String Phenomenology

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Abstract

A study of string theory as a viable model of particle physics.

Contents

Chapter 1

Introduction

The goal of string phenomenology is to predict the framework and data of particle physics using string theory. Showing that string theory is consistent with previous models of particle physics is a major step as gravitation can then be incorporated into a more fundamental framework. To understand how this is achieved we firstly need to understand an earlier model of particle physics known as the standard model, improve this model into a Grand Unified Theory and then show the problem of its incompatibility with the gravitational interaction.

After examining the mechanics of string theory we will show that not only can string theory reproduce the data requirement of the standard model and G.U.T's but improve them to become a more fundamental theory by incorporating gravity. Just as G.U.T's improved on the standard model, string phenomenology seeks to prove that string theory is yet another progression to a deeper and more fundamental model of particle physics.

1.1 The Standard Model

The standard model is a quantum field theory of the three generations of matter and three of the four known interactions that affects these elementary particles.

The elementary particles are separated into two groups;

Fermions (matter particles with spin 1/2 fields) and gauge bosons (force carriers with integer spin fields). There are three generations of fermions denoted here by i where $i = 1, 2, 3$. The fermions consist of quarks and leptons.

Quarks (up-type μ_i and down-type d_i) are defined by carrying colour charge, electric charge and weak isospin, enabling them to interact with other fermions via the electromagnetic or weak interactions.

Leptons (the charged lepton L_i and its corresponding uncharged neutrino N_i)are defined by carrying weak isospin but not the colour charge. Three of the leptons also carry the electromagnetic charge.

The interactions are represented by gauge groups known as unitary groups, each with different dimensions to reflect the number of gauge bosons that generate their force.

All of these groups are required to be gauge invariant meaning that they possess local quantum gauge symmetry. This allows all their observables to remain unchanged after transformations (i.e., their amplitudes are renormalizable). The three local symmetries in the standard model (spin 1 fields) are named; electromagnetism U(1), weak SU(2) and strong SU(3). With the group combination generally written as SU(3)XSU(2)XU(1).

Another vital component of the standard model is the spin 0 higgs field which has one gauge boson known as the higgs particle h. The SU(2) field has three gauge bosons which carry the weak force and in order to satisfy the gauge symmetry they are required to be massless. However the weak interaction is short ranged, behaving as if the gauge bosons are extremely heavy. In order to meet the gauge invariant requirement and describe the weak interaction the higgs field was required to produce heavy weak gauge bosons and still maintain an invariant system. This is known as symmetry breaking.

This completes the gauge fields of the elementary particles,

We have the spin 1 fields,

U(1) gauge field with 1 gauge boson B^μ carrying weak hypercharge.
SU(2) gauge field with 3 gauge bosons $W^{\pm}\mu$,W_3^μ carrying weak isospin.
SU(3) gauge field with 8 gauge bosons G^μ carrying color charge.

The standard model is a chiral gauge theory meaning that the two helicities behave differently. Only the left handed fermions feel the weak force, to combat this problem the right handed fermions can be thought of as left handed anti-fermions.

Note: there are no right handed neutrinos. When the standard model was formulated the neutrino was believed to be massless thus travelling at the light speed constant. Since this constant is unpassable the neutrino could never be viewed to be right handed. The neutrino is now believed to be massive so a correct model requires a right handed neutrino.

The spin 1/2 fields are,

SU(3)singlet,SU(2)doublet,U(1)weak hypercharge -1	(left-handed lepton doublets)(V_L,e_L)=L_{il}
SU(3)singlet,SU(2)singlet,U(1)weak hypercharge 2	(left-handed antileptons)=right handed ch
SU(3)triplet,SU(2)doublet,U(1)weak hypercharge 1/3	(left-handed quark doublets)(U_{iL},D_{iL})=Q_i
SU(3)triplet,SU(2)singlet,U(1)weak hypercharge -4/3	(left-handed up-type antiquark singlets)=r
SU(3)triplet,SU(2)singlet,U(1)weak hypercharge 2/3	(left-handed down-type antiquark singlets)

The inclusion of a generation index i reproduces the 3 families of fermions where i = 1,2,3.

There is only one spin 0 field, the higgs field.

SU(2) doublet H with U(1) weak hyper-charge -1

1.2 Grand Unified Theories

The three gauge groups of the standard model can be combined into a theory with a single renormalizable gauge group. These theories are known as Grand Unified Theories or G.U.T's. The group symmetry allows the particles of the standard model to be represented as different states of a single particle field.

the simplest G.U.T is labelled SU(5),

$$SU(5) \supset SU(3) X SU(2) X U(1)$$

The two smallest representations of SU(5) are 5 and 10,

$$5 = (\overline{3}, 1)_x + (1, 2)_y$$

Which contains 3 (left-handed down-type antiquark singlets) = right handed down quark singlets D_R and the left-handed lepton doublet (left-handed lepton doublets) $(V_L, e_L) = L_{il}$

$$10 = (\overline{3}, 1)_x + (3, 2)_y + (1, 1)_z$$

Which contains the 6 up-type quarks, 3 (left-handed up-type antiquark singlets) = right handed up quark singlets U_R and 3 (left-handed up-type quark singlets). Then 3 (left-handed down-type quark singlets) plus a right handed electron.

This is repeated for each generation of matter and as in the standard model right handed neutrino's are not included.

The next group which contains the standard model is labelled SO(10),

$$SO(10) \supset SU(5) \supset SU(3) X SU(2) X U(1)$$

The S0(10) model produces 16 spinors which contain the $\overline{5}$ and 10 of SU(5) plus a right handed neutrino. This is the complete particle content of the standard model unified in a single group.

1.3 Summary

Defining the standard model as three gauge groups we have shown that it is a strong representation of particles and their interactions. We then improved on the model by unifying the three gauge groups into a single group known as a G.U.T namely SU(5). However SU(5) had no quantum field representation for a right-handed neutrino. This problem was solved by embedding SU(5) into the SO(10) group which allowed a definition of a right-handed neutrino, completing the contents of all known matter and their interactions.

However, in this framework it still is not possible to incorporate the gravitational interaction into the model.

The gauge models previously discussed are so powerful and accurate because they utilise renormalizable quantum fields. This means that inherent infinities can be dealt with by absorbing them into a finite number of measurable parameters. Since gravity is nonrenormalizable it cannot be dealt with in the same way and its infinities go untamed.

If we think of the fundamental particles of SO(10) as different manifestations of a fundamental, finite string then it is possible to describe the gravitational interaction as a gauge theory. We can then unify the gravitational gauge group into the SO(10) group and produce a complete fundamental model of particle physics.

We will now look more closely at string theory with the ultimate aim of reproducing the requirements of the SO(10) model.

Chapter 2

Non-relativistic strings

To understand String Phenomenology we must first understand the mechanics of a non-relativistic string. The strings we will use have both mass and tension and are able to vibrate longitudinally and transversely. From these first principles and applied boundary conditions we will develop an equation of motion. Utilising the Lagrangian equation the action of the string as a function of time can then be calculated.

2.1 Equations of motion of a transverse oscillation

Consider a string in the (x,y) plane with one of its endpoints at the origin and the other at (a,o). As transverse movement is perpendicular to the direction along the string, the transverse oscillations will take place in the y direction (Describing transverse oscillations in higher dimensions can be easily ascertained from this example).

Our first principles are that the string has both mass and tension, the mass per unit length is denoted by μ_0 and the tension is denoted by T_0. The mass total of the string is then given by the mass per unit length multiplied by the length in question or $M = \mu_0 a$.

Consider a small element of the string from (x,y) to (x+dx,y). Now at time t the transverse oscillation of the string will have the spacetime coordinates y(t,x,y) at (x,y) and y(t,x+dx,y) at (x+dx,y) (remember transverse oscillations are only in the y direction). If we assume that the oscillations at these points are small compared to the strings length then it follows that the tension will remain constant. However as there is a different gradient at these two points the tension changes direction, generating a net force on the string. This net force can be calculated by finding the difference between the tension gradient product at both points.

$$dF_v = T_0 \frac{\partial y}{\partial x}|_{x+dx} - T_0 \frac{\partial y}{\partial x}|_x$$

As dx is arbitrarily small this reduces to,

$$dF_v = T_0 \frac{\partial^2 y}{\partial x^2} dx$$

Using Newton's law of motion $F = ma$ (were the mass is found by $dm = \mu_0 dx$) we can calculate the net force to be

$$T_0 \frac{\partial^2 y}{\partial x^2} dx = (\mu_0 dx) \frac{\partial^2 y}{\partial t^2}$$

The dx on each side cancels and then a small rearrangement leaves

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0} \frac{\partial^2 y}{\partial t^2} = 0 \tag{2.1}$$

This should look familiar as it is in fact the wave equation of motion.

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2} \frac{\partial^2 y}{\partial t^2} = 0$$

Were v_0 is the velocity of the waves, implying that

$$v_0 = \sqrt{\frac{T_0}{\mu_0}}$$

is the velocity of the vertical displacement on our non-relativistic string.

The First principles of tension and mass have led to an equation of motion in which the velocity of displacement is proportional to the tension and mass of the string. This leads to the conclusion that the lighter and more tensile the string, the faster the waves will pervade through it.

2.2 Boundary conditions, Initial conditions and acquiring a solution

The wave equation derived (2.1) contains partial derivatives so in order to produce a solution we need to set boundary and initial conditions.

There are two types of boundary conditions that can be used to model the movement of strings, these are known as Dirichlet and Neumann boundary conditions.

The Dirichlet boundary condition is defined as

$$y(t, x = 0) = y(t, x = a) = 0 \tag{2.2}$$

This fixes both ends of the string to a definite point, namely 0 and a.

The Neumann boundary condition is defined as

$$\frac{\partial y}{\partial x}(t, x = 0) = \frac{\partial y}{\partial x}(t, x = a) = 0 \tag{2.3}$$

This allows both ends of the string to slide along the y axis at designated points, namely 0 and a. (We will use both of these Boundary conditions to acquire a solution to highlight the properties of each).

With the boundary conditions defined we can now look at the initial conditions.

The solution to this wave equation will be of the standard form

$$y(t, x) = h_+(x - v_0 t) + h_-(x + v_0 t)$$

with h_+, h_- defining arbitrary single variable functions. The h_+ function plots the right moving wave and the h_- function plots the left moving wave. The superposition of the two gives the complete solution. Initial conditions at t=o can then be used to acquire a definitive solution to the wave equation.

If the string is oscillating sinusoidally then the solution will be

$$y(t, x) = y(x) sin(\omega t + \phi) \tag{2.4}$$

where ω is the angular frequency and ϕ is the phase difference. Substituting this solution (2.4) back into the acquired wave equation (2.1) gives

$$\frac{\partial^2 y(x)}{\partial x^2} + \omega^2 \frac{\mu_0}{T_0} y(x) = 0 \tag{2.5}$$

The general solution of which, will be

$$y_n(x) = A_n sin(l_n x) + B_n cos(l_n x) \tag{2.6}$$

Now we will solve this equation using both Boundary conditions.

Using the Dirichlet boundary conditions (2.2) it follows that

$$\begin{aligned}
&y_n(0) = y_n(a) = 0 \\
&y_n(0) = B_n \rightarrow B_n = 0 \\
&y_n(a) = A_n sin(l_n a) = 0 \rightarrow l_n a = n\pi \rightarrow l_n = \frac{n\pi}{a}
\end{aligned}$$

This gives the solution

$$y_n(x) = A_n sin\frac{n\pi x}{a}$$
$$n = 1, 2, 3...$$

(Note how $n \neq 0$ as this would correspnd to a stationary string). The allowed frequencies of oscillation for a Dirichlet string can now be found by inserting these solutions into the wave equation (2.5) giving,

$$\omega_n = \sqrt{\frac{T_0}{\mu_0}}\left(\frac{n\pi}{a}\right) \tag{2.7}$$

To use the Neumann boundary conditions (2.3) we have to differentiate the general solution (2.6) with respect to x. The general solution then becomes,

$$\frac{\partial y_n(x)}{\partial x} = A_n l_n cos(l_n x) - B_n l_n sin(l_n x)$$

Following that,

$$\frac{\partial y_n(0)}{\partial x} = A_n \rightarrow A_n = 0$$
$$\frac{\partial y_n(0)}{\partial x} = -B_n l_n sin(l_n a) = 0 \rightarrow l_n = \frac{n\pi}{a}$$

This gives the solution

$$y_n(x) = B_n cos\frac{n\pi x}{a}$$
$$n = 0, 1, 2, 3...$$

(Note how $n = 0$ is now allowed. The string doesn't oscillate but is rigidly translated. This is due to the allowance of movement at the endpoints inherent with the Neumann boundary condition.)

The allowed frequencies of oscillation for a Neumann string can now be found by inserting these solutions into the wave equation (2.5) giving,

$$\omega_n = \sqrt{\frac{T_0}{\mu_0}}\left(\frac{n\pi}{a}\right) \tag{2.8}$$

As you can see the allowed frequencies of oscillation are the same for both the Dirichlet and Neumann strings with the Neumann string having one extra solution due to the mobility of its endpoints.

To further understand the mechanics of a non-relativistic string we will use the Lagrangian equation.

2.3 Lagrangian dynamics and its application to a non-relativistic string

The Lagrangian is a function that summarizes the dynamics of a system. Its definition is the kinetic energy T of a system minus its potential energy V.

$$L = T - V$$

If the Lagrangian of a system is known then the systems equations of motion can be calculated by substituting the Lagrangian into the Euler-Lagrange equations. We will construct the Lagrangian of a non-relativistic particle then extend this to a string, finally using the Euler-Lagrange equations to calculate the equations of motion

The Lagrangian for a non-relativistic particle is given by,

$$L(t) = \frac{1}{2}m\left(\frac{dx(t)}{dt}\right)^2 - V(x(t))$$

The Lagrangian is a function of time but it has no explicit time dependance. Its time dependence originates from the position $x(t)$.

The action S can then be defined as,

$$S = \int_\rho L(t)dt$$

where ρ is the path of $x(t)$ from (t_i, x_i) to (t_f, x_f). So the action for any path can be calculated using,

$$S(x) = \int_{t_i}^{t_f} \frac{1}{2}m(\dot{x}(t))^2 - V(x(t))dt$$

The action can be calculated for any path and adheres to what is known as the Hamilton Principle in that the path of a system is one in which the action is stationary. This means that if the path is changed an infintesimal amount the action will not change. If we fix the endpoints at $\delta x(t_i) = \delta x(t_f) = 0$ and then change the path from $x(t)$ to $x(t)+\delta x(t)$. The resulting action of the varied path will be,

$$S(x+\delta x) = \int_{t_i}^{t_f} \left\{ \frac{m}{2} \left(\frac{d}{dt}(x(t)+\delta x(t)) \right)^2 - V(x(t)+\delta x(t)) \right\} dt$$

$$S(x+\delta x) = S(x) + \int_{t_i}^{t_f} \left\{ (m\ddot{x}^2 - \acute{V}(x))\delta x(t) \right\} dt + \varrho(\delta x^2) + ...$$

The Hamilton principle only concerns the first order so the δx^2 terms and higher can be disregarded. The action of the varied path can now be written as $S + \delta S$ implying that,

$$\delta S = \int_{t_i}^{t_f} \left\{ (-m\ddot{x}^2 - V(x))\delta x(t) \right\} dt$$

The action will only be stationary $\delta S = 0$ if the integrand is 0 and this can only happen if

$$m\ddot{x}^2 = -V(x)$$

By introducing the constraint that the action be stationary under variations we have derived an equation of motion which we know to be correct. What we have derived here is Newtons second law of motion.

We will now apply the Lagrangian equation to a non-relativistic string.

Recall that the mass of a segment is $dm = \mu_0 dx$ so the kinetic energy of an infintesimal segment will be $dT = \frac{1}{2}(\mu_0 dx)(\frac{\partial y}{\partial t})^2$. The total kinetic energy can then be calculated as the sum of the kinetic energy of all these segments along the string.

$$T = \int_0^a \frac{1}{2}(\mu_0 dx)(\frac{\partial y}{\partial t})^2$$

The other half of the Lagrangian is the potential energy which emerges from the work applied to stretch the string. If an infintesimal segment of string lies between $(x, 0)$ and $(x + dx, 0)$ and is stretched from its (x, y) position to $(x + dx, y + dy)$ its change in length is calculated by,

$$\Delta l = \sqrt{(dx)^2 + (dy)^2} - dx = dx \left(\sqrt{1 + (\frac{\partial y}{\partial x})^2} - 1 \right) \approx dx \frac{1}{2} \left(\frac{\partial y}{\partial x} \right)^2$$

The work done in strectching a segment is $T_0 \Delta l$. The total potential energy can then be calculated as the work required to stretch an infintesimal segment, summed across the whole length of string,

$$V = \int_0^a \frac{1}{2}(T_0)(\frac{\partial y}{\partial x})^2 dx$$

The Lagrangian for a non-relativistic string can then be written as,

$$L(t) = \int_0^a \left[\frac{1}{2}\mu_0 \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0 \left(\frac{\partial y}{\partial x}\right)^2\right] dx = \int_0^a \bar{L} dx$$

Where $\bar{L}$ is known as the Lagrangian density defined as,

$$\bar{L}\left(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\right) = \frac{1}{2}\mu_0 \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0 \left(\frac{\partial y}{\partial x}\right)^2$$

The action for a non-relativistic string can then be calculated,

$$S = \int_{t_i}^{t_f} L(t)dt = \int_{t_i}^{t_f} dt \int_0^a dx \left[\frac{1}{2}\mu_0 \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0 \left(\frac{\partial y}{\partial x}\right)^2\right]$$

The path is given by the field $y(t, x)$ which is defined across the region of space (t, x). To find the equations of motion we must introduce the constraint that the action is stationary under variations as was shown in the previous example.

Changing the path from $y(t, x)$ to $y(t, x) + \delta y(t, x)$ the resulting action of this varied path will be,

$$S(y + \delta y) = \int_{t_i}^{t_f} dt \int_0^a dx \left(\frac{1}{2}\mu_0 \left(\frac{\partial (y + \delta y)}{\partial t}\right)^2 - \frac{1}{2}T_0 \left(\frac{\partial (y + \delta y)}{\partial x}\right)^2\right)$$

$$S(y + \delta y) = \int_{t_i}^{t_f} dt \int_0^a dx \left(\frac{1}{2}\mu_0 \left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial y}{\partial t}\frac{\partial}{\partial t}\partial y - \frac{1}{2}T_0 \left(\frac{\partial y}{\partial x}\right)^2 - T_0 \frac{\partial y}{\partial x}\frac{\partial}{\partial x}\delta y\right) + \varrho(\delta^2) + ...$$

Again δ^2 terms and higher can be disregarded. The action of the varied path can now be written as $S(y + \delta y) - S(y)$ implying that,

$$\delta S = \int_{t_i}^{t_f} dt \int_0^a dx \left(\mu_0 \frac{\partial y}{\partial t}\frac{\partial \delta y}{\partial t} - T_0 \frac{\partial y}{\partial x}\frac{\partial \delta y}{\partial x}\right)$$

Integrating by parts gives,

$$\delta S = \int_{t_i}^{t_f} dt \int_0^a dx \left[\frac{\partial}{\partial t}\left(\mu_0 \frac{\partial y}{\partial t}\delta y\right) - \mu_0 \frac{\partial^2 y}{\partial t^2}\delta y + \frac{\partial}{\partial x}\left(-T_0 \frac{\partial y}{\partial x}\delta y\right) + T_0 \frac{\partial^2 y}{\partial x^2}\delta y\right]$$

$$\delta S = \int_0^a \left[\mu_0 \frac{\partial y}{\partial t}\delta y\right]_{t_i}^{t_f} dx + \int_{t_i}^{t_f} \left[-T_0 \frac{\partial y}{\partial x}\delta y\right]_{x=0}^{x=a} dt - \int_{t_i}^{t^f} dt \int_0^a dx \left(\mu_0 \frac{\partial^2 y}{\partial t^2} - T_0 \frac{\partial^2 y}{\partial x^2}\right)\delta y$$

For the variation to vanish each of the above three terms must cancel independently. The first term concerns the string at t_i and t_f. These can be easily fixed at zero without any effect on the underlying mechanism of the string. The third term concerns the strings motion were δy is not constrained by boundary or initial conditions. If we set δy to zero we retrieve the wave equation (2.1) we derived in the first part of this chapter. The second term can then be written as,

$$\int_{t_i}^{t_f} \left[\frac{\partial y}{\partial x}(t,a)\delta y(t,a) + \frac{\partial y}{\partial x}(t,0)\delta y(t,0) \right] dt$$

In order for this term to vanish either,

$$\delta y(t,a) = \delta y(t,0) = 0$$

which can be rewritten as

$$\frac{\partial y}{\partial t}(t,a) = \frac{\partial y}{\partial t}(t,0) = 0 \tag{2.9}$$

or

$$\frac{\partial y}{\partial x}(t,a) = \frac{\partial y}{\partial x}(t,0) = 0 \tag{2.10}$$

As you can see, these are the Dirichlet boundary condition (2.9) and the Neumann boundary condition (2.10) in their respective orders.

2.4 Summary

Starting from the first principles of mass and tension we developed a wave equation relating these values to the velocity of the transverse oscillations. From here, two types of boundary conditions and an initial condition were used to acquire a solution to the wave equation and calculate the allowed frequencies of oscillation. Developing a Lagrangian for the system allowed us to plot the action of the string. Then utilising the Euler-Lagrange equations and the Hamilton principle we retrieved the expected equations of motion.

Having gained an understanding of the mechanics of string motion we can now continue into a relativistic environment.

Chapter 3

Relativisitic string

Following on from the ground gained in the previous chapter we will now look at relativistic string. To do this we will take a detailed look at relativistic point-particles. If we calculate the action for a relativistic point-particle we can easily generalize the results and implicate them into a model of a relativistic string.

3.1 Relativistic point particles

The action of a free non-relativistic particle is easily calculated by summing its kinetic energy over a time,

$$S = \int L dt = \int \frac{1}{2} m v^2(t) dt \tag{3.1}$$

Its equation of motion is then calculable by varying the path and using Hamiltons principle to constrain the action under variation.

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = 0$$

So the equation of motion is,

$$\frac{dv}{dt} = 0$$

As the particle is 'free' no forces are acting on it. It's acceleration is zero so it is travelling at a constant velocity. This is fine for a free non-relativistic particle but it cannot be true for a free relativistic particle. There is no velocity limit inherent to the equation so the constant velocity described could be greater than c which is obviously incorrect.

A new relativistic action needs to constructed

The stage relativistic physics is played out on is spacetime and the path traced out by a particle is known as the world-line of the particle. The equations of motion derived from the action must be Lorentz invariant in that any and all Lorentz observers must calculate the same action when observing the particle. The action for a relativistic particle can be calculated by integrating over its world-line and multiplying it by the relativistic factor $-mc$,

$$[S = -mc \int_{\rho} ds \tag{3.2}$$

The action can be rearranged to be an integral over time by making a substitution for ds with the Lorentz transformation,

$$ds = cdt\sqrt{1 - \frac{v^2}{c^2}}$$

The action now becomes,

$$S = -mc^2 \int_{t_i}^{t_f} \sqrt{1 - \frac{v^2}{c^2}} dt$$

As you can see the relativistic Lagrangian is now,

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}$$

The problem that we had with the action for a free non-relativistic particle is overcome. There is a velocity limit inherent in the equation which doesn't allow velocities greater than c.

Aside from being the correct Lagrangian at high velocities the Lagrangian must produce the correct physics at low velocities. To investigate this we can expand the square root for low velocities,

$$L \approx -mc^2 \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) = \frac{1}{2}mv^2$$

$-mc^2$ can be ignored as it is a constant. As you can see the expanded low velocity Lagrangian agrees with the non-relativistic Lagrangian that can be seen in (3.1).

To calculate the Canonical momentum of the relativistic Lagrangian we just have to differentiate it with respect to velocity.

$$\Pi = \frac{\partial L}{\partial v} = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This produces the relativistic momentum of the particle.

We can now calculate the Hamiltonian,

$$H = (\Pi)(v) - L = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This also agrees with the relativistic energy of the point particle showing that the action (3.2) is correct.

3.2 Reparameterization invariance

A usefull property would be for the action to be reparameterization invarient. This means that if a different observer parametizes the particle world-line the action still yields the same result. The world-line can be assigned the parameter τ (proper time) to describe the motion of the particle, making the spacetime coordinates x^μ functions of τ.

$$\begin{aligned} x^\mu &= x^\mu(\tau) \\ x_i^\mu &= x^\mu(\tau_i) \\ x_f^\mu &= x^\mu(\tau_f) \end{aligned}$$

The action of the particle can now be wrote in terms of its proper time, τ as,

$$S = -mc \int_{t_i}^{t_f} \sqrt{-\eta_{\mu v} \frac{dx^\mu}{d\tau} \frac{dx^v}{d\tau}} d\tau$$

This action is now reparametization invarient. If a different observer chose a different paramterization then the action wouldn't differ.

3.3 Equations of motion of a relativistic point particle

Again we acquire the equations of motion by utilising the Hamilton principle. Setting $\delta S = 0$ and varying the path from $x^\mu(\tau)$ to $x^\mu(\tau) + \delta x^\mu(\tau)$. The alteration of the path is given by,

$$\delta S = -mc \int \delta(ds) = -mc \int_{\tau_i}^{\tau_f} \eta_{\mu v} \frac{d\delta x^\mu}{d\tau} \frac{dx^v}{ds} d\tau$$

Integrating this by parts,

$$\delta S = -mc \int_{\tau_i}^{\tau_f} d\tau \frac{d}{d\tau} \left(\eta_{\mu v} \delta(x^\mu(\tau)) \frac{dx^v}{dS} \right) + \int_{\tau_i}^{\tau_f} d\tau \delta \left(x^\mu(\tau) \left(mc\eta_{\mu v} \frac{d}{d\tau} \left(\frac{dx^v}{dS} \right) \right) \right)$$

Setting $\delta(x^\mu(\tau_i)) = \delta(x^\mu(\tau_f)) = 0$ removes the first term leaving,

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \delta(x^\mu(\tau)) mc\eta_{\mu v} \frac{d}{d\tau}\left(\frac{dx^v}{dS}\right)$$

subbing in $mc\frac{dx^v}{ds} = \rho^v$ yields,

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \delta(x^\mu(\tau)) \eta_{\mu v} \frac{d\rho^v}{d\tau}$$

then using $\eta_{\mu v}$ to lower the index of $d\rho^v$.

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \delta(x^\mu(\tau)) \eta_{\mu v} \frac{d\rho_\mu}{d\tau}$$

We know $\delta(x^\mu(\tau))$ is arbitrarily small so $\delta S = 0$. This also implies that $\frac{d\rho_\mu}{d\tau} = 0$ as well, which states that the momentum of the particle is constant along the world-line which is also referred to as the conservation of momentum.

3.4 Relativistic string

We have found that the relativistic point particle is characterised by the world-line that it traces out in space-time. The action can be derived from integrating over δS and assigning one parameter known as 'proper time' τ.

It follows that a relavtivistic string is characterised by the world-*sheet* that it traces out in space-time. The action can be derived from integrating over δA and assigning *two* parameters known as known as 'proper time' τ and 'proper area' σ. The action derived is known as the *Nambu-Goto* action and to find this we first have to calculate δA.

We start by looking at the two-dimensional surface in space that a string pervades. As it is two-dimensional it is dependant on two parameters η^1 and η^2, these parameters can be plotted and are described as parameter space. The three spatial dimensions the surface exists in is known as target-space. The parametiszed surface is now descrbed by the functions

$$\overline{x}(\eta^1, \eta^2) = (x^1(\eta^1, \eta^2), x^2(\eta^1, \eta^2), x^3(\eta^1, \eta^2))$$

This can now be used as a map to transfer parameters in parameter space to co-ordinates in target space and vice-versa.

To determine δA of target space we can calculate an arbitrarily small parallelogram of parameter space with sides

$$d\overline{v}_1 = \frac{\partial \overline{x}}{\partial \eta^1}\partial \eta^1 d\overline{v}_2 = \frac{\partial \overline{x}}{\partial \eta^2}\partial \eta^2 \tag{3.3}$$

The area would then be,

$$dA = |d\overline{v}_1||d\overline{v}_2||sin\theta| = |d\overline{v}_1||d\overline{v}_2|\sqrt{1 - cos^2\theta} = \sqrt{|d\overline{v}_1|^2|d\overline{v}_2|^2 - |d\overline{v}_1|^2|d\overline{v}_2|^2 cos^2\theta}$$

θ is the angle between $d\overline{v}_1$ and $d\overline{v}_2$ so the whole equation can be rewritten in terms of dot products.

$$dA = \sqrt{(d\overline{v}_1 \cdot d\overline{v}_1)(d\overline{v}_2 \cdot d\overline{v}_2) - (d\overline{v}_1 \cdot d\overline{v}_2)^2}$$

Substituting equations(3.3) into this equation yields the small element of the target space,

$$dA = d\eta^1 d\eta^2 \sqrt{\left(\frac{\partial \overline{x}}{\partial \eta^1} \cdot \frac{\partial \overline{x}}{\partial \eta^1}\right)\left(\frac{\partial \overline{x}}{\partial \eta^2} \cdot \frac{\partial \overline{x}}{\partial \eta^2}\right) - \left(\frac{\partial \overline{x}}{\partial \eta^2} \cdot \frac{\partial \overline{x}}{\partial \eta^2}\right)^2}$$

The area functional is then just,

$$A = \int d\eta^1 d\eta^2 \sqrt{\left(\frac{\partial \overline{x}}{\partial \eta^1} \cdot \frac{\partial \overline{x}}{\partial \eta^1}\right)\left(\frac{\partial \overline{x}}{\partial \eta^2} \cdot \frac{\partial \overline{x}}{\partial \eta^2}\right) - \left(\frac{\partial \overline{x}}{\partial \eta^2} \cdot \frac{\partial \overline{x}}{\partial \eta^2}\right)^2}$$

Introducing the new parameter σ (proper area) we can describe the world-sheet by the mapping functions,

$$X^\mu(\tau, \sigma)$$

This can now be included into the area functional which can be arranged to be reparameterization invarient,

$$A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}\right)^2} \tag{3.4}$$

3.5 Nambu-Goto relativistic string action

As S has units $\frac{ML^2}{T}$ and the area functional (3.4) has units L^2 we must multiply the area by a quantity which has the units $\frac{M}{T}$. We know the string tension T_0 has the dimension of force and force divided by velocity will leave the correct units $\frac{M}{T}$. Hence the area must be multiplied by a factor of $\frac{T_0}{c}$. The string action is now,

$$S = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \tag{3.5}$$

where $\sigma_1 > 0$ is constant, $\dot{X} = \frac{\partial X}{\partial \tau}$ and $X' = \frac{\partial X}{\partial \sigma}$ This is known as the Nambu-Goto action for a relativistic string.

In order for this to be reparametization invarient the integral must vanish,

$$-dS^2 = dX^\mu dX_\mu = \eta_{\mu v} dX^\mu dX_\mu = \eta_{\mu v} \frac{\partial X^\mu}{\partial \eta^\alpha} \frac{\partial X^v}{\partial \eta^\beta} d\eta^\alpha d\eta^\beta$$

Where $\eta_{\mu v}$ is the Minkowski space metric and η^α and η^β run over τ and σ. We can also define an induced metric on the world-sheet,

$$\gamma_{\alpha\beta} = \eta_{\mu v} \frac{\partial X^\mu}{\partial \eta^\alpha} \frac{\partial X^v}{\partial \eta^\beta} = \frac{\partial X}{\partial \eta^\alpha} \frac{\partial X}{\partial \eta^\beta}$$

in matrix form,

$$\gamma_{\alpha\beta} = \begin{vmatrix} (\dot{X})^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & (X')^2 \end{vmatrix}$$

As you can see the determinant of this matrix is equal to minus the square root in equation (3.5) so the Nambu-Goto action can now be written in reparametization invarient form,

$$S = -\frac{T_0}{c} \int d\tau d\sigma \sqrt{-\gamma}$$

Where

$$\gamma = det(\gamma_{\alpha\beta})$$

3.6 Equations of motion

Again we will acquire the equations of motion by introducing the constraint that the action is stationary under variations. We vary the path from x to $x + \delta x$ and rewrite the Nambu-Goto action (3.5) as a double integral of the lagrangian density.

$$S = \int_{\tau_i}^{\tau_f} d\tau L = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \overline{L}(\dot{X}^\mu, X'^\mu)$$

Where the lagrangian density is

$$\overline{L} = -\frac{T_0}{c} \sqrt{-\gamma}$$

For the action to be stationary the variation must vanish.

The variation is,

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial \overline{L}}{\partial \dot{X}^\mu} \frac{\partial(\delta X^\mu)}{\partial \tau} + \frac{\partial \overline{L}}{\partial X'^\mu} \frac{\partial(\delta X^\mu)}{\partial \sigma} \right]$$

where $\delta\dot{X}^\mu = \delta\left(\frac{\partial X^\mu}{\partial\tau}\right) = \frac{\partial\delta X^\mu}{\partial\tau}$ and likewise for $\delta\dot{X}^\mu$.

We can also define

$$\rho^\tau_\mu = \frac{\partial\overline{L}}{\partial\dot{X}^\mu}$$

$$\rho^\sigma_\mu = \frac{\partial\overline{L}}{\partial X'^\mu}$$

The variation then becomes,

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial}{\partial\tau}(\delta X^\mu \rho^\tau_\mu)\frac{\partial}{\partial\sigma}(\delta X^\mu \rho^\sigma_\mu) - \delta X^\mu \left(\frac{\partial\rho^\tau_\mu}{\partial\tau} + \frac{\partial\rho^\sigma_\mu}{\partial\sigma}\right)\right] \tag{3.6}$$

The first term on the right is a full derivative of τ. The flow of τ corresponds to the flow of time so if we set the initial and final conditions to equal zero then this term will vanish. The variation is then,

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \left[\delta X^\mu \rho^\sigma_\mu\right]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \delta X^\mu \left(\frac{\partial\rho^\tau_\mu}{\partial\tau} + \frac{\partial\rho^\sigma_\mu}{\partial\sigma}\right) \tag{3.7}$$

The second term also vanishes for all variations δX^μ leaving us with last term which must be set to equal zero.

$$\frac{\partial\rho^\tau_\mu}{\partial\tau} + \frac{\partial\rho^\sigma_\mu}{\partial\sigma} = 0$$

This is the equation of motion for a relativistic string, either open or closed.

3.7 The Polyakov action and boundary conditions

Although the Nambu-Goto action is usefull it is still burdened with a square root in its integrand. An equivalent formulation of the action which proves to be much more convenient is the polyakov action. We will again introduce fields that correspond to the actions parameters and as in the Nambu-Goto action define a world sheet metric $h_{\alpha\beta}(\tau,\sigma)$ where,

$$h_{\alpha\beta} = h^{-1}_{\alpha\beta}$$

$$h = det(h_{\alpha\beta})$$

The action is now,

$$S = -\frac{T_0}{c}\int d^2\sigma(\sqrt{-h})h^{\alpha\beta}\partial\alpha X^\mu \partial_\beta X^\mu$$

Classically this is analogous to the Nambu-Gotto action but it is more suitable for quantisation. The equation of motion retrieved from the Polyakov action is,

$$\frac{\partial^2}{\partial \tau^2} X^\mu(\sigma, \tau) - \frac{\partial^2}{\partial \sigma^2} X^\mu(\sigma, \tau) = 0$$

This is just a two-dimensional wave equation which has the general form of solution,

$$X^\mu(\sigma, \tau) = X^\mu_{left}(\sigma + \tau) + X^\mu_{right}(\sigma - \tau)$$

A superposition of left moving and right moving waves. For ease this solution can be rewritten using light-cone co-ordinates,

$$X^\mu(\sigma, \tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$$

Let us now take a closer look at the boundary conditions.

These solutions depend on the boundary conditions of which there are three possibilities,

- One is that the boundary conditions are periodic $X(\sigma, 0) = X(\sigma, 0+\pi)$ which results in a closed string and an unbounded world-sheet. The solutions to the wave equation then becomes,

  $$X^\mu = X^\mu_L + X^\mu_R = x^\mu + \ell_s^2 p^\mu \tau + \frac{i}{2}\ell_s \sum_{n \neq 0} \frac{1}{n} \overline{\alpha}^\mu_n e^{-2in(\sigma+\tau)} + \frac{i}{2}\ell_s \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n e^{-2in(\sigma-\tau)}$$

  where,

  x^μ : is the centre of mass of the string

  p^μ : is the total momentum of the string

  ℓ_s : is the string length

  Classically $\overline{\alpha}^\mu_n$ and α^μ_n would represent the amplitude of the n'th oscillation mode. However quantum mechanically they represent the creation and annihilation operators of string states.

The centre of mass of the string moves along a straight line and the momentum is always conserved. This is anologous to the behaviour of a relativistic particle. In fact we can think of the closed string as a relativistic particle which has inherent left and right moving harmonic oscillations produced by the creation and annihilation operators.

If we don't impose periodic boundary conditions then there are two other constraints available which will make the variation of action vanish.

- One is that the string has Neuman boundaries $\left[\frac{\partial \overline{L}}{\partial X'^{\mu}}\right]_{0,\sigma} = 0$. This means that the string is open and can move freely through space-time, the momentum at its ends is conserved.

- The final possible boundary condition is the dirichlet boundary condition $\delta(X)^{\mu}(\tau, 0) = \delta X^{\mu}(\tau, \sigma) = 0$, $\mu \neq 0$ meaning dirichlet boundary conditions are only possible for space dimensions. When $\mu = 0$ we must have $\rho_0^{\sigma}(\tau, 0) = \rho_0^{\sigma}(\tau, \sigma) = 0$. We can allow neumann boundary conditions in the time dimension and dirichlet boundary conditions in the space dimensions denoted by p. The ends of the string are fixed on p dimensional surfaces known as dirichlet-p branes or D-branes.

  A D-brane can pervade all of space and since the end points of a string are free to move across it then they too can percade all of space. A D-brane isn't a theoretical surface and isn't introduced into the equations of motion. They are an inherent, physical requirement of the equations of motion for a relativistic string.

As we are only interested in the closed string we will only concern ourselves with the periodic boundary conditions and the creation and annihilation operators of string states.

3.8 Summary

We investigated a relativistic point particle and discovered that the action is not the same as that of a non-relativistic particle. A lorentz factor needed to be introduced and therefore the action had to be lorentz invarient. We plotted the particle as it swept out a world-line and introduced a proper time parameter, making the action reparameterization invarient. The hamilton principle was then used to retrieve the equations of motion from the action.

We then repeated this process with a relativistic string but because the string is two-dimensional we added another parameter, proper area. Instead of sweeping out a world-line we saw that the string sweeps out a world sheet.

This resulted in a different action being calculated known as the nambu-goto action which was subsequently improved to what is known as the polyakov action. After retrieving the equations of motion we examined three types of boundary conditions; one resulting in free open strings, one resulting in open strings connected to a D-brane and one resulting in a closed string.

We will now proceed with the quantization of relativistic strings.

Chapter 4

Relativistic string quantisation

We will quantize the relativistic closed string to produce the required bosons, discovering that another model is needed to produce fermionic data. After the introduction of *relativistic superstrings* we will see that their quantization results in the required fermionic states and the arrival of a property known as *supersymmetry*.

4.1 Quantized bosonic strings

We saw that the solutions to the wave equation obtained by using periodic boundary conditions contained the four parameters x^μ, p^μ, $\overline{\alpha}_n^\mu$ and α_n^μ. These can now be thought of as quantum operators with commutation relations,

$$
\begin{aligned}
[x^\mu, p^\mu] &= i\eta^{\mu v} \\
[\alpha_m^\mu, \alpha_n^v] &= m\eta^{\mu v}\delta_{m+n,0} \\
[\overline{\alpha}_m^\mu, \overline{\alpha}_n^v] &= m\eta^{\mu v}\delta_{m+n,0} \\
[\overline{\alpha}_m^\mu, \alpha_n^v] &= 0
\end{aligned}
$$

The hermiticity of X^μ results in the relations $(x^\mu)^+ = x^\mu$, $(p^\mu)^+ = p^\mu$, $(\overline{\alpha}_n^\mu)^+ = \overline{\alpha}_{-n}^\mu$ and $(\alpha_n^\mu)^+ = \alpha_{-n}^\mu$

We can now define the operators,

$$a_m^\mu = \frac{1}{\sqrt{m}}\alpha_m^\mu$$

$$a_m^{\mu+} = \frac{1}{\sqrt{m}}\alpha_{-m}^\mu$$

If $m > 0$ we get creation operators. If $m < 0$ we get annihilation operators.

Notice if $u = v = 0$ then $[a_n^0, a_m^{0+}] = -1$ and the commutator of the time components become negative. These negative states are known as *ghosts*

The creation operators construct the space of states in what is known as *Fock space* which describes quantum states with an unknown amount of particles. It is represented as the sum of *single*-particle *Hilbert spaces* as shown below,

$$|\psi>_n = |\phi_1, \phi_2, \phi_3, ...\phi_n>$$

which describes n particles with wave functions $\phi_{1\rightarrow n}$. Each ϕ is a wave-function from the single particle hilbert space and they can be added and removed from the fock space by use of the creation and annihilation operators respectfully.

We will proceed with what is known as *light-cone quantisation.* Introducing the light-cone gauge

$$X^+(\sigma^0, \sigma^1) = x^+ + 2\alpha' p^+ \sigma^0$$

which has space-time coordinates

$$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^{D-1})$$

$$X^- = \frac{1}{\sqrt{2}}(X^0 - X^{D-1})$$

$$X^I, I = 1, 2, 3, ...D - 2$$

where I is the space-time indice running from 1 to D-2.

To produce the space of states we will start from the ground states $|h>$ defined by,

$$h = h^I$$

$$p^I|h> = h^I|h>$$

$$a_m^I|h> = \overline{a}_m^I|h> = 0$$

$$m > 0$$

ie, eigenstates of the momentum operators with no oscillations. A basis can now be set up consisting of the space of possible eigenstates produced by the creation operators,

$$B = \{a_m^I ... \overline{a}_n^J|h> |m_\ell, n_\ell > 0\}$$

Now defining number operators which identify the summation of their respective barred and unbarred operators,

$$N = \sum_{n=1}^{\infty} n a_{-n}^{I\dagger} a_{nI}$$

$$\overline{N} = \sum_{n=1}^{\infty} n \overline{a}_{-n}^{I\dagger} \overline{a}_{nI}$$

where the a and $\overline{a}$ are constant ambiguities due to the ordering of the creation and annihilation operators. Due to the constraint of lorentz invariance they adhere to the relation $a = \overline{a} = -1$. The constants also have to satisfy the equation,

$$a = \overline{a} = -\frac{D-2}{24}$$

resulting in the requirement of 26 space-time dimensions.

Adding another constraint of level matching, where the number of left moving and right moving oscillations must be equal $N = \overline{N}$ we can utilise the euler-lagrange equations dervied from the nambu-goto action to produce a mass-shell condition.

$$M^2 = -p^\mu p_\mu = \frac{2}{\alpha'}(N + a + \overline{N} + \overline{a})$$

The most basic states are one-particle states of a quantum scalar field in which $N = \overline{N} = 0$. Meaning there are no oscillations and the mass squared is $M^2 = -\frac{4}{\alpha'}$. These states with negative mass are known as *Tachyons*, purely theoretical they are thought to arise from instabilities in space-time.

The first excited states must satisfy the relation $N = \overline{N}$ meaning that two oscillators in the lowest possible mode act on the ground state from the left and right regions. This results in the mass squared reducing to zero and a possibility of $(D-2)^2$ states. These massless states are described by,

$$\sum_{1 \leq I,J \leq (D-2)} R_{I,J} a_{-1}^{I\dagger} \overline{a}_{-1}^{J\dagger} |h> \tag{4.1}$$

where $R_{I,J}$ are elements of an arbitrary square matrix of size (D-2) which can be seperated into three components.

$$R_{I,J} = \overline{S}_{IJ} + A_{IJ} + S'_{IJ}$$

A symmetric-traceless component, an antisymmetric component and a trace component respectively. This means that the states described in (4.1) can be split into three linearly independent groups.

- $$\sum_{1\leq I,J\leq D-2} \overline{S}_{IJ} a_{-1}^{I\dagger} \overline{a}_{-1}^{J\dagger}|h>$$

  The symetric-traceless component produces states that coincide with the one-particle graviton states that are found in the quantum gravitational theory.

- $$\sum_{1\leq I,J\leq D-2} A_{IJ} a_{-1}^{I\dagger} \overline{a}_{-1}^{J\dagger}|h>$$

  The antisymmetric component produces states that coincide with the one-particle *Kalib-Ramond* states. This is an antisymmetric tensor field $B_{\mu v}$ which can be seen as a tensor generalization of the Maxwell gauge field A_μ. The $B_{\mu v}$ field couples to strings in a similar way that A_μ couples to particles, resulting in strings carrying a *Kalib-Ramond charge*.

- $$S a_{-1}^{I\dagger} \overline{a}_{-1}^{I\dagger}|h>$$

  The trace component is summed over I so there are no free indices, so it only represents one state. This one-particle state of a massless scalar field is known as the *Dilaton* field.

Having examined the massless closed strings we have seen that all of their quantum states coincide with the bosonic particle states. We have seen that the bosons require 26 space-time dimensions, that the theorised graviton is an intrinsic part of this model and with a little more working we would see that the non-abelian gauge bosons would appear from this bosonic string theory.

Having acquired the bosons we now need to retrieve the fermions from the model. To do this we need *superstring* theories.

4.2 Relativistic superstrings

The classical bosonic string theory was characterised by the use of coordinates $X^\mu(\tau,\sigma)$ where the space-time variables commute. Leading into quantum bosonic string theory we saw that the coordinates became operators which were no longer commutable.

The fundamental characteristic of superstring theory is that the world-sheet variables used $\psi^{\mu}_{\alpha}(\tau,\sigma)$, $\alpha = 1,2$ are anticommuting dynamic variables.

For each value of μ these two variables describe a world-sheet fermion which after quantizing produces the particle states of spacetime fermions.

The action of a superstring is known as the *Dirac action*,

$$s_{\psi} = \frac{1}{2\pi}\int d\tau \int_0^{\pi} d\sigma \left[\psi_1^I(\partial_\tau + \partial_\sigma)\psi_1^I + \psi_2^I(\partial_\tau - \partial_\sigma)\psi_2^I\right]$$

To obtain the equations of motion we again vary the field,

$$\delta s_{\psi} = \frac{1}{2\pi}\int d\tau \int_0^{\pi} d\sigma \left[\delta\psi_1^I(\partial_\tau + \partial_\sigma)\delta\psi_1^I + \delta\psi_1^I(\partial_\tau + \partial_\sigma)\delta\psi_1^I + \delta\psi_2^I(\partial_\tau - \partial_\sigma)\delta\psi_2^I + \delta\psi_2^I(\partial_\tau - \partial_\sigma)\delta\psi_2^I\right]$$

rearranging to,

$$\delta s_{\psi} = \frac{1}{\pi}\int d\tau \int_0^{\pi} d\sigma \left[\delta\psi_1^I(\partial_\tau + \partial_\sigma)\delta\psi_1^I + \delta\psi_2^I(\partial_\tau - \partial_\sigma)\delta\psi_2^I\right] + \frac{1}{2\pi}\int d\tau [\psi_1^I\delta\psi_1^I - \psi_2^I\delta\psi_2^I]_{\sigma=0}^{\sigma=\pi}$$

The equations of motion and boundary terms can clearly be seen,

$$(\partial_\tau + \partial_\sigma)\psi_1^I = 0$$
$$(\partial_\tau - \partial_\sigma)\psi_1^I = 0$$

$$\psi_1^I(\tau,\sigma_*)\delta\psi_1^I(\tau,\sigma_*) - \psi_2^I(\tau,\sigma_*)\delta\psi_2^I(\tau,\sigma_*)$$

The equations of motion also imply that ψ_1^I and ψ_2^I represent right-moving and left-moving world-sheet fields respectively. These can be rewritten,

$$\psi_1^I(\tau,\sigma) = \psi_1^I(\tau - \sigma)$$
$$\psi_2^I(\tau,\sigma) = \psi_2^I(\tau + \sigma)$$

The condition that the boundary terms must vanish under variation results in two types of boundary conditions, the *Ramond* boundary condition and the *Neveu-Schwarz* boundary condition.

- The ramond boundary condition concerns periodic fermions as shown below,

$$\psi^I(\tau,\sigma) = +\psi^I(\tau,-\pi)$$

If we apply this condition to the periodic fermion field we can expand its equation of motion,

$$\psi^I = \sum_{n\in\mathbb{Z}} d_n^I e^{-2in(\tau-\sigma)}$$

There is also a similar formulation to represent right-moving periodic fermions $(\tau+\sigma)$ resulting in the mode-oscillator $\overline{d}_n^I$.

Here the oscillators with negative modes are creation operators $d^I_{-1,-2,-3...}$ and the oscillators with positive modes are annihilation operators $d^I_{1,2,3...}$

These ramond oscillators also comply with the anticommutation relation,

$$[d_m^I, d_n^J] = \delta_{m+n}, 0\delta^{IJ}$$

this means that the ramond creation operators can appear only once on a given state.

- The neveu-schwarz boundary condition concerns anti-periodic fermions as shown below,

$$\psi^I(\tau,\sigma) = -\psi^I(\tau,-\pi)$$

If we apply this condition to the anti-periodic fermion field we can expand its equation of motion,

$$\psi^I = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^I e^{-2in(\tau-\sigma)}$$

There is also a similar formulation to represent right-moving anti-periodic fermions $(\tau+\sigma)$ resulting in the mode-oscillator $\overline{b}_n^I$.

Here the oscillators with negative modes are creation operators $b^I_{-1,-2,-3...}$ and the oscillators with positive modes are annihilation operators $b^I_{1,2,3...}$

These neveu-schwarz oscillators also comply with the anticommutation relation,

$$[b_r^I, b_s^J] = \delta_{r+s}, 0\delta^{IJ}$$

this means that the neveu-schwarz creation operators can appear only once on a given state.

As with the bosonic model a basis can now be set up in the light cone gauge consisting of the space of possible eigenstates produced by the creation operators,

$$B = \{(a^I_{-n}), (d^I_{-m}) or (b^I_{-m}), (\overline{d}^I_n) or (\overline{d}^I_n) | h > | m, n, r > 0, I = 1, 2, ..., D-2\}$$

We can now define number operators which identify the summation of their respective barred and unbarred operators for both ramond and neveu-schwarz boundary conditions respectively,

$$N = \sum_{n=1}^{\infty} n a^{I\dagger}_{-n} a_{nI} + \sum_{r \in \mathbb{N}+\frac{1}{2}} r b^{I\dagger}_{-r} b_{rI}$$

$$\overline{N} = \sum_{n=1}^{\infty} n \overline{a}^{I\dagger}_{-n} \overline{a}_{nI} + \sum_{r \in \mathbb{N}+\frac{1}{2}} r \overline{b}^{I\dagger}_{-r} \overline{b}_{nI}$$

$$N = \sum_{n=1}^{\infty} n a^{I\dagger}_{-n} a_{nI} + \sum_{r \in \mathbb{N}+\frac{1}{2}} r d^{I\dagger}_{-r} d_{rI}$$

$$\overline{N} = \sum_{n=1}^{\infty} n \overline{a}^{I\dagger}_{-n} \overline{a}_{nI} + \sum_{r \in \mathbb{N}+\frac{1}{2}} r \overline{d}^{I\dagger}_{-r} \overline{d}_{nI}$$

where the a and $\overline{a}$ are constant ambiguities due to the ordering of the creation and annihilation operators given by,

$$a = -\frac{D-2}{24} + a_{\psi+}$$
$$\overline{a} = -\frac{D-2}{24} + a_{\psi-}$$

Using ramond boundary conditions results in,

$$a_{\psi\pm} = \frac{D-2}{24}$$

and using neveu-schwarz boundary conditions results in,

$$a_{\psi\pm} = -\frac{D-2}{48}$$

Due to the constraint of lorentz invariance we have to set $D = 10$. The constant ambiguities then become,

$$a = -\frac{1}{3} + a_{\psi+}$$

$$\overline{a} = -\frac{1}{3} + a_{\psi-}$$

Using ramond boundary conditions results in,

$$a_{\psi\pm} = \frac{1}{3}$$

and using neveu-schwarz boundary conditions results in,

$$a_{\psi\pm} = -\frac{1}{6}$$

Again adding the constraint of level matching, where the number of left moving and right moving oscillations must be equal $N + a = \overline{N} + a$ we can utilise the euler-lagrange equations dervied from the nambu-goto action to produce a mass-shell condition.

$$M^2 = -p^{\mu}p_{\mu} = \frac{2}{\alpha'}(N + a + \overline{N} + \overline{a})$$

This mass-shell condition can now be used to acquire the particle states arising from each sector.

- The particle states retrieved by applying the neveu-schwarz boundary condition are,

Neveu-Schwarz sector particle states			
N	Mass squared	Number of states	particle state type
0	-1/2	1	tachyon
1/2	0	8	boson
1	1/2	36	fermion
3/2	1	?	boson

  With the general rule that applying an integer N results in a fermionic state and applying a half integer N results in a bosonic state.

- The particle states retrieved by applying the ramond boundary condition are slightly more complex. Only integer masses are allowed resulting in sixteen allowed ground states which are split evenly at mass levels between bosonic and fermionic states.

Ramond sector particle states		
Mass squared	Bosonic states	Fermionic states
0	1	1
1	2	2
2	5	5

  The emergence of equal numbers of fermionic and bosonic states at every mass level is an indicator of what is known as supersymmetry.

4.3 Summary

We have now aquired bosons from the quantization of relativistic closed strings and fermions from the quantization of relativistic superstrings. After studying bosonic strings we discovered the requirement of twenty-six space-time dimensions and then after studying superstrings found that this reduced to ten space-time dimensions. We also found that the superstrings came with the constraint of supersymmetry and two boundary conditions. Utilising both we eventually acquired all the data neccessary to reproduce the standard model. We can now look to combine these two models into a viable string theory of particle physics.

Chapter 5

Viable string theories

We have now studied both bosonic string theories and superstring theories. Seperately these both contain all the data required by the standard model. However to reproduce the standard model data in its entirity we have to look at combining them into one coherent theory.

5.1 Five flavours of ten-dimesional supersymmetric string theories

There are two types of Supersymmetric closed string theories, known as type IIA and type IIB.

As we saw in the previous chapter, an open superstring has two sectors due to the two boundary conditions available. These are known as the ramond sector and the neveu-schwarz sector which we will now refer to as R and NS respectively. The open superstring spacetime bosons appear from the NS sector and the fermions appear from the R sector. A closed superstring theory can now be created by joining the left moving sector with the right moving sector of either R or NS.

This results in four possible closed string sectors (R, R),(NS, NS),(R, NS),(NS, R)

The bosons now appear from the (R, R),(NS, NS) sectors and the fermions appear from the (R, NS),(NS, R) sectors.

These string sectors must be consistent with supersymmetry and depending on how we ensure this results in several different theoris. We will start with labelling the left sector (NS_+, R_-) and the right sector (NS_+, R_+) which results in type IIA superstring theory.

The four sectors of type IIA superstring theory with their respective particle state makeup are then,

$$(NS_{+}, NS_{+})$$

which contains 64 bosonic states including the graviton, kalib-ramond field and the dilaton.

$$(R_{-}, R_{+})$$

which also contains 64 bosonic states including the graviton, kalib-ramond field and the dilaton.

$$(NS_{+}, R_{+}), (R_{-}, NS_{+})$$

which contains 128 fermionic states.

As you can see the requirement of supersymmetry has been met, there are 128 fermions and 128 bosons.

If we chose the left and right sectors to be the same then the type IIB string theory emerges. The left sector becomes (NS_{+}, R_{-}) and the right sector becomes (NS_{+}, R_{+}).

The four sectors of the type IIB superstring theory with their respective particle state makeup are then,

$$(NS_{+}, NS_{+})$$

which contains 64 bosonic states including the graviton, kalib-ramond field and the dilaton.

$$(R_{-}, R_{-})$$

which also contains 64 bosonic states including the graviton, kalib-ramond field and the dilaton.

$$(NS_{+}, R_{-}), (R_{-}, NS_{+})$$

which contains 128 fermionic states.

Again the requirement of supersymmetry is met as there are equal number of fermions and bosons.

The only difference between these two superstring theories is in the R-R massless fields which concerns the stability of the D-branes.

There are several other possible choices for defining the left and right sector but the resulting theories are non-supersymmetric and lead to tachyons.

Another two types of ten-dimensional supersymmetric string theories are the heterotic theories. Instead of combining left and right moving open superstrings as in the type II theories we can combine the left moving open bosonic string with the right moving open superstring. This results in two heterotic models. Only ten out of the twenty-six left moving open bosonic coordinates correlate with the right moving bosonic coordinates of the open superstring. The consequence of this is that the heterotic superstring exists in ten space-time dimensions. The two possible heterotic models are the $E_8 X E_8$ type and the S0(32) type.

The last possible ten-dimesional supersymmetric string theory is the type I theory. It combines both open and closed unoriented strings, meaning that the states that the theory contains are invarient under an operation that flips the orientation of the string.

We now have a list of five ten-dimesional supersymmetric string theories,

- Type I
- Type IIA
- Type IIB
- $E_8 X E_8$
- S0(32)

Another theory results when the typeIIA theories string coupling limits are extended to infinity, a consequence of which is eleven space-time dimensions. This is known as M-theory which is not actually a string theory but a theory of membranes.

It is believed that the five ten-dimesional supersymmetric string theories and M-theory are infact manifestations of the same fundamental theory.

5.2 String theories containing the standard model

All the above viable string theories can be used to derive the standard model and since they are all believed to be related the different avenues available are also believed to be essentially the same. The most prominent attack is based from the $E_8 X E_8$ heterotic supersymmetric string theory.

The theory operates in ten dimensions, one allocated to time, three allocated to visible space and the other six *compactified* into what is known as *Calabi-Yau space*. This is a small six-dimensional space which has discrete and continuous parameters. These parameters are what regulate the four dimensional space-time theory that emerges from this compactification. The number of generations of chiral matter is dependent on the topology of the Calabi-Yau space with viable models available that reproduce the three generations of matter neccessary to the standard model. Also, compactification is central to string phenomenology in that it allows the symmetry breaking of the group. Firstly into $E_6 X E_8$, then E_6 can be broken into the grand unified theory gauge group S0(10).

Chapter 6

Conclusion

The goal of string phenomenology is to predict the framework and data of particle physics using string theory. Showing that string theory is consistent with previous models of particle physics will then allow gravitation to be incorporated into a more fundamental framework. To understand how this is achieved we examined the standard model and showed that it could be improved by incorporating it in into a grand unified theory. We then set about recreating the framework and data of the standard model using string theory. After studying the mechanics of non-relativistic then relativistic string we developed equations of motion and boundary conditions. Relativistic closed strings were then quantized to produce the bosonic states and superstrings were quantized to produce the fermionic states. We then combined these and found that we had six models which all reproduced the correct framework and data. Continually working back towards the standard model we centered on the E_8XE_8 heterotic theory eventually finding that with the aid of Calabi-Yau spaces we could reproduce the grand unified theory S0(10) framework plus the three generations of chiral matter recquired.

Through string phenomenology we have shown that string theory can reproduce the data requirement of the standard mdoel but it also improves the standard model by defining and incorporating quantum gravity.