)}(Aq) = c_{i_1 \dots i_n} A^{n-\Sigma_k i_k} \varepsilon^n f^{(i_1)}(q) \dots f^{(i_n)}(q) \quad (45)$$

Because without substitution of $F(Aq)$, it is A^{-2}, Thus, $\Sigma_k i_k = n+2$. Then to show that either $c_1 \neq 0$ or $(q + \varepsilon f(q); q) = 0$. Define $\varepsilon^{ab}(b) := \varepsilon f(b) : b \mapsto a := \varepsilon f(b)$, think the infinitesimal mapping $v^{ab} : a \mapsto b$,$v^{bc} : b \mapsto c$, $v^{ca} : c \mapsto a$. So that

$$b = a + \varepsilon^{ba}(a) = a - \varepsilon^{ab}(b) = a - \varepsilon^{ab}(a + \varepsilon^{ba}(a)) \quad (46)$$

$$c = b + \varepsilon^{bc}(b) = b + \varepsilon^{bc}(a + \varepsilon^{ab}(a)). \quad (47)$$

From b, $\varepsilon^{ba} + \varepsilon^{ab} \circ (\mathbb{1} + \varepsilon^{ba}) = 0$, so $\varepsilon^{ba} = -\varepsilon^{ab}$. similarly for a and c. And

$$\varepsilon^{ca} = \varepsilon^{cb} + \varepsilon^{ba} = \varepsilon^{cb} - \varepsilon^{ab} \quad (48)$$

Because $(q^v; q)$ satisfies cyclic condition, when $q = q^c$,$q^v = q^a := c_1 + \varepsilon^{(3)}(q^c)$ substitute the equation of co-cyclic condition to the infinitesimal expansion,

$$\begin{aligned}(q^a; q^c) &= c_1 \varepsilon^{ac(3)}(q^c) + O^{ac}(\varepsilon^2) \\ &= (1 + \frac{d\varepsilon^{bc}(q^c)}{dq^c})^2 [c_1 \varepsilon^{ab(3)}(q^c) + O^{ab}(\varepsilon^2) - (c_1 \varepsilon^{cb(3)}(q^c) + O^{cb}(\varepsilon^2))]\end{aligned} \quad (49)$$

If $c_1 = 0$, the left terms would be

$$O^{ac} = (1 + \frac{d\varepsilon^{bc}(q^c)}{dq^c})^2 (O^{ab}(\varepsilon^2) - O^{cb}(\varepsilon^2)). \quad (50)$$

Because

$$\frac{d\varepsilon^{bc}(q^c)}{dq^c} = \frac{d(\varepsilon f(q^c))}{dq^c} = \varepsilon \frac{df(q^c)}{dq^c} \approx 0 \quad (51)$$

$$O^{ac} = O^{ab}(\varepsilon^2) - O^{cb}(\varepsilon^2) \quad (52)$$

which differs from that $\varepsilon^{ac} = \varepsilon^{ab} + \varepsilon^{bc}$. However, it is all about the second or higher order terms of ε, so for the first order term, the required relationship $\varepsilon^{ac} = \varepsilon^{ab} + \varepsilon^{bc}$ is not guaranteed (because $c_1 = 0$ directly cancels all the first order terms), and also it implies that $(q^a, q^c) = 0$.

Then, it is to show that Schwarzian Derivative is the only possible answer. The different brackets are to denote different relationship between a and b, and they are all satisfies the co-cyclic condition. The method is to include an assumed Schwarzian Derivative term in the final answer, then to check that the answer must also be a Schwarzian Derivative. Define $[a;b] := (a;b) - c_1\{a;b\}$, because both $(a;b)$ and $\{a;b\}$ satisfy only depending on the first and higher order derivatives of a. So, $[q+\varepsilon f(q);q] = c_2\varepsilon f^{(3)}(q) + O(\varepsilon^2)$. It is known that $\{q+\varepsilon f(q), q\} = c_2\varepsilon f^{(3)}(q) + O(\varepsilon^2)$. According to Faraggi &Matone (1999), $c_2 = 0$, so $[q+\varepsilon f(q);q] = 0$. Thus, $(q;q^v) = c_1\{q,q^v\} = -\frac{\beta^2}{4m}$.

Now, the Quantum Mechanics Hamilton Jacobi Equation is

$$\frac{1}{2m}(\frac{\partial S_0}{\partial q})^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0, q\} = 0 \tag{53}$$

where

$$W(q) = \frac{\beta^2}{4m}(\{S_0, q\} - \{e^{\frac{2i}{\beta}S_0}, q\} - Q(q)) \tag{54}$$

$$Q(q) = \frac{\beta^2}{4m}(\{S_0, q\} - \{e^{\frac{2i}{\beta}S_0}, q\} + \{q^0; q\}) \tag{55}$$

so that both W and Q satisfy Co-cyclic Condition.

4.4 New Wave Function

Wave function is derived from solving the Schrödinger equation. Classical Schrödinger equation can be obtained by substituting quantum momentum $i\hbar\frac{\partial}{\partial q}$. After determining that the Quantum Mechanics Hamilton-Jacobi Equation is

$$\frac{1}{2m}(\frac{\partial S_0}{\partial q})^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0, q\} = 0 \tag{56}$$

the corresponding new Schrödinger Equation is

$$[-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)]\psi = E\psi \tag{57}$$

Compare with the classical Schrödinger equation, it is derived that $\beta = \hbar$.

For classical Schrödinger Equation, the solution is often two conjugate solutions $\psi_1 = Re^{\frac{iS_0}{\hbar}}$ or $\psi_2 = Re^{-\frac{iS_0}{\hbar}}$. In Classical Mechanics, the wave function has to approach 0 when q approaches to the infinity. However, for the Quantum Mechanics Hamilton-Jacobi Equation, because the Möbius transformation works on the entire extended complex plane, it is no need to force the wave function to be 0 at infinity. So try to think of the possibility of the solution in form of $A\psi_1 + B\psi_2$.

For the solution has the form

$$\psi = \frac{1}{\sqrt{S_0'}}(Ae^{-\frac{iS_0}{\hbar}} + Be^{\frac{iS_0}{\hbar}}) \tag{58}$$

Because the solution space is closed under scale multiplication,

$$\psi_3 = \frac{A\psi_1 + B\psi_2}{C\psi_1 + D\psi_2} = \frac{A\frac{\psi_1}{\psi_2} + B}{C\frac{\psi_1}{\psi_2} + D} \tag{59}$$

is another solution. Thus $e^{\frac{2iS_0}{\hbar}} = \frac{\psi_1}{\psi_2}$. The $e^{\frac{2iS_0}{\hbar}}$ is also the same as the first term in the Schwarzian Derivative in $W = -\frac{\hbar^2}{2}\{e^{\frac{2iS_0}{\hbar}}, q\}$. Then a new wave function is obtained

$$\psi_3 = e^{\frac{2iS_1}{\hbar}} = \frac{Ae^{\frac{2iS_0}{\hbar}} + B}{Ce^{\frac{2iS_0}{\hbar}} + D} \tag{60}$$

It can be solved that

$$S_1 = \frac{\hbar}{2i} ln(\frac{Ae^{\frac{2iS_0}{\hbar}} + B}{Ce^{\frac{2iS_0}{\hbar}} + D}) \tag{61}$$

That two solutions of a second-order differential equation are linearly independent is a common case. There is a case worth consideration: one of the solutions is linearly dependent on the other solution while the other solution can also be viewed as its conjugation. For the this case where $\bar{\psi} \propto \psi$, the simplest one is that $\psi = \bar{\psi}$. In this case,

$$Ce^{\frac{iS_0}{\hbar}} = e^{-\frac{iS_0}{\hbar}} \tag{62}$$

so

$$S_0 = \sqrt[1-c]{\frac{i}{\hbar}} \tag{63}$$

is a constant.

$$Q = \frac{\hbar^2}{4m}\{S_0, q\} = \frac{\hbar^2}{4m}(ln(\frac{dS_0}{d\psi}\frac{d\psi}{dq}))'' - \frac{1}{2}ln(\frac{dS_0}{d\psi}\frac{d\psi}{dq})'^2$$
$$= \frac{\hbar^2}{4m}(ln(-R\frac{\hbar}{iS_0}\frac{d\psi}{dq}))'' - \frac{1}{2}ln(\frac{dS_0}{d\psi}\frac{d\psi}{dq})'^2) = -\frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial q^2} \tag{64}$$

Choose a simplest case, ψ is a constant, for the other solution of the second order differential equation

$$-\frac{h^2}{2m}\frac{\partial^2}{\partial q^2}\psi + (V - E + Q)\psi = 0 \tag{65}$$

The coefficient of the first order term is 0, so Wronskian

$$Wr = e^{-\int 0dx} = e^{constant} \tag{66}$$

is never 0. Hence, the two solutions are linearly independent. So the case $\psi \propto \bar{\psi}$ is unlikely to happen. For the simplest case, choose the constant=0, so

$$\bar{\psi} = R\int_{q_0}^{q}\frac{1}{R^2}dx \tag{67}$$

From equation (60),

$$S_0 = \frac{\hbar}{2i}ln\frac{\bar{\psi}}{\psi} = \frac{\hbar}{2i}ln\int_{q_0}^{q}\frac{1}{R^{-2}}dx \tag{68}$$

5 About the Trajectory

5.1 S_0 and the Reduced Action

S_0 imitates the Lagrangian when establishing the duality on the format. However, its physical meaning does not correspond to the Lagrangian, but to the Reduced Action.

Because Classical Mechanics is viewed as the limits of Quantum Mechanics, S_0^{cl} is hoped to be the same as $\lim_{\hbar\mapsto 0} S_0$. However, it is not always the case, especially for $W = 0$. a counter example is that in classical harmonic oscillator,

$$\frac{\partial S_0^{cl}}{\partial q} = \sqrt{2mT} = \sqrt{2m\frac{m\omega^2(A^2 - q^2)}{2}} = \pm m\omega\sqrt{A^2 - q^2} \tag{69}$$

where A is the amplitude. However in Quantum Mechanics $S_0 = 0$, which differs from the classical one. Additionally,

$$Q = E_n - \frac{m\omega^2 q^2}{2} = (n + \frac{1}{2})\hbar\omega - \frac{m\omega^2 q^2}{2} \tag{70}$$

also differs from the classical one $\frac{m\omega^2(A^2-q^2)}{2}$. Therefore, there is a case that S_0 does not agree with its classical limit.

For this situation, it is possible to substitute the classical limits into the solution, and the wave function has to vanish at $\pm\infty$. That is

$$\psi = \frac{1}{\sqrt{S_0^{cl}\prime}}(Ae^{\frac{-iS_0^{cl}}{\hbar}} + Be^{\frac{iS_0^{cl}}{\hbar}}) \tag{71}$$

where either A or B is 0.

5.2 Path Integral

Because the format of formulae in Classical Mechanics is actually based on the theory of trajectory, it is necessary to see what the trajectory would be like in Quantum Mechanics. In classical mechiancs, Hamilton's Principle states that the motion of the system from t_1 to t_2 is the stationary-valued line intergral $I = \int_{t_1}^{t_2} Ldt$ for the actual path of motion. According to Roncadelli & Schulman (2007), if choosing $(\hat{q}, \hat{Q})$ as the set of independent Canonical variables, and $K(\hat{Q}, \hat{P}, t) = H(\hat{q}, \hat{p}, t) + \frac{\partial w(\hat{q}, \hat{Q}, t)}{\partial t}$, where K is the transformed Hamilton and $w(\hat{q}, \hat{Q}, t)$ is the generated function, then

$$w(\hat{q}, \hat{Q}, t) = S((\hat{q}, \hat{Q}, t) - \frac{i\hbar}{2} logdet \frac{\partial^2 S}{\partial q \partial Q} \tag{72}$$

where —$S((\hat{q}, \hat{Q}, t)$ means the solution of Classical Hamilton-Jacobi Equation and the second term means the density of the path.

The method used in the paper is that for a very small time t, the generating function F is in the form $F(q, Q) = \frac{m(Q-q)^2}{2t}$. Then to guess the limit of operator

$$\hat{W} := \frac{m}{2t}(\hat{Q}^2 - \hat{q}\hat{Q} - \hat{Q}\hat{q} + \hat{q}^2) + g(t) \tag{73}$$

Hamilton-Jacobi Equation is the case when K=0 in the Canonical Transformation. Hence, $0 = \frac{m}{2t^2}[\hat{q}, \hat{Q}] + \frac{\partial g(t)}{\partial t}$ is required. It is known that the

Heisenberg Equation of motion provide the relation $\hat{q} = \hat{Q} + \hat{P}t/m$. Hence,

$$\frac{\partial g}{\partial t} = \frac{i\hbar}{2t}.g(t) = \frac{i\hbar}{2}lnt \tag{74}$$

Therefore,

$$\hat{W} = \frac{m}{2t}(\hat{Q} - \hat{q})^2 + \frac{i\hbar}{2}lnt \tag{75}$$

$$K(q, Q, t) = C\sqrt{\frac{1}{t}}e^{\frac{i}{\hbar}(\frac{m}{2t}(Q^2-2qQ+q^2)} \tag{76}$$

The solution S of the Hamilton-Jacobi Equation $S = \frac{m}{2t}(Q^2 - 2qQ + q^2)$ So,

$$w = S(\hat{q}, \hat{Q}, t) - \frac{i\hbar}{2}logdet\frac{\partial^2 S}{\partial q \partial Q} \tag{77}$$

The second term in w is the path density along a classical path and it also adjusts the difference caused by the non-commutative of operators.

However, there is also argument suggests that for compact space, there is no trajectory for the Quantum Hamilton-Jacobi Equation involving the time parameter. In the Quantum Mechanics, energy is quantised. Therefore, $t_1 - t_2 = \frac{\partial S_0}{\partial E_n}$ implies that there should also exists velocity levels with respect to energy levels (Faraggi & Matone, 2013). Considering in Quantum Mechanics wave function is related to probabilities, it may be possible that the "velocity level" has some connection with de Broglie wave. It may worth further consideration.

6 Example: Potential Well

Take potential well as an example. Assume

$$V(x) = \begin{cases} V_1 < E & |x| < L \\ V_2 > E & |x| > L \end{cases} \tag{78}$$

Schördinger equation is that

$$-\frac{\hbar^2}{2m}\psi'' + (V(x) - E)\psi = 0 \tag{79}$$

Take $k_1^2 := \frac{2m|E-V_1|}{\hbar^2}$; $k_2^2 := \frac{2m|E-V_2|}{\hbar^2}$. The classical general solution is

$$\psi(x) = \begin{cases} Ae^{k_2 x} & x < L \\ Csin(k_1 x) + Dcos(k_1 x) & |x| \leq L \\ Fe^{-k_2 x} & x > L \end{cases} \tag{80}$$

Because $V(x)$ is an even function, for $|x| < L$, it is either $\psi_1 = Csin(k_1x)$ or $\psi_2 = Dcos(k_1x)$. For $|x| > L$, it is symmetric, either $\psi_3 = Ae^{k_2x}$ or $\psi_4 = Be^{-k_2x}$. Because it is on the extended real line, the Möbius Transformation on continuity condition requires that $\frac{\tilde{\psi}(\pm\infty)}{\psi(\pm\infty)}$ should be convergent in the extended real line. So the only two possible cases are ψ_1 joining ψ_3, and ψ_2 joining ψ_3. Otherwise there will be vibrating divergence at infinity

The first case ψ_1 joining ψ_3 gives the result

$$\psi = \frac{1}{k_1}\begin{cases} -sin(k_1L)e^{k_2(x+L)} & x < -L \\ sin(k_1L) & |x| < L \\ sin(k_1L)e^{-k_2(x-L)} & x > L \end{cases} \quad (81)$$

$$k_1cot(k_1L) = -k_2 \quad (82)$$

$$\tilde{\psi} = \frac{1}{2cos(2k_1L)}\begin{cases} e^{-k_2(x+L)} + cos(2k_1L)e^{k_2(x+L)} & x < -L \\ 2cos(k_1L)cos(k_1x & |x| < L \\ e^{-k_2(x-L)} + cos(2k_1L)e^{-k_2(x-L)} & x > L \end{cases} \quad (83)$$

$$\frac{\tilde{\psi}}{\psi} = \frac{k_1}{2sin(2k_1L)}\begin{cases} -cos(2k_1L) - e^{-k_2(x+L)} & x < -L \\ 2sin(k_1L)cot(k_1x & |x| < L \\ cos(2k_1L) + e^{-k_2(x-L)} & x > L \end{cases} \quad (84)$$

$$\lim_{q\mapsto\pm\infty} \frac{\tilde{\psi}}{\psi} = \pm\infty \quad (85)$$

$\pm\infty$ in extended real line can be viewed as a limit point.

The second case gives the result:

$$\psi = \begin{cases} cos(k_1L)e^{k_2(x+L)} & x < -L \\ cos(k_1L) & |x| < L \\ cos(k_1L)e^{-k_2(x-L)} & x > L \end{cases} \quad (86)$$

$$\tilde{\psi} = \frac{1}{2cos(2k_1L)}\begin{cases} cos(2k_1L)e^{k_2(x+L)} - e^{-k_2(x+L)} & x < -L \\ 2sin(k_1L)sin(k_1x & |x| < L \\ e^{k_2(x-L)} - cos(2k_1L)e^{-k_2(x-L)} & x > L \end{cases} \quad (87)$$

$$\frac{\tilde{\psi}}{\psi} = \frac{1}{2k_1sin(2k_1L)}\begin{cases} cos(2k_1L) - e^{-k_2(x+L)} & x < -L \\ sin(2k_1L)tan(k_1x) & |x| < L \\ -cos(2k_1L) + e^{k_2(x-L)} & x > L \end{cases} \quad (88)$$

$$\lim_{q\mapsto\pm\infty} \frac{\tilde{\psi}}{\psi} = \pm\infty \quad (89)$$

The solutions contain both the exponential function and the trigonometric function, which is different from the solution in the Classical Quantum Mechanics.

7 Conclusion

The report focuses on establishing one-dimensional time-independent Equivalent Postulate in Quantum Mechanics. The reconstructing is based on the geometric structure of Legendre transformation in the Classical Mechanics. So the $p-q$ and $S_0 - T_0$ duality is obtained. The duality also generates the Canonical Potential $U(S)$ in the form of Schwarzian Derivative. Imitating the stability of the reduced action indicates that S_0 is invariant, Möbius transformation is the proper transformation because it does not change the value of Schwarzian Derivative.

Then because for Hamilton-Jacobi Equation, $W = 0$ states is isolated from connecting with other $W \neq 0$ states, which disagrees with the Equivalence Principle. So the modification on the Hamilton-Jacobi Equation is necessary. The method of modification is adding inhomogeneous terms into W so that it processes the Co-cyclic condition as $\frac{\partial S_0}{\partial q}$. As the consequence of the modification, the Quantum Hamilton-Jacobi Equation is

$$\frac{1}{2m}(\frac{\partial S_0(q)}{\partial q})^2 + W(q) + Q(q) = 0 \tag{90}$$

Because the Möbius transformation involves the entire extended complex plane, the vanishing of the wave function at infinity is not necessary, but the convergence on the extended real line at the infinity. So, the wave function can be different from that in the Classical Quantum Mechanics.

The physical meaning of S_0 is a bit ambiguous. It imitates the Lagrangian but processes some property of reduced action. In the $W = 0$ case, it differs from its Classical limit. For path integral which is the place generates the reduced action, the second term of the generating function in the Canonical Transformation corresponds to the path density. However, there are arguments about whether the trajectory exists in the Quantum

Mechanics following the Equivalence Postulate. This area may require more investigation.

8 References

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