String Phenomenology: Using String Theory to Derive a Grand Unified Theory of the Standard Model

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Abstract

In this report, I will demonstrate how the standard model can be embedded within a larger grand unified theory, $SO(10)$. I will do so by using string theory to devise a phenomenological framework based around the free fermionic construction. This framework will be realised in the form of the NAHE set of basis vectors, which will then be used to analyse the massless spectrum. By implementing string theory, I will show that this grand unified theory will also include the fourth fundamental force gravity.

Non-Technical Summary

In this report I outline a method used to derive the standard model (SM) through string theory. However, this method will also include some particles which are not present in the current SM, specifically the right-handed neutrino and the graviton. The current SM is comprised of: fermions, the matter portion, and bosons, the force carriers. The three forces present in the SM each have a corresponding gauge boson which physically mediates the forces between particles. These include the strong interaction with the gluon particle, the electromagnetic with the photon and the weak with the W and Z bosons. A third important particle is the Higgs boson, responsible for giving mass to the rest of the particles in the SM.

Once the SM was presented, I discuss some drawbacks of this current model. First, there is no unification between the three fundamental forces, which mathematically correspond to different gauge groups. Transformations under these groups correspond to a particle interacting with the associated gauge boson. However, the SM still fails to include the fourth fundamental force, gravity. The mechanics of gravity is described by general relativity rather than quantum mechanics like the rest of the SM. This highlights the disparity between the physics around us and is something I look to unify. Furthermore, the right-handed neutrino is not present in the SM. Neutrino oscillations have been observed, indicating neutrinos have some mass, thus the right-handed neutrino should exist to account for this mass. The outlined disagreements motivate us to look for a grand unified theory (GUT). To unify the force sector, we wish to find a single simple gauge group which also embeds the matter sector. The gauge group $SO(10)$ both unifies the forces and includes the right-handed neutrino, but we do not have gravity as part of the gauge group. This requires the quantisation of gravity, meaning it requires an associated gauge boson, the graviton. This is when I implement string theory. String theory follows the simple idea that particles are thought to be strings rather than point-like, meaning a particle's quantum numbers now arise from the vibrational modes of this string and a certain vibrational mode may correspond to the graviton.

Strings are realised in many more dimensions than the four dimensions (4D) we perceive. To build an agreeing theory in 4D spacetime, we must curl or fold up these extra dimensions onto a compact space, a process known as compactification. I also introduce supersymmetry, a theoretical framework in which every fermion has a bosonic counterpart and vice versa, called superpartners, adding a new symmetry between matter particles and force carriers, allowing us to construct models which unify the two sectors. Supersymmetry introduced the theory of supergravity, which includes the gravitino, the graviton's superpartner. The minimal supersymmetry we can have is $\mathcal{N} = 1$ which in supergravity, relates to one graviton and one gravitino. The method I will use is called the free fermionic formulation, we treat the extra dimensions, or degrees of freedom, as free fermions, yielding a total of 20 free left-moving fermions and 44 free right-moving fermions. I then use the ABK rules to build viable basis vectors describing the periodicity of these free fermions as they travel along the string's worldsheet. The periodicity is defined to be either Ramond (R, periodic) or Neveu-Schwarz (NS, antiperiodic), with worldsheet fermions in the R sector corresponding to ground state fermions and those in the NS sector as ground state bosons. I then performed the GSO projections of the basis vectors in different sectors, which contain a set of states, the GSO projections decide whether certain states survive, with the unphysical states being projected out of the spectrum. Adding more basis vectors and perform subsequent projections causes the large dimension gauge groups to break to a combination of smaller ones, as well as reducing the supersymmetry to $\mathcal{N} = 1$. A viable GUT emerges, $SO(10)$, which naturally embeds the matter sector of the SM along with the right-handed neutrino, but now, by applying string theory, our GUT also includes quantised gravity.

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Chapter 1

Introduction

The standard model we currently know is an object of beauty and has made many agreeable predictions for experiments; it agrees well with experimental observations at the low energy scale [1]. However, the current model still contains a few 'teething problems' which the current theory cannot explain. There are 19 or so parameters within the current standard model which are arbitrary and free, among which include: fermion and boson masses, gauge coupling constants, the three θ angles and so on. The current model also does not explain the theory of dark matter or include the cosmological constant. With one of the most serious theoretical problems of the current model being the Hierarchy problem, TeV supersymmetry may be an answer as to why the weak scale is so much smaller than the Planck scale [2]. Furthermore, the current model's neutrino masses are unevenly smaller than any physics scale in the standard model [3]. In order to reach a universal, truly applicable model these parameters and 'blind spots' must be derived and included to create a more unified fitting theory. In the 1940s quantum field theory was shown to be the correct framework for the unification of quantum mechanics and electromagnetism, and by the 1970s the weak and strong force were also shown to be explained through quantum field theory [4]. However, still some disparities loomed, the largest of these being the exclusion of the fourth fundamental force, gravity, as general relativity and quantum field theories are so far incompatible at the fundamental level; the union between the two yields a nonrenormalisable quantum field theory, implying a need for high energy physics [4]. In order to arrive at a grand unified theory (GUT) of the standard model, to incorporate all four fundamental forces, we require a theory in four dimensional flat spacetime with $\mathcal{N} = 1$ SUSY and chiral matter fields [5]. This paper is based on the argument that a GUT embedding the standard model must seamlessly include gravity amongst the other forces and derive the massless spectrum from string theory, absorbing any free parameters.

The best fitting model for unifying the standard model is string theory. Idealising the particles as strings rather than particles allows for oscillations and vibrations to give the particle its mass and charge. In the late 1960s, in an attempt to explain the strong force, string theory was born [6]. The framework for string theory includes gauge symmetries, making it a viable option for reproducing the standard model through compactification of heterotic strings. This is in aim of eliminating the 19 or so free parameters in the current standard model and create space for the theorised spin-2 graviton [4], the force carrier for gravity, so we can eventually arrive at a model that has no anomalies or free parameters and can unify quantum mechanics and general relativity, using GUTs.

Grand unified theories have long been theorised and some have been proposed such as the Georgi-Glashow model and the Pati-Salam model which can both be absorbed into the GUT $SO(10)$. Whilst these make some strong predictions such as the right-handed neutrino, they do not include quantised gravity and it seems the only viable method for doing so is implementing string theory. A grand unified theory makes many a prediction, the first important two being gauge coupling unification and proton decay as well as going on to predict Yukawa coupling unification [7]. The method I have used to do so was devised by Antoniadis, Bachas and Kounnas (ABK), in their paper on four-dimensional superstrings [5], they devised a set of simple rules to follow to derive a set of basis vectors called the NAHE set. This set of basis vectors creates a framework which can be implemented to derive realistic generation models including: the 15 generation fermions along with the 16th, the right-handed neutrino; and quantised gravity with its corresponding gauge boson, the graviton, all embedded within an anomaly free model except for one anomalous $U(1)$ gauge group [1].

But a question that arises is why are we looking to include this theorised right-handed neutrino? Right-handed neutrinos, or as they are also known, sterile neutrinos, only interact with gravity as they cannot transform under $SU(3)_C$, $SU(2)_L$ or $U(1)_Y$, given that they have no electromagnetic and colour charge as well as no left-handed chirality. We believe these particles are present as the observation of neutrino oscillations implies they have non-zero mass [8]. Given that fermions have Dirac masses, which couples a left-handed field to a right-handed one, if neutrinos are massive rather than massless, then the right-handed neutrino is required in our model. This, in part, begins the quest for physics beyond the current standard model.

The free fermionic formulation follows the idea that instead of needing to embed the superstring in ten dimensions and then further compactifying the other 6 dimensions onto a Calabi-Yau manifold. We can formulate a theory directly into four dimensions by identifying the extra degrees of freedom as either bosonic or fermionic internal degrees of freedom [1]. The ambition in doing so being that all parameters can be derived from string theory, rather than free and experimentally calculated, through one grand unified theory, such as the gauge group $SO(10)$.

Chapter 2

The Standard Model

2.1 Contents of the Standard Model

The current standard model organises all currently known fundamental particles into three key sectors: Matter, Interaction and the Higgs mechanism which contain the particles Fermions, Bosons and the Higgs boson respectively.

The standard model is built from three generations of fermions, each generation forming multiplets under the gauge symmetry of the standard model. The fermions account for the matter portion of the standard model consisting of quarks and leptons, all with spin-$\frac{1}{2}$. Each generation consist of a pair of quarks, an up-type quark and a down-type quark, and a pair of leptons, a lepton and its corresponding neutrino.

$$\begin{matrix} \begin{pmatrix} u \\ d \end{pmatrix} & \begin{pmatrix} s \\ c \end{pmatrix} & \begin{pmatrix} t \\ b \end{pmatrix} \\ \\ \begin{pmatrix} e^- \\ \nu_e \end{pmatrix} & \begin{pmatrix} \mu^- \\ \nu_\mu \end{pmatrix} & \begin{pmatrix} \tau^- \\ \nu_\tau \end{pmatrix} \end{matrix} \tag{2.1}$$

Along with their anti-particles, which, say for the first generation, also include $\bar{Q} = \bar{u}, \bar{d}$ and $\bar{L} = \bar{e}, \bar{\nu}$. These for each generation make up the 15 left-handed Weyl spinors.

Bosons are spin-1 gauge particles and are the force carriers for three of the four fundamental forces, namely strong with the gluon, electromagnetic and the photon, γ and weak with the $W^{\pm}, Z^0$ bosons. These bosons are responsible for mediating particle interactions.

Fermions follow Fermi-Dirac statistics and are mediated by Fermi fields whilst bosons follow Bose-Einstein statistics and are mediated by bosonic fields, the aim of GUTs and specifically string theory is to incorporate both into one mathematical framework.

The Higgs boson was theorised many years ago but only recently discovered and has spin-0. It is responsible for giving particles their mass through the Higgs mechanism, this is done by Yukawa couplings when particles interact with the Higgs field.

Thus, one can see, the defining distinction between these particles is their spin; $\frac{1}{2}$,1 and 0.

The Standard Model is based on the gauge group

$$SU(3)_C \times SU(2)_L \times U(1)_Y \tag{2.2}$$

Which contains Lie groups; $SU(3)_C$ corresponding to the strong interaction, $SU(2)_L$ corresponding to the weak and $U(1)_Y$ corresponding to the weak-hypercharge, where matter states transform as representations of these gauge groups.

The current matter sector of the standard model is as follows

$$3\left[(3,\, 2)_{1/6} + (\bar{3},\, 1)_{-2/3} + (\bar{3},\, 1)_{1/3} + (1,\, 2)_{-1/2} + (1,\, 1)_{+1}\right] \tag{2.3}$$

Which follows the form

$$(a, b)_c \tag{2.4}$$

where a represents the multiplet the particles transform as under $SU(3)_C$, b represents the multiplet transformed under $SU(2)_L$ and c corresponds to the hypercharge, or electric charge, of the particles.

Equation (2.3) therefore corresponds to the three generations of chiral matter, denoted by the 3 at the front, with the first term corresponding to Q_L, one generation of a left-handed quark family; the second to U_L^C, the conjugate of the left-handed up-type quarks; the third to D_L^C, the conjugate of the left-handed down-type quarks; the fourth to L_L, one generation of the left-handed leptons and the fifth to e_L^C the conjugate of the left-handed electron.

In quantum field theory, we can then differ between the fermionic fields, the bosonic fields and the Higgs field. So there is a difference in our gauge group, between abelian and non-abelian representations. The standard model is based on the general framework of second quantised renormalisable perturbative quantum field theories. The central principles that underly the standard model are gauge invariance and renormalisation. Where gauge invariance includes local phase invariance and internal symmetries, and renormalisation meaning divergences are absorbed into a finite number of parameters that are measured experimentally.

2.2 Gauge Groups

When replicating the SM through the application of string theory, it must be ensured that properties of the SM gauge groups are reflected. Gauge invariance must be preserved, as well as the presence of global symmetries like Poincaré and flavour symmetries as well as local gauge symmetries. I will first demonstrate these central principles of the gauge groups that make up the standard model in order to reflect what I am working towards.

The covariant derivative for a field ψ describes how the field interacts with the fundamental forces, as gauge symmetries are local, it preserves symmetry and ensures invariance under gauge transformations and is written as

$$D_\mu = \partial_\mu + ig_s G_\mu^\alpha T^\alpha + igW_\mu^\beta T^\beta + ig' B_\mu Y \tag{2.5}$$

Where the second term in (2.5) correlates to the strong interaction $SU(3)_C$, the third term to the weak interaction $SU(2)_L$ and the fourth to the weak-hypercharge $U(1)_Y$. We can explain T^α, T^β, Y

to be the generators of $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ respectively. Each gauge group dictates how the field will transform under symmetries and determines which bosons will couple to the field. I will now go into more detail about the mechanics and invariance of each Lie group in the standard model and expand into possible higher dimension GUT lie groups such as $SO(10)$.

$U(1)_{E.M}$ produces the theory of quantum electrodynamics, QED, the first and simplest gauge theory. Under global symmetry, we can consider a free Dirac fermion $\psi(x)$ with Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \tag{2.6}$$

with ψ being a spin-$\frac{1}{2}$ fermion (specifically, a Fermi-Dirac) field, which is invariant under a global $U(1)_{E.M}$ phase transformation [9]

$$\psi(x) \xrightarrow{U(1)} e^{iQ\theta}\psi(x) \tag{2.7}$$

$$\bar{\psi}(x) \xrightarrow{U(1)} e^{-iQ\theta}\bar{\psi}(x) \tag{2.8}$$

Where $Q\theta$ is an arbitrary constant. If we dimensionalise θ, $\theta \to \alpha(x)$, redefining the local phase, the Lagrangian is no longer invariant and so we now require local gauge symmetry to ensure invariance.

$$\psi(x) \to e^{iQ\theta(x)}\psi(x) \tag{2.9}$$

And so the covariant derivative becomes

$$\partial_\mu\psi(x) \to e^{iQ\theta(x)}\left(\partial_\mu\psi(x) + i\left(\partial_\mu Q\theta(x)\right)\psi(x)\right) \tag{2.10}$$

And here we can see the gauge invariance breaks. The 'gauge principle' looks to preserve the local phase invariance to do so we must introduce a new spin-1 gauge field, $A^\mu(x)$, and so we now have the covariant derivative

$$D_\mu = \partial_\mu + ieQA_\mu(x) \tag{2.11}$$

Where the fermion ψ has mass m and charge Q. The Lagrangian

$$\mathcal{L} = \bar{\psi}(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x) = \mathcal{L}_0 - eQA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x) \tag{2.12}$$

is now invariant under local $U(1)$ transformations. We also must look to ensure that our new gauge field, A^μ has the proper dynamics, and so we introduce the field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.13}$$

This is the electromagnetic field tensor, to describe the propagation of the field A^μ which we can now interpret to be our photon. The photon is the quantum of the electromagnetic 4-vector potential $A^\mu(x)$ [10] (i.e. the photon field), the simplest gauge field.

Therefore, the Lagrangian of QED is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \tag{2.14}$$

describing the interaction between a Dirac fermion (meaning specifically a spin-$\frac{1}{2}$ fermion) and an abelian gauge field, A^μ, specifically a spin-1 boson field. So the local gauge invariance demands the gauge field A^μ plus the interaction terms,

$$L_{\text{int}} = -eI^\mu_{EM}A_\mu \tag{2.15}$$

where

$$I^{\mu}_{EM} = \bar{\psi}\gamma^{\mu}\psi \tag{2.16}$$

is the electromagnetic current.

However, the gauge group $U(1)_{E.M}$ is not directly one of the standard model gauge groups, it instead arises from a mixture of $SU(2)_L \times U(1)_Y$, the so-called electro-weak interaction. The local phase invariance under $U(1)_{E.M}$ phase transformations generalises to the case of invariance of non-abelian internal symmetries. We will take the Lagrangian from (2.6), where we now require invariance under

$$\psi(x) \rightarrow \left[1 - ig\vec{\alpha}(x)\vec{T}\right]\psi \tag{2.17}$$

with g being the gauge coupling and $\vec{T}$ being the generators of the gauge group in the adjoint representation. $SU(3)_C$ and $SU(2)_L$ are both non-abelian gauge groups, and so their generators obey

$$[T_i, T_j] = if^{ijk}T^k \tag{2.18}$$

In other words, each group's their generators anti-commute, with f^{ijk} being the structure constants of each group.

For $SU(3)_C$, the generators are

$$T_a = \frac{1}{2}\lambda_a, \qquad a = 1, ..., 8 \tag{2.19}$$

where λ_a are the Gell-Mann matrices (see Appendix A)

And for $SU(2)_L$, the generators are

$$T_i = \frac{1}{2}\sigma_i, \qquad i = 1, 2, 3 \tag{2.20}$$

where σ_b are the Pauli matrices (see Appendix B)

The Lagrangian for the weak, $SU(2)_L$, is invariant under local gauge transformations with the covariant derivative

$$D_\mu = \partial_\mu + ig\vec{W}_\mu \cdot \vec{T} \tag{2.21}$$

With the Lagrangian of the kinetic gauge sector being

$$\mathcal{L} = -\frac{1}{4}W_{\mu\nu}W^{\mu\nu} \tag{2.22}$$

where

$$W_{i\mu\nu} = \partial_\mu W_{i\nu} - \partial_\nu W_{i\mu} - gf_{ijk}W_{j\mu}W_{k\nu} \tag{2.23}$$

is the field strength tensor. We will see how this joins with $U(1)_Y$ to form the electro-weak interaction in the following chapter.

2.3 $\mathrm{SU(2)_L \times U(1)_Y}$

We begin with the Dirac spinor

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{2.24}$$

These spinors are solutions to the Dirac equation, where the Majorana condition can be defined

$$\psi = \hat{\psi} \tag{2.25}$$

This is the reality condition [11]. In other words, the Majorana condition states that a particle is its own antiparticle. This introduces the notion of chirality, or left-handedness and right-handedness. This employs the gamma matrix, γ_5 (see Appendix C) giving the matrices

$$L = \frac{1}{2}(1 - \gamma_5) \quad R = \frac{1}{2}(1 + \gamma_5) \tag{2.26}$$

So that any spinor Ψ can be written in terms of its left-handed (chiral) and right-handed parts [11]

$$\Psi = L\Psi + R\Psi \tag{2.27}$$

where

$$L\Psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \text{ and } R\Psi = \begin{pmatrix} 0 \\ \Psi_R \end{pmatrix} \tag{2.28}$$

Thus defining our left-handed and right-handed fermions.

$SU(2)_L$, the weak isospin, only transforms particles with left-chirality, i.e. left-handed particles, as they are weak-isospin doublets, but right-handed particles are weak-isospin singlets, thus the weak isospin for $\psi_R = 0$ The Lagrangian is invariant under infinitesimal $SU(2)_L$ transformation

$$\psi_L \to e^{i\theta^i T^i} \psi_L \tag{2.29}$$

T^i being the generators mentioned in (2.20). The mass term

$$m\bar{\psi}\psi = m \left(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \right) \tag{2.30}$$

is not invariant under $SU(2)_L$ so at this stage all standard model fermion fields are massless. In addition to $SU(2)_L$ we also introduce the weak hypercharge, $U(1)_Y$

$$U(1)_Y = \left\{ e^{i\alpha Y}, \ \alpha \in \mathbb{R} \right\} \tag{2.31}$$

The generator of $U(1)_Y$ is the hypercharge operator, Y, which is a scalar. Thus the transformation of the electro-weak, $SU(2)_L \times U(1)_Y$, interaction is

$$\psi \to e^{i\theta^i T^i} e^{i\alpha Y} \psi \tag{2.32}$$

With the covariant derivative of $SU(2)_L \times U(1)_Y$ being

$$D_\mu = \partial_\mu + igW_\mu^\beta \, T^\beta + ig' B_\mu \, Y \tag{2.33}$$

Where we can see the covariant derivative for $SU(2)_L$ gauge group first mentioned in (2.21); with W^μ being the three gauge bosons and g is the associated coupling constant; now being joined with the covariant derivative for $U(1)_Y$, where B_μ is the gauge boson associated and g' is the coupling constant.

The gauge field strength for $SU(2)_L$ is defined through the interaction between the weak gauge fields and its generators

$$W \cdot T = W^+T^+ + W^-T^- + W^3T^3 \tag{2.34}$$

Where

$$T^\pm = \frac{1}{\sqrt{2}}(T_1 \pm iT_2) \tag{2.35}$$

are the raising and lowering operators of the weak isospin states and

$$W^\pm = \frac{1}{\sqrt{2}}(W_1 \mp iW_2) \tag{2.36}$$

so the electric charge is given by

$$Q_{EM} = T_3 + \frac{Y}{2} \tag{2.37}$$

Where T_3 is the weak-isospin associated with $SU(2)_L$ and Y is the hypercharge quantum number associated with $U(1)_Y$. Through the spontaneous symmetry breaking of $SU(2)_L \times U(1)_Y$, we see the electromagnetic current from (2.15, 2.16) is contained within the neutral term of the covariant derivative

$$i\left(gW_{3\mu}T_3 + g'B_\mu Y\right) \tag{2.38}$$

So W_3 and B are a linear combination of A and a new vector Z

$$\begin{pmatrix} W_3 \\ B \end{pmatrix} = \begin{pmatrix} \cos(\theta_W) & \sin(\theta_W) \\ -\sin(\theta_W) & \cos(\theta_W) \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix} \tag{2.39}$$

Introducing θ_W, the weak mixing angle, or the Weinberg angle. This now makes the neutral term

$$iA\left(g\sin(\theta_W)T_3 + g'\cos(\theta_W)Y\right) + iZ\left(g\cos(\theta_W)T_3 - g'\sin(\theta_W)Y\right) \tag{2.40}$$

Where the coefficient of A must equal $ieQ(T_3 + Y)$, so

$$g = \frac{e}{\sin(\theta_W)} \tag{2.41}$$

$$g' = \frac{e}{\cos(\theta_W)} \tag{2.42}$$

Therefore

$$\tan(\theta_W) = \frac{g'}{g} \tag{2.43}$$

We can then write an equation for the Z-term in the covariant derivative as

$$D^z_\mu = \partial_\mu - ig_Z Z_\mu\left(T_3 - Q\sin^2\theta_W\right) \tag{2.44}$$

where

$$g_Z = \frac{e}{\cos\theta_W \sin\theta_W} \tag{2.45}$$

We can see this reflects the Z^0 boson with Z_μ being the associated gauge field, and so both the Z boson and the photon, A, have been realised from the breaking of the electroweak sector. We will look into this breaking shortly.

2.4 SU(3)$_\mathrm{C}$

$SU(3)_C$ is the gauge group that governs quantum chromodynamics (QCD), which, in order to satisfy the Fermi-Dirac statistics, assumes the existence of the quantum number, colour. The gauge group is an 8 dimensional non-abelian group with the 8 generators T^a, mentioned in (2.19). The λ^a, in T^a, are the Gell-Mann matrices which are analogous with the Pauli matrices σ^i for $SU(2)_L$ but in three dimensions.

The T^a anti-commuting generators cause gluons to interact with each other.

We will denote a quark q_f^α with colour α and flavour f, the free Lagrangian for the $SU(3)_c$, describes the dynamics of quarks and gluons and is written as

$$\mathcal{L} = \sum_f \bar{q}_f \left(i\gamma^\mu D_\mu - m_q\right) q_f \tag{2.46}$$

This is invariant under global transformations [9]

$$q_f^\alpha \overset{SU(3)}{\longrightarrow} U_\beta^\alpha q_f^\alpha \tag{2.47}$$

With U being the $SU(3)_C$

$$U = e^{\frac{i}{2}T^a\theta_a} \tag{2.48}$$

We further require the Lagrangian to be locally invariant under transformations and so introduce the kinetic term

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}G_a^{\mu\nu}G_{\mu\nu}^a \tag{2.49}$$

Introducing the field strength tensor, $G_{\mu\nu}$, of $SU(3)_C$. We therefore arrive at a Lagrangian invariant under $SU(3)_C$ transformations, the Lagrangian of quantum chromodynamics

$$\mathcal{L}_{QCD} = -\frac{1}{4}G_a^{\mu\nu}G_{\mu\nu}^a + \sum_f \bar{q}_f \left(i\gamma^\mu D_\mu - m_q\right) q_f \tag{2.50}$$

Where the covariant derivative for a quark is

$$D_\mu = \partial_\mu - ig_s G_\mu^\alpha T_\alpha \tag{2.51}$$

Where we have the 8 generators, T_α, and the 8 independent gauge boson fields, G_a^μ, these correspond to our gluons, which are the quantum for the QCD potential, where colour index a runs from 1 to 8. The gluons carry the colour charge and can self-interact.

2.5 Spontaneous Symmetry Breaking of SU(2)$_\mathrm{L}$ × U(1)$_\mathrm{Y}$

In chapter 2.3, discussing the electro-weak interaction, it was so far assumed that the $W^\pm$ and Z^0 bosons were massless. However, in reality, this is false as $M_W \sim 80$ GeV and $M_Z \sim 90$ GeV. This implies the need for the Higgs mechanism, to give these bosons mass. This is done through the introduction of scalar field vacuum expectation values [12]. We will introduce the Higgs field doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{2.52}$$

With Lagrangian

$$\mathcal{L} = |D_\mu \Phi|^2 - V\left(|\Phi|^2\right) + \mathcal{L}_\Phi^F \tag{2.53}$$

The field Ψ has a non-vanishing vacuum expectation value (VEV), in other words the expectation value when evaluating Φ at its lowest energy, in the vacuum state. Expanding Φ around the vacuum

$$\langle \Phi(x) \rangle = \tfrac{1}{\sqrt{2}} e^{\frac{\xi(x)\cdot\tau}{2\nu}} \begin{pmatrix} 0 \\ \nu + H(x) \end{pmatrix} \tag{2.54}$$

With ν being the VEV, which is a constant that is responsible for the spontaneous symmetry breaking. By performing a finite gauge transformation we can eliminate the $\xi(x)$ dependence

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu + H \end{pmatrix} \tag{2.55}$$

Where there's now one degree of freedom, the Higgs boson, H, this causes the spontaneously breaking of the electroweak interaction to

$$SU(2)_L \times U(1)_Y \xrightarrow{\text{Higgs}} U(1)_{EM} \tag{2.56}$$

After this breaking we're left with the theory

$$SU(3) \times U(1)_{EM} \tag{2.57}$$

Which, as we have seen, is left invariant under gauge transformations. The VEV ν produces gauge bosons with masses, namely giving the $W^\pm$ and Z mass whilst leaving the photon massless. Note all this builds to the spontaneous symmetry breaking of the electroweak - neutral bosons W_μ^3 and B_μ mix to form the Z_μ boson and the A_μ boson (the photon) and fermion masses are then obtained from the coupling of fermions to the Higgs field. Looking at $\mathcal{L}_\Phi^F$ in (2.54) which describes the part of the Lagrangian that couples the Higgs field to the fermions, specifically the Yukawa coupling, which gives quarks and leptons chiral symmetry breaking masses [13]

$$\mathcal{L}_\Phi^F = \lambda_e \left[\left(\bar{\nu}_e,\ \bar{e}\right)_L \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R + \bar{e}_R \left(\phi^-,\ \phi^0\right) \begin{pmatrix} \nu_e \\ e \end{pmatrix} \right] \tag{2.58}$$

Where λ_e is the Yukawa coupling constant. Setting the expectation value to

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu \end{pmatrix} \tag{2.59}$$

The Lagrangian becomes

$$\mathcal{L} = \frac{\lambda_e}{\sqrt{2}} \nu \left(\bar{e}_L e_R + \bar{e}_R e_L\right) \tag{2.60}$$

Giving the mass of the electron to be

$$m_e = \frac{\lambda_e \nu}{\sqrt{2}} \tag{2.61}$$

We then reach the diagonal matrices by two unitary rotations [13]

$$U_R^{-1}\mathcal{M}^u U_L = \begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t \end{pmatrix} \tag{2.62}$$

and

$$D_R^{-1}\mathcal{M}^d U_L = \begin{pmatrix} m_d & & \\ & m_s & \\ & & m_b \end{pmatrix} \tag{2.63}$$

Where U_R, D_R, U_L and D_L are unitary matrices and $m_{u,c,t}$ and $m_{d,s,b}$ are the physical quark masses. In the charged current sector of the weak interaction we have

$$\overline{(u_1, u_2, u_3)}_L \gamma^\mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_L = (u, c, t)_L \, U_L^\dagger D_L \gamma^\mu \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{2.64}$$

So we will therefore, in general, have generation mixing of the mass eigenstates, with this we then generate masses for the charged leptons, up-type quarks and down-type quarks

$$\lambda_e \begin{pmatrix} \nu \\ e \end{pmatrix} Eh \tag{2.65}$$

$$\lambda_u \begin{pmatrix} u \\ d \end{pmatrix} U\tilde{h} \tag{2.66}$$

$$\lambda_d \begin{pmatrix} u \\ d \end{pmatrix} Dh \tag{2.67}$$

Note that it is the $\tilde{h}$ that gives the up-type quark masses, in other words $\tilde{h} = i\tau^2 h^*$. Fermion masses are then

$$m_{e,u,d} \sim \lambda_{e,u,d}\, \nu \tag{2.68}$$

In the charged-lepton sector we can always rotate to a basis in which the mass eigenstates coincide with gauge symmetries, giving the Lagrangian of Yukawa coupling

$$\mathcal{L}_{\text{Yukawa}} = \lambda_u^{ij} \bar{u}_i \, (\tilde{h}^\dagger Q_j) + \lambda_d^{ij} \bar{d}_i \, (h^\dagger Q_j) + ... \tag{2.69}$$

where λ_u and λ_d are non-diagonal matrices. From the VEVs of the Higgs doublet h and $\tilde{h}$, we reach the mass terms for the $+\frac{2}{3}$ and $-\frac{1}{3}$ charged quarks

$$(\bar{u}_1, \bar{u}_2, \bar{u}_3)_R \, M_u \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \text{h.c.} \tag{2.70}$$

$$\implies M_u^{ij} = \frac{\nu}{\sqrt{2}} \lambda_u^{ij} \tag{2.71}$$

and

$$\left(\bar{d}_1, \bar{d}_2, \bar{d}_3\right)_R M_d \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} + \text{h.c.} \tag{2.72}$$

$$\implies M_d^{ij} = \frac{\nu}{\sqrt{2}} \lambda_d^{ij} \tag{2.73}$$

This is described by the matrix

$$V = U_L^\dagger D_L \tag{2.74}$$

Which is known as Cabibbo-Kobayashi-Maskawa (CKM) matrix, this matrix gives the physical states.

$$V_{CKM} = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 c_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3^{i\delta} & c_1 c_2 s_3 + s_2 c_3^{i\delta} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix} \tag{2.75}$$

This is the mixing matrix, where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$. We are then left with four physical parameters, the three mixing angles, θ_i where $i = 1, 2, 3$ so $0 \le \theta_i \le \frac{\pi}{2}$ and one phase, δ where $0 \le \delta \le 2\pi$.

We then arrive at the total Lagrangian for the standard model

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{EW}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}} \tag{2.76}$$

where $\mathcal{L}_{\text{QCD}}$ describes the strong interaction associated with $SU(3)_C$, $\mathcal{L}_{\text{EW}}$ describe the electro-weak interaction of $SU(2)_L \times U(1)_Y$, $\mathcal{L}_{\text{Higgs}}$ describes the Higgs mechanism and $\mathcal{L}_{\text{Yukawa}}$ describes fermion mass generation.

In the current standard model, we have around 19 free and arbitrary parameters, this includes but is not limited to; the above mixing angles θ, the phase δ as well as the gauge couplings g_s, g and g'. This, in part, is where our motivation for string theory originates, which is to be able to build a model where these parameters would be derivable.

2.6 Grand Unified Theories

We are looking for a theory that incorporates the free parameters that currently exist in the standard model. A theory that can explain values such as neutrino and fermion masses, mixing values and the arbitrary Higgs sector. Additionally, looking at (2.54), we so far do not include the conjugate of the left-handed neutrino, or rather the right-handed neutrino, so we wish to pursue a theory which can naturally include this.

We look for a group, G, such that

$$\exists\, G \supset SU(3) \times SU(2) \times U(1) \tag{2.77}$$

A viable GUT is the gauge group $SO(10)$ in 4D Minkowski space. This model was first suggested by Howard Georgi in 1974 [14], the spin group includes the Georgi-Glashow model, which is another GUT $SU(5)$, and the Pati-Salam model. It is based on 3 generations, each containing 16 spinor representations. $SO(10)$ can break in different ways, either by

$$SO(10) \to SU(5) \times U(1)_Z \tag{2.78}$$

Where $SU(3)_C \times SU(2)_L \times U(1)_Y \subset SU(5)$ naturally and thus the standard model we currently know is contained within the larger unified group $SO(10)$. However, this theory does not naturally contain the right-handed neutrino.

Alternatively, $SO(10)$ may break as

$$SO(10) \to SU(4) \times SU(2)_L \times SU(2)_R \tag{2.79}$$

which contains both the left and right components of $SU(2)$ separately, therefore the right-handed neutrino can transform, but this cannot technically be considered as it is not a single simple gauge group [15]. However, both the Georgi-Glashow model and the Pati-Salam model can be embedded within the larger $SO(10)$ unification. The Pati-Salam model introduces the lepton number as a fourth colour, with the electric charge being given by [13]

$$Q = T_{3L} + T_{3R} + \frac{1}{2}(B - L) \tag{2.80}$$

With B being the baryon number and L being the lepton number, where the charge is quantised as it is embedded in a non-abelian gauge group. However, as mentioned, the Pati-Salam model is not a grand unified gauge group, but since

$$SU(4) \approx SO(6) \tag{2.81}$$

and

$$SU(2) \times SU(2) \approx SO(4) \tag{2.82}$$

we therefore have

$$SO(6) \times SO(4) \subseteq SO(10) \tag{2.83}$$

And hence we reach a viable GUT, more specifically a quark-lepton unification and gauge-coupling unification [13]. As stated, $SO(10)$ has other breaking possibilities, but the end result will be breaking to a gauge group that incorporates the standard model. The gauge group $SO(10)$ is based on 3 generations, each containing 16 spinor representations, it has the weight lattice vector

Q_1	Q_2	Q_3	Q_4	Q_5
+	+	+	+	+
+	+	+	-	-
+	+	-	+	-
+	+	-	-	+
+	-	+	+	-
+	-	+	-	+
+	-	-	+	+
-	+	-	+	+
-	-	+	+	+
-	+	+	-	+
-	+	+	+	-
+	-	-	-	-
-	+	-	-	-
-	-	+	-	-
-	-	-	+	-
-	-	-	-	+

Table 2.1: The weight lattice vector of the 16 spinor representation of $SO(10)$.

This gives the chiral representation, with $Q_{1,\ldots,5}$ being the $U(1)$ generators, where each charge is multiplied by a factor of $+\frac{1}{2}$. In this representation we have either 0, 2 or 4 negative charges to give an overall positive charge. The $\overline{16}$ representation would have either 1, 3 or 5 negative charges, to give an overall negative charge. We can see the decomposition of this representation of $SO(10)$ under $SU(5) \times U(1)$ and under $SO(6) \times SO(4)$. Under $SU(5) \times U(1)$, we see each representation either has 0, 2 or 4 negative charges, so, in combinatorial notation, 16 is made up of

$$16 = \binom{5}{0} + \binom{5}{2} + \binom{5}{4} \tag{2.84}$$

Where the top term in a given bracket is the total number of charges and the bottom term is the number of negative charges, this leads to

$$16 = \left(1,\ \frac{5}{2}\right) + \left(10,\ \frac{1}{2}\right) + \left(\bar{5},\ -\frac{3}{2}\right) \tag{2.85}$$

Denoting the multiplet of each representation under $SU(5)$ and its related charge under $U(1)$. For $SO(6) \times SO(4)$, we see from (2.82) and (2.83) that we can take this to be $SU(4) \times SU(2)_L \times SU(2)_R$, the charges are split so that $Q_{1,2,3}$ are under $SO(6)$ and $Q_{4,5}$ are under $SO(4)$, so again, under combinatorial notation, 16 is made up of

$$16 = \left[\binom{3}{0} + \binom{3}{2}\right]\left[\binom{2}{0} + \binom{2}{2}\right] + \left[\binom{3}{1} + \binom{3}{3}\right]\binom{2}{1} \tag{2.86}$$

Note that $SU(4) \subset SO(6)$ and $SU(2)_L \times SU(2)_R \subset SO(4)$, so we have

$$16 = (4,\ 2,\ 1) + (\bar{4},\ 1,\ 2) \tag{2.87}$$

Where the first term in the bracket is associated with the multiplet under $SO(4)$, the second being the multiplet under $SU(2)_L$ and the third the multiplet under $SU(2)_R$. We can see while the Pati-Salam model is naturally contained within the 16 representation of $SO(10)$, the Georgi-Glashow model on its own is not. But when we add the additional $U(1)$ group, $SO(10)$ decomposes, which is a sign that we will need an extra $U(1)$ gauge group. Last but not least, we must see whether our extended standard model, $\{SU(3)_C \times SU(2)_L \times U(1)_Y\} + U(1)$, is also embedded. For $SU(3)_C \times SU(2)_L \times U(1)^2$, we take the $SU(3)_C \times U(1)_C$ to correspond to the first three charges, $Q_{1,2,3}$, and $SU(2)_L \times U(1)_Y$ to correspond to the last two charges, $Q_{4,5}$. We then see the 16 spinor representation of $SO(10)$ dempose to

$$16 = \left[\binom{3}{0}\binom{2}{0}\right] + \left[\binom{3}{0}\binom{2}{2}\right] + \left[\binom{3}{1}\binom{2}{1}\right] + \left[\binom{3}{2}\binom{2}{0}\right] + \left[\binom{3}{2}\binom{2}{2}\right] + \left[\binom{3}{3}\binom{2}{1}\right] \tag{2.88}$$

Giving

$$16 = \left(1,\ \tfrac{3}{2},\ 1,\ 1\right) + \left(1,\ \tfrac{3}{2},\ 1,\ -1\right) + \left(3,\ \tfrac{1}{2},\ 2,\ 0\right) + \left(\bar{3},\ -\tfrac{1}{2},\ 1,\ 1\right) + \left(\bar{3},\ -\tfrac{1}{2},\ 1,\ -1\right) + \left(1,\ -\tfrac{3}{2},\ 2,\ 0\right) \tag{2.89}$$

This is where we see our the matter sector of our standard model emerge, with the first term in each bracket corresponding to the multiplet under $SU(3)_C$, the second term to the charge under $U(1)_C$, the third to the multiplet under $SU(2)_L$ and the fourth to the charge under $U(1)_L$. We can notice that the bracket terms in (2.89) correspond to e^c_L, ν^c_L, Q_L, d^c_L, u^c_L and L_L respectively. It is also noticed that our hypercharge reappears from

$$U(1)_Y = \frac{1}{3}U(1)_C + \frac{1}{2}U(1)_L \tag{2.90}$$

We can therefore now see how the 16 representation of $SO(10)$ naturally embeds the standard model, making it a viable GUT as it naturally contains all three generations of the 15 fermions in the standard model whilst also allowing for the inclusion of one more fermion, the right-handed neutrino.

Chapter 3

Implementing String Theory

3.1 What is String Theory?

In string theory, particles are thought of as one dimensional strings rather than point-like. The strings have an incredibly small length, which are only realised on the string scale, so at larger distances, even in particle detectors, they appear as point-like. As mentioned, a unique characteristic of string theory is that it does not contain any adjustable dimensionless parameters [16], the strings vibrate as they propagate, it is these oscillations that give the viewed particles its fundamental properties, such as mass and charge. This makes string theory consistent with other particle theories on a larger scale. The core idea with string theory is that different vibrational modes correspond to different fundamental properties, like mass, charge and spin, therefore allowing for the natural inclusion of the spin-2 graviton as it would merely correspond to a certain vibrational mode. The inclusion of gravity at the quantum level is a general theme of string theory, the theory, proposed by Einstein, does get modified at very high energies, but at the ordinary scale, it is presented in the exact same way Einstein proposed [6]. There are different types of string theory where the strings are realised as either open, closed or open and closed, depending on which sub-theory one is using. I will be working under the assumption that the strings are closed, so that they form loops or as they are otherwise known, heterotic strings.

String theory models are only possible in 26 dimensions for the Veneziano model of bosons and 10 dimensions for the Ramond-Neveu-Schwarz model of bosons and fermions, rather than the 4 spacetime dimensions [17]. In heterotic string theory we treat the left and right moving excitations completely separately, so in this model we will treat left-moving excitations as a bosonic string in 26 dimensions and the right moving as a superstring in 10 dimensions, we therefore must quantise these strings separately. However the differing 16 dimensions must be compactified so that the left and right movers are compatible, meaning eventually the left bosonic sector will have 10 spacetime dimensions and 16 internal degrees of freedom, and the right fermionic sector will have the same 10 spacetime dimensions.

3.2 Supersymmetry

Before we can quantise the worldsheet, it is imperative that I touch on the phenomenology of supersymmetry, which is a symmetry that governs string theory. We are already familiar with the idea of symmetry being embedded in nature around us, such as global and local symmetries [2]. The basis of supersymmetry is that it predicts a corresponding boson for every fermion and vice versa, also known as a superpartner, in other words, for every spin-0 particle there is a spin-$\frac{1}{2}$ superpartner. The aim is to bridge the gap between bosons and fermions, and moreover, their two differing statistics. The phenomenological structure I will use, which is built on string theories with fermions, crucially depends on the invariance of local supersymmetry. It is a framework required for supergravity, quantum field theories and string theory. Whilst global supersymmetry indicates a symmetry between the two types of particle across all spacetime, it does not alone explain and incorporate gravity. For this we require local supersymmetry, meaning the transformations depend on certain coordinates and the symmetry holds at independent points in spacetime. Therefore this implies the existence of a new gauge field to mediate this symmetry, introducing the gravitino, the superpartner of the graviton. Thus local supersymmetry is required for the theory of supergravity. To formulate supersymmetry algebra, we must extend the Lorentz symmetry group of 4-dimensional quantum field theory using additional fermionic generators, known as Weyl-spinors [2]. Note a Weyl spinor has components such that in 4-dimensional spacetime, it has four real components. This generator transforms bosons to fermions and vice versa, and so implementing supersymmetry. So a supersymmetry with one generator, Q_α is called $\mathcal{N} = 1$ SUSY; one with two generators, $Q^1_\alpha Q^2_\alpha$ is called $\mathcal{N} = 2$ SUSY and so on. It is seen that $\mathcal{N} = 1$ is the only supersymmetry that allows chiral representations, thus this is the one we shall look to derive in our string model. In the $\mathcal{N} = 1$ supersymmetry, we have the following three fundamental multiplets:

The chiral multiplet,

$$\begin{pmatrix} f \\ \tilde{f} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \tag{3.1}$$

With f being a fermion, with spin-$\frac{1}{2}$ and $\tilde{f}$ being its superpartner, with spin-0.

We then have the vector multiplet, corresponding to the gauge bosons

$$\begin{pmatrix} g \\ \tilde{g} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \tag{3.2}$$

With g being a gauge spin-1 boson and $\tilde{g}$ being its spin-$\frac{1}{2}$ superpartner.

We also have the gravity multiplet

$$\begin{pmatrix} G \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} \tag{3.3}$$

where G is the spin-2 graviton and $\tilde{G}$ is its spin-$\frac{3}{2}$ superpartner, the gravitino.

3.3 Parametrisation of the Worldsheet

Particles are usually treated as point-like, where the trajectory of the particle is described by its worldline $X^\mu(\tau)$, with line element

$$ds^2 = -dX^\mu \cdot dX_\mu = -g_{\mu\nu} dX^\mu dX^\nu \tag{3.4}$$

and thus the action of the particle is

$$\begin{aligned} S &= -m \int d\tau \sqrt{-g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} \\ &= -\tfrac{1}{2} \int d\tau \left(\tfrac{1}{e} \dot{X}^2 - em^2 \right) \end{aligned} \tag{3.5}$$

Where the einbein $e(\tau)$ is introduced to help with quantisation. Introducing a string as opposed to a point-like particle introduces another dimension, as strings have length

$$X^\mu(\tau) \to X^\mu(\tau, \sigma) \tag{3.6}$$

And so transforms the action

$$S = -m \int ds \to S = -T \int dA \tag{3.7}$$

As the action should now describe the proper area of the worldsheet as opposed to the length of a worldline, equivalently we must now consider the tension as opposed to just the mass, due to the extra length from the string. Consider the induced worldsheet metric, $g_{\alpha\beta}$

$$g_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \tag{3.8}$$

Where the worldsheet can be thought of as a flat space and thus we're using the Minkowski metric, η. It should be noted that we will use

$$\dot{X} = \frac{\partial X}{\partial \tau} \quad \text{and} \quad X' = \frac{\partial X}{\partial \sigma} \tag{3.9}$$

And so the metric becomes

$$g_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ X' \cdot \dot{X} & X'^2 \end{pmatrix} \tag{3.10}$$

Which gives the following action

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det g_{\alpha\beta}}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{3.11}$$

This is known as the Polyakov action, where it is inferred that the tension, $T = \frac{1}{4\pi\alpha'}$. The Nambu-Goto action can then be derived, as $g_{\alpha\beta}$ is an auxiliary field which can be integrated out, giving

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det g_{\alpha\beta}} \tag{3.12}$$

and so we then have two constraints on S:

1. A reparameterisation invariance on S,

$$\tau \to \tilde{\tau}(\tau) \tag{3.13}$$

meaning the parameters become

$$\sigma \to \tilde{\sigma}(\tau, \sigma), \quad \tau \to \tilde{\tau}(\tau, \sigma) \tag{3.14}$$

This ensures the action is only dependent on the geometric path of the string, this symmetry is crucial for string theory, where the Polyakov action has 2-dimensional diffeomorphism invariance.

2. Weyl rescaling, such that the metric can be rescaled locally without affecting the physical content,

$$g_{\alpha\beta} \to e^{f(\tau,\sigma)} g_{\alpha\beta} \tag{3.15}$$

So in two dimensions, the metric has three independent components, reparameterisation invariance allows us to fix two of them and Weyl rescaling fixes the third. Therefore, the worldsheet metric can be reduced to the Minkowski flat space metric, $g_{\alpha\beta} \to \eta_{\alpha\beta}$. So the intrinsic geometry of the worldsheet becomes unimportant, only its conformal structure now matters.

Let us consider the complex coordinates

$$z = \tau + i\sigma \quad \bar{z} = \tau - i\sigma \tag{3.16}$$

The Liouville mode drops from the action, so the action becomes

$$S = \frac{1}{4\pi\alpha'} \int dz d\bar{z} \partial_z X^\mu \partial_{\bar{z}} X_\mu = 0 \tag{3.17}$$

This is the flat gauge action, a 2-dimensional conformal field theory of D-free bosons. Where the equation of motion is a 2-dimensional wave equation

$$\partial_z \partial_{\bar{z}} X^\mu(z, \bar{z}) = 0 \tag{3.18}$$

We can split X^μ into left and right moving components

$$X^\mu(z, \bar{z}) = X^\mu_R(z) + X^\mu_L(\bar{z}) \tag{3.19}$$

Where we have the modes of expansion

$$\partial_z X^\mu \sim \textstyle\sum_n z^{(-n-1)} \alpha^\mu_n \tag{3.20}$$

and

$$\partial_{\bar{z}} X^\mu \sim \textstyle\sum_n \bar{z}^{(-n-1)} \tilde{\alpha}^\mu_n \tag{3.21}$$

In quantum field theories, α^μ_n and $\tilde{\alpha}^\mu_n$ are realised as the creation and annihilation operators respectively. It is these operators that define a Hilbert space. The Hilbert space is the space of all possible quantum states of a single particle, which are then used together to construct a Fock space, for multi-particle states.

3.4 Quantisation of the String

3.4.1 Quantising the Bosonic String

To quantise the string, we must introduce the light-cone gauge. The light-cone coordinates [16] are

$$x^{+} = \frac{1}{\sqrt{2}}\left(x^{0} + x^{1}\right) \tag{3.22}$$

and

$$x^{-} = \frac{1}{\sqrt{2}}\left(x^{0} - x^{1}\right) \tag{3.23}$$

which in a D-dimension spacetime become

$$x^{+} = \frac{1}{\sqrt{2}}\left(x^{0} + x^{D-1}\right) \tag{3.24}$$

and

$$x^{-} = \frac{1}{\sqrt{2}}\left(x^{0} - x^{D-1}\right) \tag{3.25}$$

This uses the canonical approach, by imposing equal time commutation relations and varying the action with respect to the worldsheet metric, we reach the constraint that

$$T_{z\bar{z}} = 0 \tag{3.26}$$

This follows from the traceless energy-momentum tensor, in other words there is conformal symmetry. Leading to the two constraint operators

$$T(z) = T_{zz} = \frac{1}{2}\partial_z X^{\mu}\partial_z X_{\mu} \tag{3.27a}$$

$$\bar{T}(\bar{z}) = T_{\bar{z}z} = -\frac{1}{2}\partial_{\bar{z}} X^{\mu}\partial_{\bar{z}} X_{\mu} \tag{3.27b}$$

This can be expanded in terms of Fourier modes

$$T(z) = \sum_{n} z^{-(n+2)} L_n \tag{3.28}$$

Where L_n satisfies the Virasoro algebra

$$[L_n,\, L_m] = (n-m)L_{m+n} \tag{3.29}$$

Classically, the L_n Fourier modes represent the normal modes of vibration of the string, describing how the string oscillates at different frequencies. Therefore, L_n and $\tilde{L}_n$ become the harmonic oscillator's creation/annihilation operators,

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{n-m}\cdot\alpha_n \tag{3.30}$$

It follows that we then must consider an anomaly in Virasoro algebra, as in reality the energy-momentum tensor is not truly traceless. However, in conformal field theories we require it to be traceless, as classical symmetries, such as Weyl invariance, can be broken by quantum anomalies which arise when the trace of $T_{\alpha\beta}$ has a non-zero expectation value. This anomaly arises when introducing another term to the Virasoro algebra

$$[L_n,\, L_m] = (n-m)L_{m+n} + \frac{c}{12}(n(n^2+1)\delta_{n+m} \tag{3.31}$$

Where c is the central charge, our anomaly, of the Virasoro algebra. This anomaly arises as the Liouville mode does not decouple when quantised, in other words, $g_{\mu\nu} \to \eta_{\mu\nu}$ cannot be recovered, signifying the breaking of conformal invariance. In order to reduce this anomaly to zero, it is found that only at $D = 26$ does $c = 0$.

The operator L_0 has a normal ordering ambiguity, in other words we introduce a constant a, the normal ordering constant, to ensure we avoid infinite vacuum energy. So the constraint operators in (3.27a) and (3.27b) become the condition that L_n and $\tilde{L}_n$ annihilate the physical state. For left-movers, this is

$$L_n|\text{Phys}\rangle = 0, \;\; \text{n} > 0 \tag{3.32a}$$

$$(L_0 - a)|\text{Phys}\rangle = 0 \tag{3.32b}$$

$$(L_0 - \tilde{L}_0)|\text{Phys}\rangle = 0 \tag{3.32c}$$

and similarly for right-movers. In the light-cone gauge, the spectrum is free of negative-norm states and is Lorentz invariant for $D = 26$ and $a = 1$.

3.4.2 Quantising the Superstring

A similar method can be implemented using the Fadeev-Popov approach, introducing ghost fields. This introduces another contribution to the anomaly. The ghosts fields are an unphysical auxiliary field which are introduced to correctly handle gauge fixing and to ensure we are correctly counting the physical degrees of freedom. They maintain consistent quantisation, but they must later be removed to ensure unitarity and avoid negative-probability states and tachyons. It is found that the ghost contributions are $c_{gh} = -26$ for the bosonic strings. Therefore giving the total anomaly as, $c_B = D - 26$, with c_B being the total contribution from the bosonic string.

Introducing supersymmetry allows us to move from bosonic strings to heterotic strings, this is done by introducing superpartners. This is where each boson has a fermionic counterpart and vice versa,

$$X^\mu \to X^\mu,\, \psi^\mu \tag{3.33}$$

Where ψ^μ are D-Majorana worldsheet fermions, this transforms the action to

$$S = \int d^2\sigma \left[\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right] \tag{3.34}$$

The gauge fixed action has global supersymmetry, to have local supersymmetry we can use the ungauged fixed action. Taking the Majorana fermions as its two Weyl components,

$$\psi^\mu(z, \bar{z}) = \begin{pmatrix} \psi^\mu(z) \\ \bar{\psi}^\mu(\bar{z}) \end{pmatrix} \tag{3.35}$$

In light-cone coordinates, the action becomes

$$S = \frac{i}{\pi} \int d^2z \left(\psi_z \partial_{\bar{z}} \psi_z + \bar{\psi}_{\bar{z}} \partial_z \bar{\psi}_{\bar{z}} \right) \tag{3.36}$$

So the left- and right-handed degrees of freedom decouple. Similarly to the bosonic case, fermion modes give rise to negative-norm states. The total fermionic ghost contribution is $c_{fg} = 11$, while each Majorana fermion contributes $c = \frac{1}{2}D$ and therefore giving the total fermionic anomaly contribution as $c_F = 11 = \frac{D}{2}$. Therefore the total anomaly for the superstring is

$$c_{\text{total}} = c_B + c_F = D - 26 + 11 + \frac{D}{2} \tag{3.37}$$

So in order for the anomaly to vanish, we require $c_{\text{total}} = 0$ and therefore this occurs for $D = 10$, reaching the required ten dimensions for fermionic superstring theory. We then must define the behaviour of the fermions under a 2π phase shift, this leads to our Ramond (R) and Neveu-Schwarz (NS) sectors, which describe periodic and antiperiodic fermions respectively. Where the R and NS therefore represent different boundary conditions imposed on fermions.

$$\text{Ramond (R)} : \psi^\mu(\sigma + 2\pi) = +\psi^\mu(\sigma) \tag{3.38a}$$

$$\text{Neveu} - \text{Schwarz (NS)} : \psi^\mu(\sigma + 2\pi) = -\psi^\mu(\sigma) \tag{3.38b}$$

Majorana fermions have the boundary conditions apply to the left- and right-movers separately, giving rise to mode expansions

$$\psi^\mu(\sigma + 2\pi) = \exp(2\pi i v)\ \psi^\mu(\sigma) \tag{3.39a}$$

$$\tilde{\psi}^\mu(\sigma + 2\pi) = \exp(2\pi i \tilde{v})\ \tilde{\psi}^\mu(\bar{\sigma}) \tag{3.39b}$$

Where ν and $\tilde{\nu}$ take values 0 and $\frac{1}{2}$ [18].

For bosonic closed strings, they have the boundary conditions

$$X(\sigma, \tau) = X(\sigma + \pi, \tau) \tag{3.40}$$

In supersymmetry we also have the supercurrent, which serves a similar purpose to the momentum-field tensor in general relativity. In the bosonic case, the energy-momentum tensor is realised by varying the action with respect to the worldsheet metric. In superstring theory, this idea is extended to supersymmetric variations; the worldsheet gravitino ψ_a is the supersymmetric partner of the metric $g_{\alpha\beta}$ in two-dimensional supergravity. Therefore, the supercurrent is obtained by varying the action, S, with respect to the two-dimensional gravitino,

$$T_F(z) = \psi^\mu \partial_z X_\mu(z) \tag{3.41a}$$

$$\tilde{T}_F(\bar{z}) = \bar{\psi}^\mu \partial_z X_\mu(\bar{z}) \tag{3.41b}$$

The supercurrent is necessary for maintaining supersymmetry when transforming bosonic and fermionic fields.

We can then impose worldsheet supersymmetry in the left sector, while leaving the right sector bosonic. This leads to the central charges, $c_L = 0$ giving rise to $D = 10$ and $c_R = 0$ giving rise to $D = 26$. This is heterotic string construction.

In order to cancel out these anomalies, we must compactify dimensions, namely we must compactify the differing 16 dimensions. We do so by mounting the 16 right-moving components onto a flat torus with a fixed radius. This compactification is necessary as it is unphysical to have differing spacetime dimensions between left and right sectors. Only two groups permit this

$$E(8) \times E(8) \;\text{ or }\; SO(32) \tag{3.42}$$

These are both anomaly free gauge groups which couple to the $\mathcal{N} = 1$ supergravity in 10 dimensions. Therefore the 16 extra degrees of freedom are compactified onto a 10D heterotic string. Compactifying on a torus allows us to fix certain dimensions, as the torus has two fixed radii, corresponding to each circular direction. The free fermion is formulated at a fixed point in a moduli space, the parameters of the torus allow us to define this space. In Polyakov's action, string theory is formulated as a perturbative sum over the path integral of the string worldsheet, which therefore defines a genus-g Riemann surface. The genus-g describes the number of 'holes' in the toroid with $g = 0$ being tree-level, describing a sphere, $g = 1$ giving a loop, the torus shape, and so on. We shall be compactifying onto a one-loop torus. This one-loop torus leads onto the one-loop vacuum amplitude, which is present in the GSO projection of basis vectors which we will come onto, which corresponds to a torus worldsheet integral that sums over all possible quantum fluctuations in the string.

The free worldsheet fermions, f, can then propagate around the non-contractible loops of the torus and so has the phase, $\alpha(f)$,

$$f \to e^{-i\pi\alpha(f)} f \tag{3.43}$$

This defines the boundary conditions imposed on the fermions as now they have a periodicity associated with each of the two non-contractible loops.

These 10 dimensions can then be further compactified further onto a Calabi-Yau manifold to preserve the $\mathcal{N} = 1$ SUSY in 4D spacetime, giving the effective gauge groups $E(6)$ or $SO(16)$ in 4D. However in the free fermionic formulation that we will be using, we are able to construct directly into four dimensional spacetime, where our extra dimensions will be internalised bosonic or fermionic degrees of freedom. We then reach a point where all the string's quantum numbers are incorporated, either through the extra dimensions on flat compactified tori, creating $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds or by the free periodic and antiperiodic fermions, where the worldsheet supersymmetry is non-linearly realised [5].

We therefore have, in the light-cone gauge, X^μ with $\mu = 1, 2$, as we're working in the light-cone gauge, being the bosonic coordinates; ψ^μ being their fermionic partners; χ_i, y_i, w_i being the 18 internal fermions all corresponding to the left sector. In the right sector, it follows that $\bar{X}^\mu$ corresponds to the bosonic coordinates but, as said, for the right-movers and $\bar{\psi}_a^{1,\cdots,44}$ being the 44 internal right-moving fermions.

To ensure the stability of the construction of the string vacua, we must construct the one-loop partition function which represents all vacuum fluctuations of the string, mathematically, this sums over all contributions from each sector's Hilbert space. The partition function Z is expressed as the one loop vacuum to vacuum amplitude, meaning it sums over all possible closed string states propagating our torus. It is built under the assumption that we require modular invariance, meaning it is required that the partition function is invariant under the modular transformations

$$\tau \to -\frac{1}{\tau} \tag{3.44a}$$

$$\tau \to 1 + \tau \tag{3.44b}$$

It is required that the partition function is independent of the reparameterisation of the tori, as the action is not dependent on the reparameterisation of the worldsheet. These both lead to a set of constraints on the boundary conditions, which are the consistency constraints derived by the ABK rules. This then leads to the free fermionic construction, where the extra degrees of freedom are treated as free fermions through

$$c_L = -26 + 11 + D + \frac{D}{2} + \frac{1}{2}N_{f_L} = 0 \tag{3.45}$$

and

$$c_R = -26 + D + \frac{1}{2}N_{f_R} = 0 \tag{3.46}$$

Where N_{f_L} and N_{f_R} are the total number of free fermions in the left and right sectors respectively, which both have a $\frac{1}{2}$ contribution as Majorana fermions contribute $\frac{D}{2}$. As we require to work in 4-dimensional spacetime, setting $D = 4$ yields

$$N_{f_L} = 18 \text{ and } N_{f_R} = 44 \tag{3.47}$$

In order to cancel the left and right-moving conformal anomalies, we require 18 real left-moving Majorana fermions (as the left-movers must have mass) and 44 right-moving Majorana-Weyl fermions (to include both massive and massless particles).

Chapter 4

The Free Fermionic Model and the ABK Rules

The ABK rules were formulated by I. Antoniadis, C.P. Bachas and C. Kounnas in their paper on four-dimensional superstrings [5]. They lay out constraints on modular invariance for closed string theories where all the internal quantum numbers of the strings are carried by free periodic or antiperiodic worldsheet fermions. This is in order to reach a set of rules, the basis vectors, which, under the Gliozzi–Scherk–Olive (GSO) projections on the Hilbert space, demonstrate sensible particle interpretations at the one-loop level. It should be noted these so called 'ABK' rules were built up on the earlier work of Kawai, Lewellen and Tye (KLT), which expressed the tree level closed loop amplitude as a sum over squares of open tree loop amplitudes [19].

We then have our total composition of free fermions, a total of 20 real left-movers and 44 real right-movers. Our real left movers are comprised of $\{\psi^\mu, \chi^i, y^i, w^i\}$ where $i = 1, ..., 6$. The right movers are comprised of both real and complex fermions, where $\{\bar{y}^i, \bar{w}^i\}$ are the real right movers and $\{\bar{\psi}^{1...5}, \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3, \bar{\phi}^{1...8}\}$ are the complex right movers. We pair two real fermions to give one complex fermion, given that the two real fermions have the same boundary conditions [1].

We have discussed the real left-movers in chapter 3.4.2, but we have now stated the explicit 44 right-movers, which in turn correspond to different gauge groups. We first have our real right-movers $\bar{y}^i$ and $\bar{w}^i$ corresponding to 12 of the compactified bosonic degrees of freedom. As for the complex right-movers, there is $\bar{\psi}^{1,...,5}$ being the superpartners of our light-cone spacetime fermions, and so giving the representation $S0(10)$. We then have $\bar{\eta}^1$, $\bar{\eta}^2$ and $\bar{\eta}^3$, these correspond to $SO(6)^3$, and finally we have $\bar{\phi}^{1,...,8}$ which represent 16 real fermions and so correspond to the group $SO(16)$. The free fermionic construction allows us to naturally reach the three generations of the standard model embedded in $SO(10)$, this is because the construction is formulated from a highly symmetric point in the compactification space [20].

The general basis vector will be governed by the phases of each respective spacetime coordinate and will take the form

$$b_i \left\{\alpha\left(\psi^\mu_{1,2}\right), ..., \alpha\left(w_6\right) \middle| \alpha\left(\bar{y}^1\right), ..., \alpha\left(\bar{\phi}^8\right)\right\} \tag{4.1}$$

The left half of (4.1) contributes to the effective spacetime properties and supersymmetry and corresponds to the real left movers. The right half contributes to the compactification structure and gauge symmetry and corresponds to the right movers, both real and complex. With $\alpha(f)$ being the phase factor where

$$\begin{cases} \alpha(f) = 0 & \rightarrow R \\ \alpha(f) = 1 & \rightarrow NS \end{cases} \tag{4.2}$$

The basis vectors, which the ABK rules derive, are part of an additive set Ξ

$$\Xi = \sum_{i=1}^{n} m_i b_i, \quad m_i = 0, \, ..., N_i - 1 \tag{4.3}$$

where $N_i b_i = 0 \mod 2$. The sum over all the elements of Ξ in the partition function must satisfy modular invariance and so allowed elements in Ξ dictate which states survive the GSO projection.

1. $\sum m_i b_i = 0$ if and only if $\forall\, i,\ m_i = 0 \mod N_i$
2. $N_{ij} b_i b_j = 0 \mod 4$
3. $N_i b_i b_i = 0 \mod 8$
4. The number of real fermions must be even
5. $b_1 = 1$, and more generally, $1 \in \Xi$

There are also rules to be followed on the one-loop phases

1. $C\begin{pmatrix} b_i \\ b_j \end{pmatrix} = \delta_{b_i} \exp\left(\frac{2\pi i}{N_j} n\right) = \delta_{b_j} \exp\left(\frac{2\pi i}{N_i} m\right) \cdot \exp\left(\frac{i\pi}{2} b_i \cdot b_j\right)$
2. $C\begin{pmatrix} b_i \\ b_i \end{pmatrix} = -\exp\left(\frac{i\pi}{4} b_i \cdot b_i\right) C\begin{pmatrix} b_i \\ 1 \end{pmatrix}$
3. $C\begin{pmatrix} b_i \\ b_j \end{pmatrix} = \exp\left(\frac{i\pi}{2} b_i \cdot b_j\right) C\begin{pmatrix} b_j \\ b_i \end{pmatrix}^*$
4. $C\begin{pmatrix} b_i \\ b_j + b_k \end{pmatrix} = \delta_{b_i} C\begin{pmatrix} b_i \\ b_j \end{pmatrix} C\begin{pmatrix} b_i \\ b_k \end{pmatrix}$

where

$$\delta_{b_i} = \exp\left(i\pi b_i(\psi^\mu)\right) = \begin{cases} +1, & b_i(\psi^\mu) = 0 \quad (NS) \\ -1, & b_i(\psi^\mu) = 1 \quad (R) \end{cases} \tag{4.4}$$

For a given sector $\alpha \in \Xi$, the physical states present in the Hilbert space are derived by acting on the vacuum state, $|0\rangle_\alpha$, in that sector. We do so by acting with bosonic and fermionic operators, with frequencies ν_f and ν_{f^*}, and by requiring modular invariance through the application of GSO projections.

$$\nu_f = \frac{1 + \alpha(f)}{2} \quad \text{and} \quad \nu_{f^*} = \frac{1 - \alpha(f)}{2} \tag{4.5}$$

for a given fermion f and its complex conjugate f^*.

These frequencies are used to count the number of oscillation modes, where N_L is the total number of left-moving fermionic and bosonic excitations and N_R is the total number of right moving excitations.

$$N_L = \sum \nu_L = \sum_{f_{L-\text{osc}}} \nu_f + \sum_{{f^*}_{L-\text{osc}}} \nu_{f^*} \tag{4.6}$$

$$N_R = \sum \nu_R = \sum_{f_{R-\text{osc}}} \nu_f + \sum_{{f^*}_{R-\text{osc}}} \nu_{f^*} \tag{4.7}$$

To reach the required massless spectrum, a given state must satisfy the Virasoro, or matching, condition

$${M_L}^2 = -\frac{1}{2} + \frac{\alpha_L \cdot \alpha_L}{8} + N_L = -1 + \frac{\alpha_R \cdot \alpha_R}{8} + N_R = {M_R}^2 \tag{4.8}$$

We also have the fermionic number $F(f)$ which indicates the parity of the number of fermions in a state

$$F(f) = \begin{cases} +1, & f \\ -1, & f^* \end{cases} \tag{4.9}$$

Which is used in the $U(1)$ charges where the gauge group is realised by a free fermion and we have the corresponding charge

$$Q(f) = \tfrac{1}{2}\alpha(f) + F(f) \tag{4.10}$$

This leads onto the GSO projection which arises from modular invariance and, when applied, indicates which states survive and are thus physical and which are projected out of the set and are therefore unphysical. The projection is defined by the equation

$$e^{i\pi b_i \cdot F_\alpha} |s\rangle_\alpha = \delta_\alpha C \begin{pmatrix} \alpha \\ b_j \end{pmatrix}^* |s\rangle_\alpha \tag{4.11}$$

Where $|s\rangle_\alpha$ is some state in the sector $\alpha \in \Xi$, so as said above the states $|s\rangle$ that don't satisfy this equation are projected out and no longer contribute to the partition function. The partition function is therefore simplified and now simply counts the spectrum at all masses. When we expand the partition function for a sector we can take it as a sum over its intersection with other sectors

$$Z = \sum_\alpha \sum_\beta C \begin{pmatrix} \alpha \\ \beta \end{pmatrix} Z \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4.12}$$

In the summation there may arise cancellations between different parts and these cancellations will be reflected in the spectrum. In other words, when constructing the Hilbert space (of the models) by using basis vectors, the GSO projections incorporate those cancellations, they are just a result of the modular invariance of the partition function. It then arises that $\alpha, \beta = 0$ or 1, corresponding to the Neveu-Schwarz or Ramond boundary conditions.

In the aim of seeing the $SO(10)$ group emerge, the worldsheet supersymmetry must also be conserved, leading to a unique supercurrent, T_F. This must be uniquely defined under the transformation of the worldsheet fermions and under the specified boundary conditions of each sector

$$T_F = \psi^\mu \partial X_\mu + \sum_{i=1}^{6} \chi_i y_i w_i \tag{4.13}$$

This ensures the 18 real internal fermions are transformed with the same sign as $\psi^\mu \partial X_\mu$.

Chapter 5

Deriving the Basis Vectors in Aim of the NAHE Set

5.1 The Massless Spectrum

We can take the first basis vector 1,

$$1 = \{1,\ ...\,(\times 20),\ 1|1,\ ...\,(\times 44),\ 1\} \tag{5.1}$$

Portraying the vector that specifies the periodic boundary conditions for all the worldsheet fermion degrees of freedom [21], defining the full spectrum and gives the basis $B = \{1\}$.

Checking against the ABK rules:

$$N_1 \cdot 1 = 0 \mod 2 \text{ for } N_1 = 2 \tag{5.2}$$

$$N_{11} \cdot 1 \cdot 1 = -24 = 0 \mod 8\ \checkmark \tag{5.3}$$

This gives rise to the sectors $\{1,\ NS\}$ Checking against the Virasoro condition

$${M_L}^2 = -\frac{1}{2} + \frac{10}{8} + N_L = -1 + \frac{22}{8} + N_R = {M_R}^2 \tag{5.4}$$

As $N_L, N_R > 0$ we have ${M_L}^2, {M_R}^2 > 0$ and therefore no massless states are present in the 1 sector. As all the fermions are included in the basis vector 1, we have no fermions in the NS sector so for the NS vacuum, $|0\rangle_{NS}$, we have $\alpha(f) = 0$. This gives

$$\nu_f = \nu_{f*} = \frac{1}{2} \tag{5.5}$$

and

$${M_L}^2 = -\frac{1}{2} + \frac{0}{8} + N_L = -1 + \frac{0}{8} + N_R = {M_R}^2 \tag{5.6}$$

Our massless states must therefore be constructed from a combination of one left-moving fermionic oscillator, $N_L = \frac{1}{2}$, and either two right-moving fermionic oscillators, $N_R = 2 \cdot \frac{1}{2} = 1$, or one right-moving bosonic oscillator, $N_R = 1$. There is also currently, a tachyonic state, as we can also use one fermionic right-moving oscillator. This gives rise to the following massless states in the NS sector:

$$\psi^{\mu}_{1/2} \partial \bar{X}^{\nu}_1 |0\rangle_{NS} \tag{5.7}$$

Where the bosonic states correspond to the graviton, dilation and antisymmetric tensor.

$$\psi^{\mu}_{1/2} \bar{\phi}^{a}_{1/2} \bar{\phi}^{b}_{1/2} |0\rangle_{NS}, \quad a, b = 1, ..., 44 \tag{5.8}$$

Where the right movers correspond to the gauge bosons in the adjoint representation of $SO(44)$.

$$\left\{\chi^i,\ y^i,\ w^i\right\} \partial \bar{X}^{\mu}_1 |0\rangle_{NS}, \ \ i = 1, ..., 6 \tag{5.9}$$

This state correspond to the gauge bosons in the adjoint representation of $SU(2)^6$

$$\left\{\chi^i,\ y^i,\ w^i\right\} \bar{\phi}^{a}_{1/2} \bar{\phi}^{b}_{1/2} |0\rangle_{NS} \tag{5.10}$$

This state correspond to the scalars in the adjoint representation of the gauge group $SU(2)^6 \times SO(44)$.

$$\bar{\phi}^{a}_{1/2} |0\rangle_{NS} \tag{5.11}$$

This is our tachyon, which has $M^2 = -\frac{1}{2}$. Using rule 1 and rule 4 of the ABK rules, we find

$$C \binom{NS}{NS} = \delta_{NS} = +1 \tag{5.12}$$

and

$$C \binom{NS}{b_j} = \delta_{b_j} \tag{5.13}$$

A general GSO projection on a massless state, a state in the NS sector,

$$e^{i\pi\ 1 \cdot F_{NS}} |s\rangle_{NS} = \delta_1 |s\rangle_{NS} = -|s\rangle_{NS} \tag{5.14}$$

Computing the GSO projection for the massless states we see all states survive, including the tachyon, as 1 contains the boundary conditions. Notably, for the state

$$e^{i\pi\ 1 \cdot F_{NS}} : \psi^{\mu} \bar{\phi}^{a} \bar{\phi}^{b} |0\rangle_{NS} = -1 \tag{5.15}$$

As $\psi^{\mu}, \bar{\phi}^a$ and $\bar{\phi}^b$ are periodic with basis vector 1. Therefore the adjoint of the $SO(44)$ gauge group survives. So we currently have $SO(44)$, which has 946 dimensions, this is too large and so we look to reduce these dimensions. This is done by adding more basis vectors.

5.2 Adding the $\vec{S}$ Basis Vector

I now wish to add another basis vector,

$$S = \left\{\psi^{\mu}_{1,2},\ \chi_{1,2},\ \chi_{3,4},\ \chi_{5,6}\right\} = 1 \tag{5.16}$$

The condition set $= 1$, meaning ψ and χ have Ramond boundary conditions, they're periodic.

$$\Leftrightarrow S = (1,\ ...,\ 1,\ 0,\ ...,\ 0|0,\ ...,\ 0) \tag{5.17}$$

The basis now becomes $B = \{1,\ S\}$.

Then we must check this basis vector against the ABK rules to ensure it is a viable basis vector. It should be noted that N_i will always equal 2 as we are working in $\Xi = \mathbb{Z}_2 \oplus ... \oplus \mathbb{Z}_2$, meaning we are working in the $\mathbb{Z}_2$ orbifold.

Rule one is satisfied, arbitrarily. Rule 2, $\vec{S}\cdot\vec{S} = 4$ and $\vec{S}\cdot 1 = 4$, where $1 \cdot 1$ is already assumed, so

$$N_{S,S} \cdot S \cdot S = 8 = 0 \mod 4 \checkmark \tag{5.18}$$

$$N_{S,1} \cdot S \cdot 1 = 4 = 0 \mod 4 \checkmark \tag{5.19}$$

Rule 3,

$$N_{S,S} \cdot S \cdot S = 8 = 0 \mod 8 \checkmark \tag{5.20}$$

And rules 4 and 5 are trivial. So we now have a viable basis, where we have four sectors in the additive group Ξ:

$$\{1,\ S,\ 1+S,\ NS\} \tag{5.21}$$

The Virasoro condition in the $\{1 + S\}$ sector follows as

$${M_L}^2 = -\frac{1}{2} + \frac{6}{8} + 0 = -1 + \frac{22}{8} + 0 = {M_R}^2 \tag{5.22}$$

Again, we have no massless state. Note, here we have no fermionic oscillators $N_L$and N_R as the sector is purely a Ramond vacua.

For a general GSO projection in the NS sector

$$e^{i\pi\ S\cdot F_{NS}}|s\rangle_{NS} = \delta_{NS}\delta_S|s\rangle_{NS} = -|s\rangle_{NS} \tag{5.23}$$

Performing the GSO projection first on the graviton state

$$e^{i\pi\ S\cdot F_{NS}} : \psi^{\mu}_{1/2}\partial\bar{X}^{\nu}_1|0\rangle_{NS} = -1 \tag{5.24}$$

Therefore the graviton state survives projection. The adjoint representation of $SO(44)$ also survives projection

$$e^{i\pi\ S\cdot F_{NS}} : \psi^{\mu}_{1/2}\bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}|0\rangle_{NS} = -1 \tag{5.25}$$

We see the gauge bosons of the adjoint representation of $SU(2)^6$ break as y^i and w^i are antiperiodic with the basis vector S and so only

$$\chi^i_{1/2}\partial\bar{X}^{\mu}_1|0\rangle_{NS} \tag{5.26}$$

Survives and $SU(2)^6$ breaks. We see similar results for the scalars of the adjoint representation of $SU(2)^6 \times SO(44)$, the y^i and w^i states are projected out and only

$$\chi^i_{1/2}\bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}|0\rangle_{NS} \tag{5.27}$$

Survives the projection and $SU(2)^6 \times SO(44)$ breaks. Applying GSO projection to the tachyon state

$$e^{i\pi\ 0\cdot 1}\left\{\bar{\phi}^a_{1/2}\right\} = +\left\{\bar{\phi}^a_{1/2}\right\} \tag{5.28}$$

As $\bar{\phi}$ is antiperiodic the tachyon state doesn't survive projection and is cast out. Having no tachyons helps envisage a viable physical model.

Next is to look at the S sector, the Virasoro condition follows as

$$M_L{}^2 = -\frac{1}{2} + \frac{4}{8} + N_L \implies N_L = 0 \tag{5.29}$$

and

$$M_R{}^2 = -1 + 0 + N_R \implies N_R = 1 \tag{5.30}$$

Therefore we require $N_L = 0$ and $N_R = 1$ for the massless spectrum in the sector. So we either require two right-moving fermionic oscillators or 1 right-moving bosonic oscillator. Constructing the S vacuum as

$$|s\rangle_L = \begin{matrix} |\pm\rangle & |\pm\rangle & |\pm\rangle & |\pm\rangle & |0\rangle_L \\ \psi^\mu_{1,2} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6} & \end{matrix} \tag{5.31}$$

in spinorial, spin-$\frac{1}{2}$, representation, means we are no longer denoting the individual fermions but instead the parity of the fermionic modes. This means the fermionic number becomes

$$F : |\pm\rangle = \begin{cases} 0, & |+\rangle \\ -1, & |-\rangle \end{cases} \tag{5.32}$$

Counting the total number of negative states

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 16 \tag{5.33}$$

there are 16 possible choices of state in this sector. The states in the S vacuum are

$$|s\rangle_L \partial \bar{X}^\mu_1 |0\rangle_R \tag{5.34}$$

Which have spin-$\frac{3}{2}$ and are interpreted as gravitinos

$$|s\rangle_L \bar{\phi}^a_{1/2} \bar{\phi}^b_{1/2} |0\rangle_R \tag{5.35}$$

Which have spin-$\frac{1}{2}$. When computing the one-loop amplitudes, the matrix that comprises

$$\begin{matrix} & 1 & S \\ \begin{matrix} 1 \\ S \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} -1 & \pm 1 \\ \pm 1 & \pm 1 \end{pmatrix}} \end{matrix} \tag{5.36}$$

Only $C\left(\begin{smallmatrix} S \\ 1 \end{smallmatrix}\right)$ or $C\left(\begin{smallmatrix} 1 \\ S \end{smallmatrix}\right)$ are independent the rest are chosen to ensure modular invariance, therefore we choose

$$\begin{matrix} & 1 & S \\ \begin{matrix} 1 \\ S \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}} \end{matrix} \tag{5.37}$$

Projections of basis vectors 1 and S act the same way in the S-sector. The general GSO projection of basis vector 1 is

$$e^{i\pi \, 1 \cdot F_S} |s\rangle_S = +|s\rangle_S \tag{5.38}$$

$$e^{i\pi \, 1 \cdot F_S} \left\{ |s\rangle_L \bar{\phi}^a \bar{\phi}^b |0\rangle_R \right\} = e^{i\pi \, 1 \cdot \#|-\rangle} |s\rangle_S \tag{5.39}$$

It is seen that the basis of survival of this projection is the number of $|-\rangle$ boundary conditions in the state. Only the states with an even number of $|-\rangle$ survive and the rest are projected out. The only remaining states are

$$\left[\binom{4}{0}+\binom{4}{2}+\binom{4}{4}\right]\bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}|0\rangle_R \tag{5.40}$$

a spin-$\frac{1}{2}$ state that corresponds to the gaugino and

$$\left[\binom{4}{0}+\binom{4}{2}+\binom{4}{4}\right]\partial\bar{X}^\mu_1|0\rangle_R \tag{5.41}$$

a spin-$\frac{3}{2}$ state that corresponds to the gravitino. Splitting the state

$$\left[\binom{4}{0}+\binom{4}{2}+\binom{4}{4}\right]\longrightarrow\binom{1}{0}\left[\binom{3}{0}+\binom{3}{2}\right]+\binom{1}{1}\left[\binom{3}{1}+\binom{3}{3}\right] \tag{5.42}$$

we see we have the ψ^μ fermions and the $\chi_{1,\dots,6}$ fermions, these together are the two components of a spacetime Weyl spinor. We also have four gravitinos, this corresponds to $\mathcal{N}=4$ SUSY, and currently a gauge group $SO(44)$. We now look to reduce this gauge group by adding another basis vector.

5.3 Adding the b_1 Basis Vector

We look to add another basis vector, b_1

$$b_1=\left\{\psi^\mu_{1,2},\ \chi_{1,2},\ y_{3,4},\ y_{5,6}\middle|\overline{y_{3,4}},\ \overline{y_{5,6}},\ \bar{\psi}^\mu_{1,\dots,5},\ \bar{\eta}^1\right\} \tag{5.43}$$

Giving a new basis

$$B=\{1,S,b_1\} \tag{5.44}$$

With the additive group

$$\Xi=\{1,\ S,\ b_1,\ 1+S,\ 1+b_1,\ S+b_1,\ 1+S+b_1,\ NS\} \tag{5.45}$$

Next we again check this basis vector against the ABK rules

$$N_{1,b_1}\cdot 1\cdot b_1=-8=0 \mod 4\ \checkmark \tag{5.46}$$

$$N_{S,b_1}\cdot S\cdot b_1=4=0 \mod 4\ \checkmark \tag{5.47}$$

$$N_{b_1}\cdot b_1\cdot b_1=-8=0 \mod 8\ \checkmark \tag{5.48}$$

The matching condition for b_1 follows as

$${M_L}^2=-\frac{1}{2}+\frac{4}{8}+0=-1\frac{8}{8}+0={M_R}^2 \tag{5.49}$$

Where now there are massless states and no tachyons. We now have two new independent one-loop amplitudes, $C\binom{1}{b_1}$ and $C\binom{S}{b_1}$, so the one-loop amplitude matrix is found to be

$$\begin{array}{c} \\ 1 \\ S \\ b_1 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & S & b_1 \end{array} \\ \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & +1 \\ -1 & +1 & \cdot \end{pmatrix}\end{array} \tag{5.50}$$

First we perform the GSO projection of basis vector b_1 in the NS sector, our general GSO projection will be

$$e^{i\pi\, b_1\cdot F_{NS}}|s\rangle_{NS} = \delta_{NS} C \begin{pmatrix} NS \\ b_1 \end{pmatrix}^* |s\rangle_{NS} = \delta_{NS}\delta_{b_1}|s\rangle_{NS} = -|s\rangle_{NS} \tag{5.51}$$

Once again, $\delta_{NS} = +1$ as the NS sector is antiperiodic and $\delta_{b_1} = -1$ as b_1 is constructed in a purely Ramond vacua and thus periodic. We will consider the massless states with the χ^i fermions, as these are the superpartners of the compactified internal degrees of freedom and relate to the supercurrent

$$e^{i\pi b_1\cdot F_{NS}} : \left\{\chi^{1,2}_{1/2},\ \chi^{3,4}_{1/2},\ \chi^{5,6}_{1/2}\right\}\partial\bar{X}^\mu_1|0\rangle_{NS} \tag{5.52}$$

We see the only state that survives is $\chi^{1,2}_{1/2}\partial\bar{X}^\mu_1|0\rangle_{NS}$ as $\chi^{1,2}_{1/2}$ is periodic with b_1 whereas $\chi^{3,4}_{1/2}$ and $\chi^{5,6}_{1/2}$ are antiperiodic with b_1 and so these states are projected out. Next we consider

$$e^{i\pi b_1\cdot F_{NS}} : \psi^\mu_{1/2}\bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}|0\rangle_{NS} \tag{5.53}$$

ψ^μ is periodic with b_1, but we have four possible combinations of boundary conditions for $\bar{\phi}^a\bar{\phi}^b$ as each ϕ can be either periodic or antiperiodic with b_1

$$\bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2} : \{1,\ 1\},\quad \{1,\ 0\},\quad \{0,\ 1\},\quad \{0,\ 0\} \tag{5.54}$$

so the GSO projections are as follows

$$\{1,\ 1\} : e^{i\pi b_1\cdot F_{NS}} : \psi^\mu_{1/2}\left\{\bar{\psi}^{1,\dots5},\ \bar{\eta}^1,\ \overline{y_{3,4}},\ \overline{y_{5,6}}\right\}\left\{\bar{\psi}^{1,\dots5},\ \bar{\eta}^1,\ \overline{y_{3,4}},\ \overline{y_{5,6}}\right\}|0\rangle_{NS} = -1 \quad \checkmark \tag{5.55a}$$

$$\{1,\ 0\} : e^{i\pi b_1\cdot F_{NS}} : \psi^\mu_{1/2}\left\{\bar{\psi}^{1,\dots5},\ \bar{\eta}^1,\ \overline{y_{3,4}},\ \overline{y_{5,6}}\right\}\left\{\bar{\eta}^2,\ \bar{\eta}^3,\ \overline{y_{1,2}},\ \overline{w_{1,\dots,6}},\bar{\phi}^{1,\dots8}\right\}|0\rangle_{NS} = +1 \quad \text{x} \tag{5.55b}$$

$$\{0,\ 1\} : e^{i\pi b_1\cdot F_{NS}} : \psi^\mu_{1/2}\left\{\bar{\eta}^2,\ \bar{\eta}^3,\ \overline{y_{1,2}},\ \overline{w_{1,\dots,6}},\bar{\phi}^{1,\dots8}\right\}\left\{\bar{\psi}^{1,\dots5},\ \bar{\eta}^1,\ \overline{y_{3,4}},\ \overline{y_{5,6}}\right\}|0\rangle_{NS} = +1 \quad \text{x} \tag{5.55c}$$

$$\{0,\ 0\} : e^{i\pi b_1\cdot F_{NS}} : \psi^\mu_{1/2}\left\{\bar{\eta}^2,\ \bar{\eta}^3,\ \overline{y_{1,2}},\ \overline{w_{1,\dots,6}},\bar{\phi}^{1,\dots8}\right\}\left\{\bar{\eta}^2,\ \bar{\eta}^3,\ \overline{y_{1,2}},\ \overline{w_{1,\dots,6}},\bar{\phi}^{1,\dots8}\right\}|0\rangle_{NS} = -1 \quad \checkmark \tag{5.55d}$$

Therefore only the states $\psi^\mu_{1/2}\{1,\ 1\}|0\rangle_{NS}$ and $\psi^\mu_{1/2}\{0,\ 0\}|0\rangle_{NS}$ survive and the other two are projected out. This causes our $SO(44)$ gauge group to break to

$$SO(44) \overset{\text{breaks}}{\longrightarrow} SO(16)\times SO(28) \tag{5.56}$$

As there are 8 fermions in the $\{1,1\}$ massless state, giving $SO(16)$ and 14 fermions in the $\{0,0\}$ massless state, giving $SO(28)$. We also see a similar breaking for our next massless state,

$$e^{i\pi b_1\cdot F_{NS}} : \left\{\chi^{1,2}_{1/2},\ \chi^{3,4}_{1/2},\ \chi^{5,6}_{1/2}\right\}\bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}|0\rangle_{NS} \tag{5.57}$$

However, this time, the states $\chi^{1,2}_{1/2}\{1,\ 1\}$, $\chi^{1,2}_{1/2}\{0,\ 0\}$, $\chi^{3,4,5,6}_{1/2}\{1,\ 0\}$ and $\chi^{3,4,5,6}_{1/2}\{0,\ 1\}$ survive and the other states are projected out. Next is to analyse the b_1 basis vector in the S-sector. We will use the notation defined in (5.31) as we look to satisfy the general GSO projection

$$e^{i\pi\, b_1\cdot F_S}|s\rangle_S = \delta_S C \begin{pmatrix} S \\ b_1 \end{pmatrix}^* |s\rangle_S = -|s\rangle_S \tag{5.58}$$

Where we have used our matrix (5.50) with $\delta_S = -1$. First looking at our gravitino state in (5.41),

$$|s\rangle_L\partial\bar{X}^\mu_1|0\rangle_R \tag{5.59}$$

comparing b_1 and S, we can see that they are only periodic for $\psi^\mu_{1/2}$ and $\chi_{1,2}$ and antiperiodic for $\chi_{3,4}$ and $\chi_{5,6}$. This leaves us states with

$$|+\rangle|+\rangle|-\rangle|-\rangle = \binom{4}{2} \tag{5.60}$$

After the GSO projections we see this break

$$\binom{4}{2} \xrightarrow{\text{breaks}} \binom{2}{1}\binom{2}{1} = \left[\binom{1}{0}\binom{1}{1}\right]\binom{2}{1} + \left[\binom{1}{0}\binom{1}{1}\right]\binom{2}{1} \tag{5.61}$$

Before we had $\mathcal{N} = 4$ SUSY, we now see the two $\binom{2}{1}$ spin-$\frac{3}{2}$ states corresponding to two gravitinos and so our SUSY has gone down

$$\mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \tag{5.62}$$

And the breaking specifically is

$$\{\psi^\mu, \chi_{1,2}, \chi_{3,4}, \chi_{5,6}\} \rightarrow \{\psi^\mu, \chi_{1,2}\}\{\chi_{3,4}, \chi_{5,6}\} \tag{5.63}$$

Finally we look at the gauginos in (5.40), again we must only consider the $\binom{4}{2}$ state.

$$\binom{4}{2} \bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}|0\rangle_R \tag{5.64}$$

When performing the GSO projection on the left handed part, meaning on $\binom{4}{2}$ we find

$$\binom{4}{2} \longrightarrow \left[\binom{2}{0} + \binom{2}{2}\right]_+ \text{ or } \left[\binom{2}{1} + \binom{2}{1}\right]_- \tag{5.65}$$

So we either have even states, where the + means positive chirality, or odd states, where the - means negative chirality. Performing the GSO projection on the right handed part, meaning on $\bar{\phi}^a\bar{\phi}^b$ we find the surviving states are either periodic boundary conditions, $\{1, 1\}$, antiperiodic boundary conditions, {0,0}, or one periodic and one antiperiodic boundary condition, i.e. {1,0}, I will denote these as A and B respectively, so

$$\begin{aligned} A = &\left[\{\bar{\psi}^{1,\ldots 5}, \bar{\eta}^1, \overline{y_{3,4}}, \overline{y_{5,6}}\}\{\bar{\psi}^{1,\ldots 5}, \bar{\eta}^1, \overline{y_{3,4}}, \overline{y_{5,6}}\}\right]_+, \\ &\left[\{\bar{\eta}^2, \bar{\eta}^3, \overline{y_{1,2}}, \overline{w_{1,\ldots,6}}, \bar{\phi}^{1,\ldots 8}\}\{\bar{\eta}^2, \bar{\eta}^3, \overline{y_{1,2}}, \overline{w_{1,\ldots,6}}, \bar{\phi}^{1,\ldots 8}\}\right]_+ \end{aligned} \tag{5.66}$$

So the overall chirality of A will be positive.

$$B = \left[\{\bar{\psi}^{1,\ldots 5}, \bar{\eta}^1, \overline{y_{3,4}}, \overline{y_{5,6}}\}_-\{\bar{\eta}^2, \bar{\eta}^3, \overline{y_{1,2}}, \overline{w_{1,\ldots,6}}, \bar{\phi}^{1,\ldots 8}\}_+\right] \tag{5.67}$$

And the overall chirality of B will be positive. So we look to have states that are a combination of positive and negative chirality to give neutral chirality states, this leaves

$$\left[\binom{2}{0} + \binom{2}{2}\right] \otimes B \text{ and } \left[\binom{2}{1} + \binom{2}{1}\right] \otimes A \tag{5.68}$$

Are the invariant states, these complete the $\mathcal{N} = 2$ supersymmetric representations. From the GSO projections of b_1 in the S-sector, it is clear that the S-sector is the supersymmetry generator. So all we need do to find the number of super symmetries is count the number of gravitinos left in the massless spectrum as if (5.5) corresponds to gravitons, then $|s\rangle_L$ counts the number of gravitons.

Last but not least, we must find the new states that arise from the b_1 sector after GSO projection of the other basis vectors. Writing b_1 in the combinatorial form introduced in (5.31) we have

$$b_1 = \sum_{i=0}^{12} \binom{12}{i} \tag{5.69}$$

With i counting the number of $|-\rangle$ boundary conditions. For basis vector 1, we have the general GSO projection

$$e^{i\pi\ 1\cdot F_{b_1}}|s\rangle_{b_1} = \delta_{b_1} C \begin{pmatrix} b_1 \\ 1 \end{pmatrix} |s\rangle_{b_1} = +|s\rangle_{b_1} \tag{5.70}$$

So we require an even number of $|-\rangle$ in the exponential for the state to survive, and so all states with an odd number of $|-\rangle$ are projected out, leaving

$$\left[\binom{12}{0} + \binom{12}{2} + \binom{12}{4} + \binom{12}{6} + \binom{12}{8} + \binom{12}{10} + \binom{12}{12}\right] \tag{5.71}$$

For basis vector S, we choose the general GSO projection to be

$$e^{i\pi\ S\cdot F_{b_1}}|s\rangle_{b_1} = \delta_{b_1} C \begin{pmatrix} b_1 \\ S \end{pmatrix} |s\rangle_{b_1} = +|s\rangle_{b_1} \tag{5.72}$$

So the surviving states are

$$S: \quad \left[\overset{1}{\binom{2}{0}} + \overset{1}{\binom{2}{2}}\right] \left[\overset{10\ (0)'s}{\binom{10}{\text{even}}}\right] \tag{5.73}$$

And for basis vector b_1, we choose the general GSO projection to be

$$e^{i\pi\ b_1\cdot F_{b_1}}|s\rangle_{b_1} = \delta_{b_1} C \begin{pmatrix} b_1 \\ b_1 \end{pmatrix} |s\rangle_{b_1} = +|s\rangle_{b_1} \tag{5.74}$$

So the surviving states are

$$b_1: \quad \left[\overset{1}{\binom{2}{0}} + \overset{1}{\binom{2}{2}}\right] \left[\overset{10\ (1)'s}{\binom{10}{\text{even}}}\right] \tag{5.75}$$

So far we have reduced the large $SO(44)$ gauge group, and reached $\mathcal{N} = \in$ SUSY. But we still only have massless states. To go on to obtain a chiral theory that is viable as a theory for the standard model, we next look to reduce these groups further and to see chiral matter appear, to do so we must add more basis vectors to our basis.

5.4 Adding Basis Vectors b_2 and b_3

So going forward, to have the minimum $\mathcal{N} = 1$ SUSY, we can simply check We now introduce two more basis vectors

$$b_2 = \{\psi^{\mu}_{1,2}, \chi_{34}, y_{12}, w_{56} | \overline{y}_{12}, \overline{w}_{56}, \bar{\psi}^{1\ldots5}, \bar{\eta}^2\} \tag{5.76}$$

$$b_3 = \{\psi^\mu_{1,2}, \chi_{5,6}, w_{1,2}, w_{3,4} | \overline{w}_{1,2}, \overline{w}_{3,4}, \bar{\psi}^{1,\ldots,5}, \bar{\eta}^3\} \tag{5.77}$$

To expand our basis to

$$B = \{1,\ S,\ b_1,\ b_2,\ b_3\} \tag{5.78}$$

This is the full NAHE set.

As we have seen that S is the the SUSY generator, going forward, to have the minimum $\mathcal{N} = 1$ SUSY, we can simply check the spectrum from $\{1,\ ...,\ b_1,\ ...,\ b_n\}$ and the sectors $S + \{1,\ ...,\ b_1,\ ...,\ b_n\}$ to give the superpartners. First we must, again, check the new basis vectors against the ABK rules:

$$b_2 \cdot \vec{1} = \frac{8}{2} - 8 = -4 = 0 \mod 4\ \checkmark \tag{5.79}$$

$$b_2 \cdot b_1 = \frac{2}{2} - 5 = -4 = 0 \mod 4\ \checkmark \tag{5.80}$$

$$N_{b_2} \cdot b_2 \cdot S = 2\left(\frac{4}{2} - 0\right) = 4 = 0 \mod 4\ \checkmark \tag{5.81}$$

$$N_{b_2} \cdot b_2 \cdot b_2 = 2\left(\frac{8}{2} - 8\right) = -8 = 0 \mod 8\ \checkmark \tag{5.82}$$

$$b_3 \cdot \vec{1} = \frac{8}{2} - 8 = -4 = 0 \mod 4\ \checkmark \tag{5.83}$$

$$b_3 \cdot b_1 = \frac{2}{2} - 5 = -4 = 0 \mod 4\ \checkmark \tag{5.84}$$

$$N_{b_3} \cdot b_3 \cdot S = 2\left(\frac{4}{2} - 0\right) = 4 = 0 \mod 4\ \checkmark \tag{5.85}$$

$$b_3 \cdot b_2 = \frac{2}{2} - 5 = -4 = 0 \mod 4\ \checkmark \tag{5.86}$$

$$N_{b_3} \cdot b_3 \cdot b_3 = 2\left(\frac{8}{2} - 8\right) = -8 = 0 \mod 8\ \checkmark \tag{5.87}$$

Additionally, we will check the basis vectors for the Virasoro condition,

$$M_L{}^2 = -\frac{1}{2} + \frac{4}{8} + N_L = -1 + \frac{8}{8} + N_R = M_R{}^2 \tag{5.88}$$

we reach the same condition for both b_2 and b_3 as they have the same number of respective left and right fermions. Again we have no tachyons present and N_L and N_R will be zero, meaning again we are working in a purely Ramond vacua. Next is to compute the one-loop vacuum amplitudes, to give the coefficients for the GSO projection

$$\begin{array}{c} \\ 1 \\ S \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & S & b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} -1 & +1 & -1 & \pm1 & \pm1 \\ +1 & +1 & +1 & \pm1 & \pm1 \\ -1 & +1 & -1 & \pm1 & \pm1 \\ \pm1 & \mp1 & \pm1 & \pm1 & \pm1 \\ \pm1 & \mp1 & \pm1 & \pm1 & \pm1 \end{pmatrix} \end{array} \tag{5.89}$$

We have introduced six new phases to check against the massless spectrum, specifically $C\binom{1}{b_2}$, $C\binom{1}{b_3}$, $C\binom{S}{b_2}$, $C\binom{S}{b_3}$, $C\binom{b_1}{b_2}$ and $C\binom{b_1}{b_3}$ First we look at basis vectors b_2 and b_3 in the NS sector, we will start with the vectors bosons

$$\psi^{\mu}_{1/2}\bar{\phi}^{a}_{1/2}\bar{\phi}^{b}_{1/2}|0\rangle_{NS} \tag{5.90}$$

Which, after the b_1 projections we performed in chapter 5.3, became

$$\psi^{\mu}_{1/2}\left\{\bar{\psi}^{1,\ldots 5},\ \bar{\eta}^{1},\ \overline{y_{3,4}},\ \overline{y_{5,6}}\right\}\left\{\bar{\psi}^{1,\ldots 5},\ \bar{\eta}^{1},\ \overline{y_{3,4}},\ \overline{y_{5,6}}\right\}|0\rangle_{NS} \tag{5.91}$$

and

$$\psi^{\mu}_{1/2}\left\{\bar{\eta}^{2},\ \bar{\eta}^{3},\ \overline{y_{1,2}},\ \overline{w_{1,\ldots,6}},\bar{\phi}^{1,\ldots 8}\right\}\left\{\bar{\eta}^{2},\ \bar{\eta}^{3},\ \overline{y_{1,2}},\ \overline{w_{1,\ldots,6}},\bar{\phi}^{1,\ldots 8}\right\}|0\rangle_{NS} \tag{5.92}$$

After applying b_2 and b_3 GSO projections, these vector bosons become

$$\begin{array}{cccccc} \psi^{\mu}_{1/2} & \left\{\bar{\psi}^{1,\ldots,5},\bar{\psi}^{1,\ldots,5}\right\} & \left\{\bar{\eta}^{1},\bar{y}^{3,\ldots,6}\right\} & \left\{\bar{\eta}^{2},\bar{y}^{1,2},\bar{w}^{5,6}\right\} & \left\{\bar{\eta}^{3},\bar{w}^{1,\ldots,4}\right\} & \left\{\bar{\phi}^{1,\ldots,8}\right\} \\ & SO(10) & SO(6)_1 & SO(6)_2 & SO(6)_3 & SO(16) \end{array} \tag{5.93}$$

with their corresponding gauge groups. This is our first notion of $SO(10)$, where $\bar{\psi}^{1,\ldots,5}$ generate the $SO(10)$ by each contributing an $SU(2)$ subgroup.

We see next that, after the GSO projection with b_2 and b_3, all the $\left\{\chi^{i},\ y^{i},\ w^{i}\right\}\partial\bar{X}^{\mu}_{1}|0\rangle_{NS}$ fermions are projected out as $\{\chi^{i},\ y^{i},\ w^{i}\}$ are all antiperiodic under the combination of b_2 and b_3. Then we look to the scalars of the adjoint representation of $SU(2)^6\times SO(44)$ in (5.10), after the projection with , we were left with the states

$$\chi_{1,2}\left\{\bar{\eta}_2,\bar{\eta}_3,\overline{w}_{1,\ldots,6},\overline{y}_{1,2},\bar{\phi}_{1,\ldots,8}\right\}\left\{\bar{\eta}_2,\bar{\eta}_3,\overline{w}_{1,\ldots,6},\overline{y}_{1,2},\bar{\phi}_{1,\ldots,8}\right\} \tag{5.94a}$$

$$\chi_{1,2}\left\{\bar{\phi}_{1,\ldots,5},\bar{\eta}_1,\overline{y}_{3,4},\overline{y}_{5,6}\right\}\left\{\bar{\phi}_{1,\ldots,5},\bar{\eta}_1,\overline{y}_{3,4},\overline{y}_{5,6}\right\} \tag{5.94b}$$

$$\chi_{3,4}\ ,\ \chi_{5,6}\left\{\bar{\psi}_{1,\ldots,5},\bar{\eta}_1,\overline{y}_{3,\ldots,6}\right\}\left\{\overline{w}_{1,\ldots,6},\overline{y}_{1,2},\bar{\eta}_2,\bar{\eta}_3,\bar{\phi}_{1,\ldots,8}\right\} \tag{5.94c}$$

For (5.94a), I denote the periodicity of each fermion with respect to b_2 and b_3, as follows

$$\begin{array}{ccccccccccc} & \chi_{1,2} & \{\bar{\eta}_2, & \bar{\eta}_3, & \overline{w}_{1,2}, & \overline{w}_{3,4}, & \overline{w}_{5,6}, & \overline{y}_{1,2}, & \bar{\phi}_{1,\ldots,8}\} & \left\{\bar{\eta}_2,\ \bar{\eta}_3,\ \overline{w}_{1,2},\ \overline{w}_{3,4},\ \overline{w}_{5,6},\ \overline{y}_{1,2},\ \bar{\phi}_{1,\ldots,8}\right\} \\ b_2: & +1 & -1 & +1 & +1 & +1 & -1 & -1 & +1 & \\ & & & & & & & & & \text{Repeat} \\ b_3: & +1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & \end{array} \tag{5.95}$$

so after projection we are left with the state

$$\chi_{1,2}\left\{\bar{\eta}_2,\overline{w}_{5,6},\overline{y}_{1,2}\right\}\left\{\bar{\eta}_3,\overline{w}_{1,\ldots,4}\right\} \tag{5.96}$$

With the first bracket being periodic with b_2 and the second bracket being periodic with b_3. Using the same format for (5.94b), after projection, we are left with the state

$$\chi_{1,2}\left\{\bar{\psi}_{1,\ldots,5}\right\}\left\{\bar{\eta}_1,\overline{y}_{3,\ldots,6}\right\} \tag{5.97}$$

I continue this for (5.94c) for both $\chi_{3,4}$ and $\chi_{5,6}$, after projection the states left are

$$\chi_{3,4}\left\{\bar{\eta}_1,\overline{y}_{3,\ldots,6}\right\}\left\{\overline{w}_{1,\ldots,4},\bar{\eta}_3\right\} \tag{5.98}$$

$$\chi_{3,4}\left\{\bar{\psi}_{1,...,5}\right\}\left\{\overline{w}_{1,...,4},\bar{\eta}_3\right\} \tag{5.99}$$

$$\chi_{5,6}\left\{\bar{\eta}_1,\overline{y}_{3,...,6}\right\}\left\{\overline{w}_{5,6},\overline{y}_{1,2},\bar{\eta}_2\right\} \tag{5.100}$$

$$\chi_{5,6}\left\{\bar{\psi}_{1,...,5}\right\}\left\{\overline{w}_{1,...,4},\bar{\eta}_3\right\} \tag{5.101}$$

The expressions in (5.96-5.101) therefore state the massless spectrum of the NS sector.

Secondly, we must analyse the b_2 and b_3 basis vectors in the S-sector. After b_1, we were left with the state (5.61) with two gravitinos. Comparing basis vector b_2 with the gravitino state

$$\begin{array}{rcccccc} & \psi^{\mu}_{1,2} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6} & \partial\bar{X}^{\mu}_1 & |0\rangle_L \\ |s\rangle_L = & |\pm\rangle & |\pm\rangle & |\pm\rangle & |\pm\rangle & & \\ b_2: & 1 & 0 & 1 & 0 & & \end{array} \tag{5.102}$$

our state breaks

$$\left[\binom{1}{0}\binom{1}{1}\right]\binom{2}{1}+\left[\binom{1}{0}\binom{1}{1}\right]\binom{2}{1}\rightarrow\binom{1}{1}\binom{1}{0}\binom{2}{0}+\binom{1}{0}\binom{1}{1}\binom{2}{2} \tag{5.103}$$

and we are left with just one gravitino, giving $\mathcal{N}=1$ SUSY. As for the projection of this single gravitino state for b_3, depending on the one-loop vacuum amplitude, this state either survives or is projected out

$$C\begin{pmatrix} S \\ b_3 \end{pmatrix}=\begin{cases} +1 & \text{survives} \\ -1 & \text{projected out} \end{cases} \tag{5.104}$$

This means we can decide whether or not we wish to keep this single gravitino. So we have set up a $\mathcal{N}=1$ SUSY model with gauge group from (5.93),

$$SO(10)\times SO(6)^3\times SO(16) \tag{5.105}$$

We see the three chiral 16 spinor representations of SO(10) when we perform the GSO projections from the b_2 and b_3 basis vectors in the b_1 sector. It is important to note that there is cyclic permutation symmetry between the three basis vectors

$$b_1\rightarrow b_2\rightarrow b_3\rightarrow b_1 \tag{5.106}$$

$$\begin{array}{ccccc} \left\{\bar{\eta}_1,\overline{y}_{3,...,6}\right\} & & \left\{\bar{\eta}_2,\overline{y}_{1,2},\overline{w}_{5,6}\right\} & & \left\{\bar{\eta}_3,\overline{w}_{1,...,4}\right\} \\ & \rightarrow & & \rightarrow & \\ \left\{\chi_{1,2}y_{3,...,6}\right\} & & \left\{\chi_{3,4},y_{1,2},w_{5,6}\right\} & & \left\{\chi_{5,6},w_{1,...,4}\right\} \end{array} \tag{5.107}$$

This means the GSO projections from b_2 and b_3 on the b_1-sector generate the 16 spinor representations of the b_1-sector, the same is said for b_1 and b_2 on the b_3-sector and b_1 and b_3 on the b_2-sector, thus giving the three generations of chiral matter. We therefore look to perform the GSO projection of b_2 on the states in the b_1-sector in (5.75), the surviving states after this are

$$\begin{aligned} &\binom{2}{0}\left[\binom{4}{0}+\binom{4}{2}+\binom{4}{4}\right]\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right]\binom{1}{0}+ \\ &\binom{2}{0}\left[\binom{4}{1}+\binom{4}{3}\right]\left[\binom{4}{1}+\binom{4}{3}\right]\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right] \end{aligned} \tag{5.108}$$

+ other components due to CPT invariance.

The CPT invariance ensures that for each of our fermionic states, we have an associated anti-fermion, it does not mean they form representations but are rather reflected versions, to ensure we derive the entire physical spectrum. The term

$$\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right] \tag{5.109}$$

Can be interpreted as the 16 generations of SO(10). We can also interpret

$$\left[\binom{5}{1}+\binom{5}{3}+\binom{5}{5}\right] \tag{5.110}$$

As the $\overline{16}$ generations of SO(10), which have been projected out of our states above, but is included in the components due to CPT invariance, to ensure we also have the opposite-chirality fermions in our spectrum. However, as the representation is chiral, in reality there's only the left-handed 16 generations present. Due to the chiral permutation symmetry signalled in (5.106, 5.107) we also have 16 generations from the b_2-sector and another 16 generations from the b_3-sector, therefore producing our three generations of 16 chiral spinor representations, of the standard model, now embedded in $SO(10)$.

Continuing down this path, we can add more basis vectors to the NAHE set, this is the set that is required to then build viable and realistic generation models. We have reached a point where we see $\mathcal{N}=1$ SUSY and an observable gauge group, $SO(10)$ with 48 generations, introducing further basis vectors would break down our gauge group further, to eventually see the known standard model emerge. As stated in (5.93), $\bar{\psi}^{1,\cdots,5}$ have the gauge group $SO(10)$, but further GSO projections would see these states break

$$\psi^\mu\bar{\psi}^{1\ldots5}\bar{\psi}^{1\ldots5}|0\rangle_{NS} \overset{\text{breaks}}{\longrightarrow} \left\{\psi^\mu\bar{\psi}^{1\ldots3}\bar{\psi}^{1\ldots3}\right\}\left\{\psi^\mu\bar{\psi}^{4,5}\bar{\psi}^{4,5}\right\}|0\rangle_{NS} \tag{5.111}$$

With the gauge group breaking as

$$SO(10) \overset{\text{breaks}}{\longrightarrow} SO(6)\times SO(4) \tag{5.112}$$

This is one example of how a realistic model is derived in the free fermionic formulation. Implementing the NAHE set allows one to construct models with gauge groups such as

$$SU(5)\times U(1) \tag{5.113a}$$
$$SO(6)\times SO(4) \tag{5.113b}$$
$$SU(3)\times SU(2)\times U(1)^2 \tag{5.113c}$$

which are all contained within $SO(10)$. The (5.113c) gauge group resembles the standard model we are trying to reach. The $U(1)^2$ are the two unitary groups, one for the weak hypercharge $U(1)_Y$ already present in the standard model, the other is an anomalous $U(1)$, added as an extension to the existing standard model to aid unification, this helps to make it phenomenologically realistic. It may manifest as $U(1)_{B-L}$ or $U(1)_Z$.

Chapter 6

Conclusion

Throughout the process of building the NAHE set we saw inklings that may lead us back to the standard model, whilst always keeping in mind our main aim - a possible grand unified theory. The discussion of the current standard model reminded us what it was we were building towards, as well as highlighting the disparity between the matter sector and the force sector. Whilst the current standard model makes for a great basis which agrees well with experiments and predictions, it does not provide us with the full picture nor help us to fully explain the universe around us. It was clear to us that the 'Achilles heel' of the standard model is the exception of gravity, highlighting the disparity between quantum mechanics and general relativity.

This is where we began our search for a grand unified theory, to unify the three quantised forces and to allow for the presence of the right-handed neutrino, which we saw could naturally be embedded in $SO(10)$. However, this did not quite cut it, we still lacked the presence of gravity in our model and had no symmetry between the force and matter sectors. To be a fully grand unified theory, we surely wished to combine gravity, and thus general relativity, with the other three fundamental forces.

For these requirements, I believe string theory fits the bill, where we took point-particles and reimagined them to be small strings. Using string theory as a base to build a phenomenological framework in the form of the NAHE set of basis vectors allowed us to demonstrate the mathematical derivation of some of the key properties of the standard model. We were also then able to see the embedding of the current group makeup of the standard model within a single simple irreducible group, our $SO(10)$, but this time with the spin-$\frac{3}{2}$ gravitino present. By implementing strings and supersymmetry, we further displayed a symmetry between bosons and fermions, aiding our unification.

Whilst this is not yet a complete grand unified theory, I hope I have constructed a convincing argument for the capabilities that string theory, and by extension string phenomenology, holds in building a viable grand unified theory. The derived gravitino; the chiral 16 spinor representation of $SO(10)$; and the symmetry displayed between matter and forces through the implementation of both string theory and supersymmetry, are all sanguine signs that we are on the right path, in the hopes of eventually leading to a theory of everything.

However, any experimental proof of strings may not be realised for a long while yet, if ever. As mentioned, string theory is a high energy physics, energies of which we are not yet capable of reproducing. This new energy scale would be required to reveal an overlying symmetry and to realise the 10 or more dimensions in which strings theoretically occur; to eventually reveal these little strings ‘unfold’ from the point-particle picture we are currently seeing. The hopes of observing any evidence for the seemingly ’magic’ $SO(10)$ group may be wishful thinking. But the idea of a grand unified theory puppeteering the universe around us through the manipulation of its strings, is certainly a intriguing one.

Appendix A

Gell-Mann Matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.1}$$

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.2}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.3}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{A.4}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{A.5}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{A.6}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{A.7}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{A.8}$$

These generate the elements of SU(3).

Appendix B

Pauli Matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.1}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{B.2}$$

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.3}$$

We also have the rule

$${\sigma_1}^2 = {\sigma_2}^2 = {\sigma_3}^2 = -i\sigma_1\sigma_2\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{B.4}$$

These matrices generate the elements of SU(2)

Appendix C

Gamma Matrices

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{C.1}$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{C.2}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{C.3}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{C.4}$$

Where

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{C.5}$$

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