String Phenomenology TP6

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Abstract

The success of the standard model serves as a guide to any extentions beyond the model in that it should arrive at the same phenomenology at appropriate energy scales. We study in this thesis the phenomenological aspects of realistic string models under the free fermionic construction.

Starting from a brief review of the standard model, we introduce string theory, concentrating on the necessary tools to construct a realistic free fermionic model. We then develop a model producing the observable $SU(3)_C \times SU(2)_L \times U(1)_y$ symmetries. This is free of the custodial nonabelian $SU(2)_c$ gauge group, a common feature of a class of realistic string models in which only the leptons or quarks transform.

Contents

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1 Introduction

In the Standard Model the fermion masses are treated as free parameters. The values are obtained by confronting experiments. One important aspect of the superstring derived standard-like models in the free fermionic formulation is its potential ability to calculate fermion masses. In particular, it offers a possible explanation to the top quark mass hierarchy problem [1]. Certain models were even successful in predicting the top quark mass before its experimental observation [2].

For the purpose of calculating the top quark mass, realistic string models are constructed. In this thesis we discuss development of an existing model, which has an observable gauge group of $SU(3)_C \times SU(2)_L \times SU(2)_c$. The $SU(2)_c$ is a non-Abelian custodial gauge symmetry in which only the leptons tranform. It is enhanced from a $U(1)$ symmetry as a consequence of two extra spacetime bosons in the model. The aim of this thesis is to remove this extra symmetry to recover the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_y$.

The thesis begins by introducing the idea of gauge symmetry in particle physics. We relate the standard model gauge group to three forces observed in nature. In chapter 3 the basic idea of bosonic string theory is introduced. Conformal anomaly, which airses as a result of the quantizaition of classical strings, is described. We explain how the anomaly can be removed by introducing extra degrees of freedom. Superstrings, a theory which describe fermions by incorporating supersymmetry to the bosonic string theory, is introduced. Then in chapter 4 the free fermionic formulation is discussed. In this formulation the extra degrees of freedom are interpreted as free fermions propagating on the string worldsheet. By specifying the boundary conditions of these free fermions, subject to certain consistency constraints, a model is defined. In chapter 5 we present a set of rules which encompass these consistency constraints. It serves as our primary tool to extract the physical states of a model. A model which possesses the standard model gauge group is then defined. The fermion mass texture of the model is discussed, and an explanation to the missing of the custodial symmetry is suggested.

2 The Standard Model

The success of the standard model in describing three fundamental forces of nature: the electromagnetic, weak and strong forces, is a major triumph of the 20th century physics. So far no experimental observations shows contradiction to the predictions of the standard model.

Its theoretical description rests on the principle of local non-Abelian gauge invariance. This is the invariance under change of fields that does not change the corresponding observables. Tensor equations are used to maintain such invariance and connection coefficients are introduced for parallel transport, in a way analogues to Einstein's theory of gravitation. Any departure from zero connection coefficients are interpreted as a force observed in nature.

Associated with every force is a particular symmetry and hence a symmetry group. It is possible to turn it around, that is to identify a symmetry group and then investigate the effects of the corresponding symmetry transformation in the internal space. The form of the covariant derivative and the gauge field are determined by the symmetry. This in turn gives the coupling of the field to material particles.

A generic precription to how this is done is as follows. Consider the action for a free particle:

$$S_{free} = \int d^4x\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{1}$$

We subject the particle wavefunction to a certain symmetry transformation. In order to maintain gauge invariance, instead of ∂_μ, a covariant derivative D_μ, which includes the field associated with the symmetry is used. Once this is determined, the transformation of the field is determined by the requirement of the invariance of the lagrangian. An action associated with the energy density of the field can then be constructed.

A brief introduction to the symmetry groups and the gauge transformations of the strong and electroweak interactions are given below. Other important aspects of particle theory such as renormalization are spared.

2.1 Strong Interaction

The quark model suggests baryons are composed of three quarks. However there exists several problems, one of which being the violation of the Pauli exclusion priciple in structures like Δ^{++}. In this example, Δ^{++} is made up of 3 u quarks, and has intrinsic angular momentum 3/2, so the spins of the three quarks are parallel. The product of the space, spin and flavour part of the wavefunction is therefore totally symmetric.

Antisymmetry of the total wavefunction is restored if a fourth term, ψ_{colour} is introduced and be anitsymmetric under quark exchange. Since three quarks constitute a baryon the simplest choice for the colour symmetry group is $SU(3)$.

The total wavefunction now becomes:

$$\psi = \psi_{space} \times \psi_{spin} \times \psi_{flavour} \times \psi_{colour}. \tag{2}$$

This solves a well-known problem involving the electron-positron annihilation to generate quark-antiquark pairs which subsequently fragment into hadrons. The ratio of the probabilities for forming hadrons or muons is given by the well known equality:

$$R = \frac{\sigma(e^+e^- \longrightarrow \ hadrons)}{\sigma(e^+e^- \longrightarrow \ \mu^+\mu^-)} = N_c \sum_f Q_f^2, \tag{3}$$

where the sum is over all quark species with mass below the available energy. Put the number of colour species, N_c, equal to 3, together with the electric charges of the quarks involve, we find the prediction agrees well with experiments.

Since the colour force is characterised by the $SU(3)$ group, an infinitesimal local colour transformation takes the form:

$$\psi \rightarrow \exp[ig_s F_a n_a \theta(x)]\psi = (1 + ig_s F_a n_a \theta(x))\psi, \tag{4}$$

where $\theta(x)$ is the rotation angle in colour space at a point x in spacetime and g_s is the strength of the colour charge. F_a's are the generators of $SU(3)$ and summation over a is implied. The covariant derivative is given by:

$$\mathrm{D}_\mu = \partial_\mu + ig_s F_a G_{a\mu}(x), \tag{5}$$

where $G_{a\mu}(x)$ is the gauge field and carries colour. It can be shown that the lagrangian is invariant if the colour field transforms as:

$$G_{a\mu}(x) \rightarrow G_{a\mu} - n_a \partial_\mu \theta(x) - g_s f_{abc} n_b G_{c\mu}(x)\theta(x). \tag{6}$$

The energy density of the gauge field is known to be:

$$G_{a\mu\nu} G_a^{\mu\nu}, \tag{7}$$

where $G_{a\mu\nu} = \partial_\mu G_{a\nu} - \partial_\nu G_{a\mu} - g_s f_{abc} G_{a\mu} G_{c\nu}$.

The lagrangian so constructed is the starting point of the theory of quantum chromodynamics (QCD).

One observation from the above equations is that there are cubic and quartic terms in the gauge field, which correspond to interactions between the gauge bosons, gluons in this case. Interactions between gauge bosons is a unique property of non-Abelian gauge theories such as $SU(3)$ and is related to colour confinement and asymptotic freedom. This is out of the scope of our present study and interested readers are referred to [3][4]. We note that the above foundation serves as a powerful tool in anaylsing strong interactions and its theoretical predictions agree well with experimental observations.

2.2 Electroweak Interaction

Particle colliding experiments reveal the leptons and quarks undergo weak interactions and they appear to form pairs, in which one fermion transforms to its partner in the same doublet. Therefore the natural choice for the underlying symmetry for weak interaction is $SU(2)$, also known as the weak isospin symmetry. It is responsible for reactions such as beta decay. Experiements show that at sufficiently high energy only left-handed particles or their right-handed antiparticles feel the weak force, and so the symmetry is restricted to $SU(2)_L$. Right-handed fermions are singlets and do not transform under weak interaction. Electromagnetic force [5] is assumed to be familiar to readers and we do not go into details.

The Fermi theory of weak interactions, while successful at low energy scales, poses problems when momentum transfer is large. In particular the cross sections grow indefinitely as the energy of the colliding particles increases. As a result the theory is non-renormalizable. In the 1960s Glashow, Weinberg and Salam proposed a gauge theory which is fully renormalizable and it describes both weak and electromagnetic forces within one single construct. This is known as the electroweak unification. It is characterized by the group $SU(2)_L \times U(1)_y$, where the $U(1)$ is known as weak hypercharge symmetry. The electromagnetic charge (Q) is related to the weak hypercharge (y) and weak isospin by [6]:

$$Q = t_3 + \frac{y}{2}. \tag{8}$$

Under the given group structure, there are three $SU(2)$ generators and one $U(1)$ generators. Thus an infinitesimal transformation in the weak-hypercharge space is given by:

$$\psi \to \Big(1 + ig\alpha_i(x)t_i^L + i\frac{1}{2}g'y\theta(x)\Big)\psi. \tag{9}$$

Again summation over i from 1 to 3 is implied. The covariant derivative in this case is:

$$\mathrm{D}_\mu = \partial_\mu + igt_i^L W_{i\mu}(x) + i\frac{1}{2}g'yB_\mu(x). \tag{10}$$

Here the W and B are gauge fields corresponding to the $SU(2)_L$ and the $U(1)$ repectively. They transform as:

$$\mathrm{W}_{i\mu} \to W_{i\mu} - \partial_\mu\alpha_i(x) - g\epsilon_{ijk}\alpha_j(x)W_{k\mu}, \tag{11}$$

$$\mathrm{B}_\mu \to B_\mu - \partial_\mu\theta(x). \tag{12}$$

The energy dessity is given by terms proportional to:

$$W_{i\mu\nu}W_i^{\mu\nu} \qquad \text{and} \qquad F_{\mu\nu}F^{\mu\nu}, \tag{13}$$

where they take the forms:

$$W_{i\mu\nu} = \partial_\mu W_{i\nu} - \partial_\nu W_{i\mu} - g\epsilon_{ijk} W_{j\mu} W_{k\nu} \qquad \text{and} \tag{14}$$

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{15}$$

We note here that in all gauge theories the gauge bosons are massless if there is only one gauge field presents. In order for the weak bosons to acquire mass, which is what we observe experimentally, there must exist additional gauge field. This is accomplished by the Higgs mechanism in which the Higgs field has a non-vanishing vaccum field. It generates spontaneous symmetry breaking and the weak bosons acquire mass as a consequence. This is out of the scope of this thesis and readers who are interested may consult [7].

2.3 Extensions Beyond the Standard Model

We have seen that all known fundamental forces of nature, except gravity, are captured by the gauge group $SU(3)_C \times SU(2)_L \times U(1)y$, which underlie the symmetry of strong and electroweak interactions. The $U(1)_y$ charge is defined in (8). We could now group all the known particles in each generation according to the way they transform under the standard model gauge group. This is summarized in the table below:

		$SU(3)$	$SU(2)$	$U(1)$
left handed quarks	Q	3	2	$\frac{1}{6}$
right handed up quarks	$\bar{u}$	$\bar{3}$	1	-$\frac{2}{3}$
right handed down quarks	$\bar{d}$	$\bar{3}$	1	$\frac{1}{3}$
left handed leptons	L	1	2	-$\frac{1}{2}$
charge conjugate of LH electron	e_L^c	1	1	1

Together with the two other generations of particles having the same group structure, they constitute the foundation of the standard model.

It is possible to extend the standard model gauge group to larger groups which contain $SU(3) \times SU(2) \times U(1)$ [8]. In particular, $SU(5)$ is the mimimal extension. Further enlargement can be made if we extend the group structure to say, $SO(10)$. A point worthwhile to note is the addtion of a singlet to the existing particles, which may be identified as the right handed neutrino, makes the whole family fits into 16 of $SO(10)$. The larger group structure has new generators in addition to those in the standard model. They correspond to bosons that mediate quarks and leptons and therefore all unified theories predict physical process such as proton decay. Again our emphasis is on the excellent agreement between experimental results and theoretic predictions based on the standard gauge group. It is thus important to have a mechanism for a unified theory to reveal the same group structure in the low energy limit. We will show later that this is the case for a class of realistic free fermionic model in string theory we consider later.

3 Introduction to String Theory

The standard model, while proved to be extremely successful as a working theory, leaves nevertheless many problems unanswered. A partial list includes the large number of free parameters, such as particle masses, which we choose to fit with experiments, and also the exclusion of gravity.

String theory attempts to solve the above problems by supposing higher dimensional objects, instead of point particles, as fundamental constituents of nature. As the theory is relevant only at extremely high energy scale, the theory is at present not testable by experiments. Demand of mathematical consistencies is the primary tool in investigating the theory.

As string theory is a vast and extremely technical subject it is not possible to give a full account of it here. What is intended is to address two key ideas, in which the free fermionic model we will discuss later rely on, through a lightest sketch of the free bosonic string. The same ideas can be applied to the case of supersymmetric strings but is out of the scope of this text, so only the results will be quoted. The key ideas are:

i	Various particle species might be identified as different vibrational states of the string
ii	Extra dimensions, or degrees of freedom, are required for mathematical consistency

3.1 Classical Bosonic Strings

When a point particle moves in spacetime its trajectory is described by a worldline. Similarly for a string to travel in spacetime, its trajectory is described by a worldsheet. A string can be open or closed. We will only concentrate on closed strings but the treatment for open strings is analogues.

The world sheet coordinates can be specified by $X^\mu(\tau, \sigma)$, where a curve of constant τ, whose points are labelled by σ, runs along the length of the string.

The string action which is known to be mathematically consistent is the Polyakov action [9], and is given by:

$$S = -\frac{1}{4\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^l d\sigma (-\gamma)^{\frac{1}{2}} \gamma^{ab} \partial_a X_\mu \partial_b X^\mu. \qquad (16)$$

The indices of the worldsheet metric ,γ_{ab}, run from 0 to 1, with $\sigma^0 = \tau$, $\sigma^1 = \sigma$. The string tension is given by $1/2\pi\alpha'$.

The Euler-Lagrange equation, obtained by varying X_μ is:

$$\gamma^{ab} \nabla_a \nabla_b X^\mu = 0, \qquad (17)$$

and the energy-momentum (EM) tensor is obtained by varying the metric.

The EM tensor is defined by:

$$T^{ab} = -4\pi(-\gamma)^{-\frac{1}{2}} \frac{\delta S}{\delta \gamma_{ab}} = -\frac{1}{\alpha'}[\partial^a X_\mu \partial^b X^\mu - \frac{1}{2}\gamma^{ab}\partial_c X_\mu \partial^c X_\mu]. \quad (18)$$

It can be shown that it obeys:

$$\nabla_a T^{ab} = 0. \quad (19)$$

The conjugate momentum $\Pi^\mu(\tau, \sigma)$ of the field $X_\mu(\tau, \sigma)$ is defined by:

$$\Pi^\mu = -\frac{\delta S}{\delta \dot{X}_\mu} = \frac{1}{2\pi\alpha'}(-\gamma)^{\frac{1}{2}}\gamma^{0a}\partial_a X^\mu. \quad (20)$$

Because the Polyakov action does not depend explicitly on X^μ, we obtain a version of conservation of spacetime momentum carried by the string, in which the momentum is given by:

$$P^\mu = \int_0^l d\sigma \Pi^\mu(\tau, \sigma). \quad (21)$$

It can be shown that a Weyl transformation [7] on the worldsheet metric, defined by:

$$\gamma'_{ab}(\tau, \sigma) = \exp[\omega(\tau, \sigma)]\gamma_{ab}(\tau, \sigma), \quad (22)$$

where $\omega(\tau, \sigma)$ is a position dependent arbitrary function, leaves the action invariant. It is also true that reparametrizing the worldsheet coordinates by:

$$\tau \to \tau'(\tau, \sigma) \qquad \sigma \to \sigma'(\tau, \sigma), \quad (23)$$

does not change the Polyakov action. Since the worldsheet metric is two dimensional, it has 3 degrees of freedom. The three degrees of freedom given by the above transformation thus allow us to fix the gauge to obtain a Minkowskian metric. From this (17) becomes:

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0, \quad (24)$$

and we obtain a standard wave equation which can be decomposed into Fourier components.

Conformal invariance

The idea of conformal invariance is too important in string theory to be overlooked. We will show shortly that it can be looked upon as a combination of reparametrization and Weyl transformations.

First of all we replace the Minkowskian metric on the worldsheet with a Euclidean one by complexifying τ. Write $\tau = -i\sigma^2$, we define:

$$\omega \equiv \sigma^1 + i\sigma^2 = \sigma - \tau, \qquad \bar{\omega} \equiv \sigma^1 - i\sigma^2 = \sigma + \tau. \quad (25)$$

It is straightforward to show that:

$$\partial \equiv \frac{\partial}{\partial \omega} = \frac{1}{2}\Big(\frac{\partial}{\partial \sigma} - \frac{\partial}{\partial \tau}\Big), \qquad \text{and} \qquad \bar{\partial} \equiv \frac{\partial}{\partial \bar{\omega}} = \frac{1}{2}\Big(\frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau}\Big). \tag{26}$$

The worldsheet metric, in terms of the new coordinates, becomes:

$$\begin{pmatrix} \gamma_{\omega\omega} & \gamma_{\omega\bar{\omega}} \\ \gamma_{\bar{\omega}\omega} & \gamma_{\bar{\omega}\bar{\omega}} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{27}$$

while the EM tensor is calculated to be:

$$\begin{pmatrix} T_{\omega\omega} & T_{\omega\bar{\omega}} \\ T_{\bar{\omega}\omega} & T_{\bar{\omega}\bar{\omega}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha'}\partial X_\mu \partial X^\mu & 0 \\ 0 & -\frac{1}{\alpha'}\bar{\partial} X_\mu \bar{\partial} X^\mu \end{pmatrix}. \tag{28}$$

The proper worldsheet distance now becomes:

$$ds^2 = -d\tau_{WS}^2 = -d\tau^2 + d\sigma^2 = d\omega d\bar{\omega}. \tag{29}$$

The string action can be shown to equal:

$$S = -\frac{1}{2\pi\alpha'}\int d\omega d\bar{\omega} \partial X_\mu \bar{\partial} X_\mu. \tag{30}$$

Reparametrizing the fields $X^\mu(\omega, \bar{\omega})$ by $X^\mu(\omega, \bar{\omega}) \to X'^\mu(f(\omega), \bar{f}(\bar{\omega}))$ leaves the action invariant. This is known as the conformal invariance. The proper distance is changed but can be fixed by a Weyl transformation, thus confirming the claim that conformal invariance is a special combination of reparameterization and Weyl transformation.

It is useful to calculate the Poisson brackets of X^μ, Π^μ and the EM tensors. Using the standard definition of the Poisson braket, it is easy to show that:

$$\{X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\} = -\eta^{\mu\nu}\delta(\sigma - \sigma'), \tag{31}$$

$$\{X^\mu(\tau,\sigma), T(\tau,\sigma')\} = -2\pi\delta(\sigma - \sigma')\partial X^\mu(\omega,\omega'), \tag{32}$$

$$\{X^\mu(\tau,\sigma), \widetilde{T}(\tau,\sigma')\} = -2\pi\delta(\sigma - \sigma')\bar{\partial} X^\mu(\omega,\omega'). \tag{33}$$

Mode expansions

The gauge-fixed equation in (24) can be expanded in Fourier modes:

$$X^\mu(\tau,\sigma) = x^\mu + \alpha' p^\mu \tau + i\Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\Big[\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}\Big]. \tag{34}$$

Since $X^\mu(\tau,\sigma)$ is real, we have two relations between α and $\tilde{\alpha}$, namely:

$$\alpha_n^{\mu *} = \alpha_{-n}^\mu \qquad \text{and} \qquad \tilde{\alpha}_n^{\mu *} = \tilde{\alpha}_{-n}^\mu. \tag{35}$$

By integrating (34) over the space-like coordinate σ, we find that $x^\mu + \alpha' p^\mu \tau$ can be thought as the centre of mass of the string. Using (31), and integrating over σ and σ', we obtain:

$$\{x^\mu, p^\nu\} = -\eta^{\mu\nu}. \tag{36}$$

At this point the relation between X^μ and Π^μ can be used to obtain equations for α_m^μ and $\tilde{\alpha}_m^\mu$ in terms of these two quantities. We can then make use of these equations and also (31) to obtain the Poisson brackets between the mode operators. They are found to be:

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} = im\eta^{\mu\nu}\delta_{m,-n} \quad \text{and} \quad \{\alpha_m^\mu, \tilde{\alpha}_n^\nu\} = 0. \tag{37}$$

We now rewrite the mode expansion in the Euclidean coordinates ω and $\bar{\omega}$:

$$X^\mu(\omega, \bar{\omega}) = x^\mu + \frac{1}{2}\alpha' p^\mu(\omega - \bar{\omega}) + i\Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\Big[\alpha_n^\mu \mathrm{e}^{in\omega} + \tilde{\alpha}_n^\mu \mathrm{e}^{-in\bar{\omega}}\Big]. \tag{38}$$

Evidently it is a sum of a function of ω and a function of $\bar{\omega}$. We can simplify the EM tensor and put it into a compact form. If we define:

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}} p^\mu, \tag{39}$$

then the components of the EM tensor take the forms:

$$T_{\omega\omega} \equiv T(\omega) = \sum_{n=-\infty}^{\infty} L_n \mathrm{e}^{in\omega}, \qquad T_{\bar{\omega}\bar{\omega}} \equiv \widetilde{T}(\bar{\omega}) = \sum_{n=-\infty}^{\infty} \widetilde{L}_n \mathrm{e}^{-in\bar{\omega}}, \tag{40}$$

where L_n and $\widetilde{L}_n$ are the generators of conformal transformations. They are given by:

$$L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{m\mu}\alpha_{n-m}^\mu \qquad \text{and} \qquad \widetilde{L}_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty} \tilde{\alpha}_{m\mu}\tilde{\alpha}_{n-m}^\mu. \tag{41}$$

The Poission brackets of the conformal generators are evaluated to be:

$$\{L_m, L_n\} = -i(m-n)L_{m+n}, \tag{42}$$
$$\{\widetilde{L}_m, \widetilde{L}_n\} = -i(m-n)\widetilde{L}_{m+n} \qquad \text{and} \tag{43}$$
$$\{L_m, \widetilde{L}_n\} = 0. \tag{44}$$

These relations are known as the Virasoro algebra.

3.2 String Quantization

The standard quantization procedure is followed here, namely the cooefficients in the mode expansion are promoted to operators. The commutation relations correspond to (37) are:

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = -m\eta^{\mu\nu}\delta_{m,-n} \qquad \text{and} \qquad [\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0. \tag{45}$$

It would make sense for the quantized version of the theory to maintain conformal invariance and also the commutation relations as in the classical theory. This proves to be non-trivial because in all quantized theories the order of field operators matters. A way to get over the problem is outlined as follows.

First of all we identify the creation and annihilation operators by looking at the commutation relations between α_n^μ and the hamiltonian. The hamiltonian is given by:

$$\begin{aligned} H &= -\int_0^{2\pi} d\sigma \Pi_\mu(\tau,\sigma)\partial_\tau X^\mu(\tau,\sigma) - L \\ &= L_0 + \widetilde{L}_0. \end{aligned} \tag{46}$$

Using this we obtain:

$$[\alpha_n^\mu, H] = n\alpha_n^\mu, \tag{47}$$

which shows that α_n^μ are annihilation operators when $n > 0$, and creation operators when $n < 0$.

It can also be shown easily that the momentum takes the form:

$$P = L_0 - \widetilde{L}_0. \tag{48}$$

Quantum Virasoro algebra

We would also like to maintain the commutation relation:

$$[L_m, L_n] = (m-n)L_{m+n}, \tag{49}$$

but as they are dependent on α_n^μ's care is needed to be taken. We see from (45) and (41) that only L_0 has ordering problem. Using (39), we can write it as:

$$L_0 = -\frac{\alpha'}{4}p_\mu p^\mu - \sum_{m=1}^{\infty} \alpha_{-m\ \mu}\alpha_m^\mu + a. \tag{50}$$

This expression is in normal ordering, in the sense that the creation operators are on the left of the annihilation operators. Using this, together with (41), we obtain:

$$[L_m, L_n] = (m-n)L_{m+n}, \tag{51}$$

for $m \neq n$, and:

$$[L_m, L_{-m}] = 2m(L_0 - a) + m\delta^\mu_\mu \sum_{r=1}^{\infty} r, \tag{52}$$

where δ^μ_μ is the dimensionality of spacetime and is denoted by d from now on. This is not sensible as we have an infinite value. However, careful analysis on the effect of $[L_m, L_{-m}] - 2m(L_0 - a)$ on the ground state of the oscillators, which are the normal modes of string vibrations, reveals:

$$[L_m, L_{-m}] - 2m(L_0 - a) = \frac{d}{12} m(m^2 - 1). \tag{53}$$

Together with (51) and (52) we obtain the following relation:

$$[L_m, L_n] = (m - n)L_{m+n} + \Big[\frac{d}{12} m(m^2 - 1) - 2ma\Big]\delta_{m,-n}. \tag{54}$$

The Fadeev-Popov method

We have seen that by quantizing the bosonic string, an extra term is obtained in the commutation relation of the conformal generators. This anomaly can be removed by the Fadeev-Popov method, in which ghost fields are introduced in addition to the original fields X^μ. This involves lots of technicalities, and is very hard to give an adequate justification to it in the short space here. Rather a few important results are quoted [10].

The ghost action associated with the ghost fields (b, c) can be expressed in the form:

$$S_g(b, c) = \frac{1}{4\pi} \int d\tau d\sigma (-\gamma)^{\frac{1}{2}} b_{ab} \Big[\nabla^a c^b + \nabla^b c^a - \gamma^{ab} \nabla_c c^c\Big]. \tag{55}$$

¿From this the EM tensor of the ghost fields can be calculated using the same definition of the EM tensor for X^μ. Indeed the previous procedure used to calculate various commutation relations can be applied here. The conformal generators can be shown to be:

$$L_n^{(g)} = \sum_{m=-\infty}^{\infty} (sn - m) : b_m c_{n-m} : - \delta_{n,0}. \tag{56}$$

The combined result is given by:

$$\begin{aligned}[L_m, L_n] \;&=\; \Big[\frac{1}{12} m(m^2 - 1)(c^{(X)} + c^{(g)}) - 2m(a^{(X)} + a^{(g)})\Big]\delta_{m,-n} \\ &\quad + (m - n)L_{m+n} \quad ,\end{aligned} \tag{57}$$

where $L_n = {L_n}^{(X)} + {L_n}^{(g)}$. The ghost charge $c^{(g)}$ is known to be -26. We see that the anomalous term vanishes if we take $c^{(X)} + c^{(g)}$ and $a^{(X)} + a^{(g)}$ both equal zero. The first term means that we take $c^{(X)} = 26$, while the second term can be set zero by a suitable definition.

The mass spectrum

There is a constraint which states that for any physical states in the Hilbert space, $(L_0 - a)|\psi\rangle = 0$, where L_0 is the Virasoro generators for the combined X^μ and the ghost fields [11]. Armed with this relation, together with (50) and (56), we obtain:

$$L_0 = -\frac{\alpha'}{4} p_\mu p^\mu + \sum_{n=1}^{\infty} n\Big(b_{-n}c_n + c_{-n}b_n - n^{-1}\alpha_{-n\ \mu}\alpha_n^\alpha\Big) - 1 + a, \quad \text{or} \tag{58}$$

$$M^2 = \frac{4}{\alpha'}(N-1), \tag{59}$$

where N is the terms in the summation. The ground state, where $N = 0$, gives a negative value and therefore violate causality. It is known as the tachyon and is required to be projected out of the mass spectrum of any realistic string model.

3.3 Superstrings

We saw that tachyon is present in bosonic strings. Also all degrees of freedom are bosonic and so the theory does not describe fermions. It turns out that fermionic fields can be inserted into the bosonic string action, and if the fermionic and bosonic fields are related by supersymmetry a lot of amazing results come out naturally.

The generalized action which includes fermionic fields on a Minkowskian worldsheet is given by [10]:

$$S = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \int_0^l \Big[\frac{2}{\alpha'}\partial X_\mu \bar{\partial} X^\mu + i\psi_\mu \bar{\partial}\psi^\mu - i\tilde{\psi}_\mu \partial \tilde{\psi}^\mu\Big] \tag{60}$$

where $X^\mu(\tau, \sigma)$ are identified as spacetime coordinates of the worldsheet, and the fermionic fields are components of D-Majorana spinors. The condition of the Majorana spinor forces ψ^μ and $\tilde{\psi}^\mu$, its left- and right-moving components, to be real.

To maintain the commutation relations in this supersymmetric version of string theory, we introduce fermionic ghosts. The fermionic ghost charge turns out to be ${c_f}^{(g)} = 11$, while each Majorana fermion gives ${c_f}^{(X)} = c^{(X)}/2$. We obtain the following relation:

$$c_{total} = c^{(g)} + {c_f}^{(g)} + c^{(X)} + {c_f}^{(X)} = -15 + \frac{3}{2}c^{(X)}, \tag{61}$$

which gives spacetime dimension 10 for $c_{total} = 0$.

We demand the value of the Lagrangian density to be unique at each spacetime point. This transpires into the internal degrees of freedom, ψ^μ and $\tilde{\psi}^\mu$, to be either periodic or antiperiodic for a closed string:

$$\psi^\mu(\tau, \sigma + l) = \pm\psi^\mu(\tau, \sigma) \tag{62}$$

and similarly for $\tilde{\psi}^\mu$. We therefore arrive at two boundary conditions [12][13]. The Ramaond (R) boundary condition is periodic, while the antiperiodic boundary condition is termed Neveu-Schwarz (NS) boundary condition.

GSO projection

A consequence of these different boundary conditions is that the full Hilbert space of the superstring contains several topological sectors [3]. It turns out that there are consistency requirements such that only certain combinations of sectors appear in the physical Hilbert space. The extraction of the physical states is realized by the GSO projection, which is an important idea in constructing realistic string models. We will come back to this later.

Summary

This is by no means an adequate account of string theory. It is hoped that through this rather hand-waiving introduction, it becomes clear how extra degrees of freedom and mass spectrum come about as consequences of requiring mathematical consistencies. Readers are referred to other texts [14][15][16] for further information.

It is worthy to note some incredible results for superstring theory [14]. First of all, it is completely anomaly free if the theory is goverened by groups $SO(32)$ or $E_8 \times E_8$ [17]. The theory is finite to all orders in perturbation theory, and therefore no renormalization is required [18]. Also spin-2 bosons always appear in the theory which we interpret as graviton and so it leads to quantum gravity naturally. Perhaps the most appealing of all is that the theory is so tightly constrained that it is free of adjustable parameters. This is compared with the standard model.

Another important point is that the large groups of $SO(32)$ or $E_8 \times E_8$ makes phenomenology possible [3] because we may break the large group sturcture into smaller groups and hope to recover the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_y$ in the process. This is the goal of the thesis and we will continue to develop the idea in subsequent chapters.

4 The Free Fermionic Model

In the last chapter some basic ideas in string theory were introduced. Now we turn to the construction of realistic string models.

In this chapter we introduce the heterotic string, which is regarded as the most promising [19] among all string theories to be relevant to the nature we are living in. Then we turn to the free fermionic formulation [20], in which we interpret the degrees of freedom required to cancel the conformal anomaly as internal degrees of freedom on the worldsheet. We will introduce a string partition function, and sketch how the implementation of modular invariance to the partition function will lead to a set of rules related to the GSO projection which we will rely on in our actual development of a realistic string model.

Most substance in this chapter follows the lecture notes on string phenomenology by Dr. A. Faraggi [11].

4.1 Heterotic Strings

¿From (60) we see that the two components of the Majorana spinor are decoupled. This means that the left- and right-moving modes of the string are independent. We can therefore impose supersymmetry on the left-moving modes only, while the right-moving modes remain purely bosonic. In this case, the conformal anomalies cancel seperately.

Now we have the left-moving modes living in 10-dimensions, while the right-moving modes living in 26-dimensions. The latter can be labelled by X^μ which has been compactified to 10 dimensions, leaving a compact 16-dimensional space. Each compactified coordinate generates a vertex operator for a $U(1)$ spacetime current. As it turns out, there are only two group structures possible in this construction, namely the $SO(32)$ and the $E_8 \times E_8$.

4.2 The Fermionic Construction

In the free fermionic formulation, instead of interpreting the degrees of freedom required for conformal anomaly cancellations as spacetime dimensions, they are interpreted as internal degrees of freedom, and can be thought of as free internal fermions propagating on the worldsheet. The theory can then be formulated on a 4-dimensional spacetime.

In this context, anomaly cancellations give:

$$C_L = 0 = -26 + 11 + D + \frac{D}{2} + n_{b_L} + \frac{n_{f_L}}{2} \quad \text{and} \tag{63}$$

$$C_R = 0 = -26 + D + n_{b_R}. \tag{64}$$

In supersymmetry, there is a one-to-one correspondence between bosons and their fermionic superpartners, so we have $n_{b_L} = n_{f_L}$. Also there is a mechanism which allows bosons to transform into fermions in 2 dimensions, defined by:

$$\exp^{iX} = y + iw \qquad \text{and} \qquad \exp^{-iX} = y - iw. \tag{65}$$

Together with $D = 4$, and the two superpartners of the left-moving worldsheet coordinates, we obtain:

20 left-moving real fermions
44 right-moving real fermions.

Qualitatively we can explain this result as follows: In the free-fermionic construction, the extra degrees of freedom are interpreted as internal degrees of freedom. Therefore for the right-moving bosonic part there are 22 extra bosonic degrees of freedom. The mechanism to transform bosons into fermions is employed to generate the 44 real fermions. For the left-moving supersymmetric part, there are 6 extra dimensions, giving rise to 6 bosons and 6 fermions. The same mechanism is applied here to transform the 6 bosons into 12 real fermions. On top of the 18 left-moving real fermions, we also include the 2 superpartners of the worldsheet coordinates, thus generating 20 left-moving fermions in total. One expects a correspondence between 12 left-moving fermions and 12 right-moving fermions from the construction of the heterotic string.

4.3 Partition function

The partition function is defined as the one loop vacuum-to-vacuum-to-vacuum amplitude. It is clear that by a suitable conformal transformation, an one-loop can be transformed into a torus. The partition function is then the sum over all physically inequivalent tori.

It is easy to see that in a torus there are two non-contractible loops. As a world-sheet fermion propagates around these loops it picks up a phase, defined by:

$$f \to -e^{-i\pi\alpha(f)} f. \tag{66}$$

The 64 $\alpha(f)$'s specify the boundary conditions of the free fermions and can be represented by a basis vector with 64 entries. A set of basis vectors then generates worldsheet fermions.

When we calculate the partition fuction, we sum over all the physically inequivalent tori. The partition function is given by:

$$Z_i \begin{bmatrix} \theta \\ \beta \end{bmatrix} = \sum_{s \in \mathcal{H}} \langle s | e^{i\theta P} e^{-s\pi\beta H} | s \rangle \tag{67}$$

where $\mathcal{H}$ represents the Hilbert space, $e^{i\theta P}$ represents spatial propagation, and $e^{-2\pi\beta H}$ represents time propagation of the states $|s\rangle$. It can be shown to be equal to:

$$Z_i \left[\begin{array}{c} \theta \\ \beta \end{array} \right] = \mathrm{Tr}_{\mathcal{H}}(e^{i\theta P} e^{-2\pi\beta H}) \tag{68}$$

It was shown in (46) and (48) that classically the Hamiltonian and the momentum generator can be expressed in terms of L_0 and $\tilde{L}_0$. We state without proof that the quantum-mechanical versions are given by:

$$H = L_0 + \tilde{L}_0 - \frac{1}{24} \qquad \text{and} \tag{69}$$

$$P = L_0 - \tilde{L}_0. \tag{70}$$

The 1/24 arises from the conformal anomaly when we quantize the theory. If we write $\tau = i\beta - \frac{\theta}{2\pi}$ and $g = e^{i2\pi\tau}$, the partition function can be rewritten as:

$$Z_i(\tau) = g^{-\frac{1}{48}} \tilde{g}^{-\frac{1}{48}} \mathrm{Tr}(g^{L_0} \tilde{g}^{\tilde{L}_0}). \tag{71}$$

The total partition function is then the product of partition functions for the worldsheet fermions. The form of the total partition function is therefore:

$$Z_F(\tau) = \prod_{i=1}^{64} Z_i \left[\begin{array}{c} \theta \\ \beta \end{array} \right] (\tau). \tag{72}$$

By convention, θ and β represent the 'spatial' and 'time' boundary conditions repectively.

4.4 Modular Invariance

A torus can be mapped onto a complex plane by cutting along its two non-contractible loops. Define a complex parameter $z = \sigma_1 + i\sigma_2$ where the two coordinates are periodic with lengths λ_1 and λ_2 respectively. We then see that $z + n_1\lambda_1 + in_2\lambda_2$ is equivalent to z, if n_i are integers. Also, the same torus is obtained by substituting $\lambda_1{}' = a\lambda_1$ and $\lambda_2{}' = a\lambda_2$.

We define a parameter which specifies inequivalent tori by:

$$\kappa = \frac{\lambda_2}{\lambda_1}. \tag{73}$$

Now make a linear transformation on the λ's:

$$\left(\begin{array}{c} \lambda_2{}' \\ \lambda_1{}' \end{array} \right) = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{c} \lambda_2 \\ \lambda_1 \end{array} \right) \tag{74}$$

where a, b, c and d are arbitrary. We then obtain:

$$\kappa \to \kappa' = \frac{a\kappa + b}{c\kappa + d}. \tag{75}$$

The new torus is now given by $z + n_1{'}\lambda_1{'} + n_2{'}\lambda_2{'}$ where $n_i{'} \in Z$. This is equal to $z + (n_1{'}d + n_2{'}b)\lambda_1 + (n_1{'}c + n_2{'}a)\lambda_2$, and so we get back the same torus if the following relation is satisfied:

$$\begin{pmatrix} n_2 \\ n_1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} n_2{'} \\ n_1{'} \end{pmatrix}, \tag{76}$$

where again $n_i \in Z$.

Therefore we have an unambiguous way to determine the physically inequivalent states. In other words modular invariance can be seen as a procedure to identify physically equivalent tori.

Next we require the partition function be invariant under modular transformations. The partition function is also reparametrization invariant as the string worldsheet is invariant under reparametrization. The result of the two invariances is that we demand the partition function to be invariant under:

$$\kappa \to -\frac{1}{\kappa} \qquad \text{and} \tag{77}$$

$$\kappa \to \kappa + 1. \tag{78}$$

By requiring invariance under the above transformations a set of rules govening the allowed boundary conditions is obtained. This set of rules is the primary tool for our construction of a realistic string model.

Spin structures

The boundary conditions of real fermions around the two non-contractible loops can be periodic (R) or anti-periodic (NS). The four possibilities are:

$$Z_f \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad Z_f \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad Z_f \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad Z_f \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{79}$$

where the phase is defined in (66), and so:

$$0 \leftrightarrow \text{NS} \qquad 1 \leftrightarrow \text{R}. \tag{80}$$

These are called the spin-structures of the fermions on the torus.

A complex fermion can be built from two real fermions. It takes the form:

$$f = \frac{1}{\sqrt{2}}(f_1 + if_2) \qquad \bar{f} = \frac{1}{\sqrt{2}}(f_1 - if_2), \tag{81}$$

so we expect $\alpha(f)$ for complex fermions may take values between ± 1.

The spin structures can be expressed in the form:

$$Z_f \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{\theta_3^{1/2}}{\eta^{1/2}}, \quad Z_f \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\theta_4^{1/2}}{\eta^{1/2}}, \quad Z_f \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\theta_2^{1/2}}{\eta^{1/2}}, \quad Z_f \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\theta_1^{1/2}}{\eta^{1/2}}. \tag{82}$$

We state without proof that upon modular transformation $\kappa \to \kappa + 1$ the spin structures transform according to:

$$\eta \to e^{i\frac{\pi}{12}}\eta, \qquad \theta_1 \to e^{i\frac{\pi}{4}}\theta_1, \qquad \theta_2 \to e^{i\frac{\pi}{4}}\theta_2, \qquad \theta_3 \leftrightarrow \theta_4, \tag{83}$$

and upon $\kappa \to -\frac{1}{\kappa}$ the functions transform as:

$$\eta \to (-i\kappa)^{\frac{1}{2}}\eta, \qquad \frac{\theta_1}{\eta} \to e^{-i\frac{\pi}{2}}\frac{\theta_1}{\eta}, \qquad \frac{\theta_2}{\eta} \leftrightarrow \frac{\theta_4}{\eta}, \qquad \frac{\theta_3}{\eta} \to \frac{\theta_3}{\eta}. \tag{84}$$

In this formulation, the partition function is a product of left and right moving theta functions. It can be generated by specifying a set of boundary condition basis vectors:

$$B = \{\vec{b}_1, \vec{b}_2 \cdots \vec{b}_n\}. \tag{85}$$

The one-loop fermionic partition function is then given by:

$$Z = \sum_{\substack{spin \\ structure}} C\binom{\vec{\alpha}}{\vec{\beta}} Z_F \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix}, \tag{86}$$

where

$$\vec{\alpha} = \sum_{i=1}^{n} \alpha_i \vec{b}_i \qquad \vec{\beta} = \sum_{i=1}^{n} \beta_i \vec{b}_i. \tag{87}$$

The sum is over all spin structures allowed by modular invariance. Evidently, the phases $C\binom{\vec{\alpha}}{\vec{\beta}}$ related by modular transformation have to be the same in order to maintain the invariance of the partition function. All these constraints culminate in a set of rules derived by Antoniadis, Bachas and Kounnas, and is known as the ABK [20] rules. It specifies the conditions between different basis vectors and also the rules on the one-loop phases. We will state the rules explicitly at the beginning of the next chapter.

4.5 Other Key Ideas

Constraint on worldsheet supersymmetry

Recall that in our constructuion supersymmetry is imposed on the left-moving modes. The associated supercurrent is given by:

$$T_F = \psi^\mu \partial X_\mu + f_{abc}\psi^a\psi^b\psi^c. \tag{88}$$

Here a, b and c runs from 1 to 18 and correspond to the 18 free fermions in the 6 compactified dimensions.

If the 18 fermions are grouped into six $SU(2)$ groups, then the supercurrent takes the form:

$$T_F = \psi^\mu \partial X_\mu + \sum_{i=1}^{6} \chi_i y_i w_i. \tag{89}$$

Evidently the transformation of $\chi_i y_i w_i$ must be the same as ψ^μ in order the supercurrent is well-defined. Therefore if we have $\psi^\mu \to -\psi^\mu$, *i.e.* $\alpha(\psi^\mu) = 0$, we need:

$$\Big(\alpha(\chi), \alpha(y), \alpha(w)\Big) : \quad (0,0,0) \quad (0,1,1) \quad (1,1,0) \quad (1,0,1). \tag{90}$$

Similarly if we have $\psi^\mu \to \psi^\mu$, *ie* $\alpha(\psi^\mu) = 1$, then we require:

$$\Big(\alpha(\chi), \alpha(y), \alpha(w)\Big) : \quad (0,0,1) \quad (0,1,0) \quad (1,0,0) \quad (1,1,1). \tag{91}$$

Worldsheet current

Each complex fermions generates a worldsheet current. The charge corresponds to each current is defined by:

$$Q(f) = \frac{1}{2}\alpha(f) + F(f), \tag{92}$$

where $\alpha(f)$ is the boundary condition of a complex fermion and $F(f)$ is the fermion number.

For the non-degenerate NS vacuum, the fermion number is given by $+1$, and it is -1 if its complex conjugate acts on the NS vacuum.

For the R vacua there are two degenerate vacua, denoted by $|\pm\rangle$. The fermion numbers for complex fermions acting on these vacua are given by:

$$\begin{aligned} F|+\rangle &= (0)|+\rangle, \\ F|-\rangle &= (-1)|-\rangle. \end{aligned} \tag{93}$$

Ising operators

An Ising operator is obtained by combining a real left-moving fermion and a real right-moving fermion. The Ising operators define the Ising model in statistical mechanics. The idea is beyond the scope of this thesis. We only note that such combinations are important in constructing a realistic string model.

Summary

In this chapter, we introduce the free fermionic model which formulates string theory on a four dimensional spacetime, and interpret the extra degrees of freedom as free fermions propagating on the worldsheet. We outline that by demanding modular invariance [21] of the string partition function, a set of rules emerges naturally. It governs the possible boundary conditions of the worldsheet fermions, and also the phases between different spin sturctures. The constraint on supercurrent is also described, and the charges associated with complex fermion currents are defined. The Ising operator is introduced. These ideas will be implemented in the next chapter when we build a realistic string model.

5 Realistic String Model

In the previous chapters some ideas behind the free fermionic formulation were introduced. Constraints arise from various mathematical consistencies were also described. In this chapter we apply these constraints to construct a realistic string model.

From the previous chapter we know that a model is defined by specifying:

1)	A set of boundary condition basis vectors $\vec{b}_i$.
2)	The one-loop phases $C\binom{b_i}{b_j}$ for all intersections of basis vectors.

We will show that with the choice of basis vectors and phases decribed below the model constructed possesses the observable $SU(3)_C \times SU(2)_L \times U(1)_y$ gauge symmetries. We then compare with a class of model which has a non-Abelian custodial $SU(2)_c$ symmetry, in which only leptons transform [22], and explain how this extra symmetry is absent in our model.

5.1 The ABK Rules

The constraints imposed by modular invariance on the partition function is realized as a set of rules derived by Antoniadis, Bachas and Kounnas [20]. The rules are stated below.

1) The basis vectors, b_k, span a finite additive group $\Xi = \sum_k n_k b_k$ where $n_k = 0, \cdots, N_k - 1$, and $N_k b_k = 0 \bmod 2$.

2) Rules on basis vectors

i)	$\sum m_i b_i = 0$	iff $\forall m_i = 0 \bmod N_i$
ii)	$N_{ij} b_i \cdot b_j = 0 \bmod 4$	N_{ij} is the LCM of b_i and b_j
iii)	$N_i b_i \cdot b_i = 0 \bmod 8$	
iv)	Even no. of real fermions	
v)	$b_1 = \vec{1}$	

3) Rules on one-loop phases

i)	$C\binom{b_i}{b_j} = \delta_{b_i} e^{i\frac{i2\pi n_i}{N_j}} = \delta_{b_j} e^{i\frac{2\pi n_j}{N_i}} e^{i\frac{\pi b_i \cdot b_j}{2}}$
ii)	$C\binom{b_i}{b_i} = -e^{i\frac{\pi b_i \cdot b_j}{4}} C\binom{b_i}{1}$
iii)	$C\binom{b_i}{b_j} = e^{i\frac{\pi b_i \cdot b_j}{2}} C\binom{b_j}{b_i}^*$
iv)	$C\binom{b_i}{b_j + b_k} = \delta_{b_i} C\binom{b_i}{b_j} C\binom{b_i}{b_k}$

In the above notations, $b_i \cdot b_j$ is the Lorentzian product given by:

$$b_i \cdot b_j = \left\{ \left(\sum_{\substack{complex \\ left}} + \frac{1}{2} \sum_{\substack{real \\ left}} \right) - \left(\sum_{\substack{complex \\ right}} + \frac{1}{2} \sum_{\substack{real \\ right}} \right) \right\} b_i(f) b_j(f). \tag{94}$$

The δ_{b_i} is defined as:

$$\delta_{b_i} = e^{ib_i(\psi^\mu)\pi} = \begin{cases} -1 & b_i(\psi^\mu) = 1 \\ +1 & b_i(\psi^\mu) = 0 \end{cases} \qquad (95)$$

5.2 Basis Vectors and Phase Choices

A realistic model in the free fermionic formulation is generated by a basis of boundary condition vectors for all worldsheet fermions [2, 22, 23, 24, 25, 26, 27].

Notations

In section 4.2, it is shown that there are 20 left-moving and 44 right-moving real fermions. We also recall that two real fermions can be combined to form a complex fermion. With the benefit from hindsight we denote the 64 real fermions with different symbols and complexify certain combinations.

In our convention, the left- and right-moving fermions are distinguished by bars on top of the latter modes. The 2 real superpartners of the worldsheet coordinates are denoted by $\{\psi^\mu_{1,2}\}$. $\{y^{1,\cdots,6}\}$ and $\{w^{1,\cdots,6}\}$ reresent the 12 real fermions arising from the 6 compactified dimensions, and their superpartners are expressed as $\{\chi^{1,\cdots,6}\}$.

For the right-moving modes, we denote $\{\bar{y}^{1,\cdots,6}, \bar{w}^{1,\cdots,6}\}$ as the 12 real fermions correspond to $\{y^{1,\cdots,6}, w^{1,\cdots,6}\}$. The observable gauge group is represented by 5 complex fermions $\{\bar{\psi}^{1,\cdots,5}\}$. $\{\bar{\eta}^{1,2,3}\}$ are three complex fermions, and generate $U(1)$ currents. Finally, we write $\{\bar{\phi}^{1,\cdots,8}\}$, another 8 complex fermions correspond to the hidden gauge group.

The real fermions then pair up to form complex fermions or Ising operators. In our model the pairings are defined by:

1)	w^2w^4	$w^1\bar{w}^1$	$w^3\bar{w}^3$	$\bar{w}^2\bar{w}^4$
2)	y^1w^5	$y^2\bar{y}^2$	$w^6\bar{w}^6$	$\bar{y}^1\bar{w}^5$
3)	y^3y^6	$y^4\bar{y}^4$	$y^5\bar{y}^5$	$\bar{y}^3\bar{y}^6$

Note that the pairings are divided into 3 groups, each consists of 2 $U(1)$ currents and 2 Ising operators. The division is related to the 3 generations of particles we observe, which will become clear later.

Using the above notations we now define the 8 basis vectors used in the model. They are denoted by $B = \{1, S, b_1, b_2, b_3, \alpha, \beta, \gamma\}$. The explicit values of the vectors are given in appendix A. It can be shown that the basis vectors conform to the ABK rules as well as the symmetry of the worldsheet supercurrents.

Phase choices

¿From the ABK rules on phase choices, it can be seen that the phase choices are related to one another. In particular it can be shown that if we set up a matrix specifying the phases $C\binom{\alpha}{\beta}$, we can choose to assign values to half of the off-diagonal elements. The values of the remaining phases are then uniquely determined by the basis vectors. A table showing the general structure of phases with our choice of basis vectors, together with the particular choice of phases we use in the realistic model are shown in appendix B.

5.3 Model Building

Having chosen the basis vectors and phase choices, the resulting mass spectrum is determined. The basis vectors defines all sectors of the mass spectrum. The mass of a string in a given sector is related to its boundary conditions and also the number of oscillators that acts on the vacuum. After choosing a string mass, GSO projection is applied which serves to extract all the physical states in the Hilbert space defined in that sector.

Virasoro condition

As we are investigating the low energy limit of string theory only the massless states are studied in our model. The mass of a string is given by:

$$M_L^2 = -\frac{1}{2} + \frac{\alpha_L \cdot \alpha_L}{8} + N_L = -1 + \frac{\alpha_R \cdot \alpha_R}{8} + N_R = M_R^2. \tag{96}$$

The N_L and N_R are the sum of left-moving and right-moving oscillators acting on the vacuum, which take the values $N_L = \sum \nu_L$ and $N_R = \sum \nu_R$ respectively. The value attached to an oscillator is defined by:

$$\nu_f = \frac{1+\alpha(f)}{2} \qquad \nu_{f^*} = \frac{1-\alpha(f)}{2}. \tag{97}$$

Fermion number $+1$ is assigned to each fermion (f), and -1 is assigned to its complex conjugate (f^*). The fermion number of the vacua is given in page 22.

GSO projection

The GSO projection is defined by :

$$e^{i\pi b_j \cdot F_\xi}|s\rangle_\xi = \delta_\xi C\begin{pmatrix}\xi \\ b_j\end{pmatrix}^* |s\rangle_\xi, \tag{98}$$

where ξ belongs to the additive group Ξ, and b_j belongs to B. F_ξ represents the fermion numbers of the sector ξ. Here $|s\rangle_\xi$ is a state in the sector ξ. The dot product $b_j \cdot F_\xi$ is defined the same way as (94).

The GSO projection is to perform over all the basis vectors. The projection rule is a result of the modular invariance applied to the string partition function. Its purpose is to take care of cancellations that appear in the sum over all spin structures. Interested readers are referred to [11]. From the form of (98), it is clear that the states that survive all GSO projections, and hence the final mass spectrum, are uniquely determined by the choice of basis vectors and phases.

5.4 Results

All the necessary tools and the initial conditions are in place to construct a realistic string model. The model building is divided into 2 levels. At the first level the NAHE set [25] is discussed, which is a starting point of all realistic string models. The second level extend the NAHE set to include three more basis vectors $\{\alpha, \beta, \gamma\}$. They are reponsible for the breaking down of the $SO(10)$ group into the standard model gauge group.

Appendix D gives an example of how the extraction of physical states is performed.

The NAHE set

The NAHE set consists of $B = \{1, S, b_1, b_2, b_3\}$. The vector 1 is required by consistency. From the values of the boundary conditions of S, it is easy to see that it is the supersymmetry generator, therefore the superpartner of some sector α is given by $S + \alpha$. The NS (0) sector produces, on top of gravitons and dilatons [14], gauge bosons correspond to the group $SO(10) \times SO(6)^3 \times SO(16)$. The $SO(10)$ correspond to the observable gauge group $\{\bar{\psi}^{1\cdots5}\}$, the three $SO(6)$'s correspond to the groupings $\{\bar{\eta}^1, \bar{y}^{3,\cdots,6}\}$, $\{\bar{\eta}^2, \bar{y}^{1,2}\bar{w}^{5,6}\}$, $\{\bar{\eta}^3, \bar{w}^{1,\cdots,4}\}$, and the $SO(16)$ corresponds to the hidden symmetry. The sector $1 + b_1 + b_2 + b_3$ produces a spinorial 128 in the hidden gauge group and thus the hidden symmetry is enhanced to E_8 [28].

The sectors b_1, b_2, and b_3 each produces 16 spinorial 16 representations of $SO(10)$. The $\bar{\eta}_1, \bar{\eta}_2$ and $\bar{\eta}_3$ produces three $U(1)$ symmetrires. Corresponding to these are the three $U(1)$ supercurrents $\{\chi^{12}, \chi^{34}, \chi^{56}\}$. Also the $\{y^{3,\cdots,6}\}$, $\{y^{1,2}w^{5,6}\}$, $\{w^{1,\cdots,4}\}$ are the left-moving counterparts of $\{\bar{y}^{3,\cdots,6}\}$, $\{\bar{y}^{1,2}\bar{w}^{5,6}\}$, $\{\bar{w}^{1,\cdots,4}\}$, a reflection of the left-right symmetric conformal field theory of the heterotic string [22].

The additional vectors

The three additional basis vectors $\{\alpha, \beta, \gamma\}$ reduce the number of generations of particles to three, one from each sector b_1, b_2, b_3. The vector γ contains half integral boundary conditions, and are required to reduce $SO(2n)$ to $SU(n) \times U(1)$. The symmetry of the observable gauge group is broken down by α and β:

$$SO(10) \rightarrow SO(6) \times SO(4) \tag{99}$$

The vector γ breaks down the group futher into:

$$SO(6) \rightarrow SU(3) \times U(1), \qquad SO(4) \rightarrow SU(2) \times U(1) \tag{100}$$

thus producing the standard model gauge symmetry.

The hidden symmetry is reduced to $SO(8) \times SO(8)$ by α and β. It is further reduced by γ into $U(1)^3 \times SO(4) \times SU(3) \times U(1)_{\bar{\phi}^{567}}$. The three horizontal $SO(6)$ symmetries are broken into U(1) symmetries.

The NS sector produces the following generators:

$$\begin{aligned} SU(3)_C \times SU(2)_L \times U(1)_{B-L} \times U(1)_{t_3} \times \\ U(1)_{\bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3} \times U(1)_{\bar{y}^1\bar{w}^5, \bar{w}^2\bar{w}^4, \bar{y}^3\bar{y}^6} \times \\ U(1)_{\bar{\phi}^1, \bar{\phi}^2, \bar{\phi}^8} \times SO(4)_{\bar{\phi}^{34}} \times SU(3)_{\bar{\phi}^{567}} \times U(1)_{\bar{\phi}^{567}}. \end{aligned} \tag{101}$$

The first four terms are the observable gauge groups, the second six are the horizontal U(1) currents, while the remaining terms correspond to the hidden symmetry. The subscripts of the obersvable gauge group take their conventional meaning, where in particular $B - L$ is the difference between baryon and lepton number. This will be justified shortly. The subscripts of the remaining groups signify the free fermions responsible for the particular symmetries.

Group enhancement

Apart from the NS sector, the sector $1 + b_1 + b_2 + b_3$ also produces spacetime vector bosons. We note the isomorphism of $SO(4) \sim SU(2) \times SU(2)$. The vector bosons produces the following representations in $SU(3)_{\bar{\phi}^{567}} \times SU(2)_{\bar{\phi}^{34}}$:

$$(3, 2) \quad (\bar{3}, 2) \quad (1, 2) \quad (1, 2). \tag{102}$$

Therefore $1 + b_1 + b_2 + b_3$ gives rise to 6, $\bar{6}$, 2 and 2 representations in the hidden gauge group. Together with the spacetime bosons in the NS sector, which have representations 8 in the $SU(3)$,two 3's in the $SO(4) \sim SU(2) \times SU(2)$, and four 1's in the $U(1)$'s, we find that the hidden gauge group is enhanced to:

$$SU(5) \times SU(3) \times U(1)^2 \tag{103}$$

where the 24 of SU(5) arises from:

$$24 = 8 + 6 + \bar{6} + 3 + 1, \tag{104}$$

and the 8 of SU(3) arises from:

$$8 = 3 + 2 + 2 + 1 \tag{105}$$

according to [29]. The $U(1)$ in $SU(3)$ is given by:

$$U(1)_{h3} = U(1)_{\bar{\phi}^1} + U(1)_{\bar{\phi}^2} + U(1)_{\bar{\phi}^{567}} + U(1)_{\bar{\phi}^8}, \tag{106}$$

and the $U(1)$ in the $SU(5)$ is given by:

$$U(1)_{h5} = -3U(1)_{\bar{\phi}^1} + 3U(1)_{\bar{\phi}^2} + U(1)_{\bar{\phi}^{567}} - 3U(1)_{\bar{\phi}^8}. \tag{107}$$

The remaining two $U(1)$'s are given by the orthogonal combinations to $U(1)_{h3}$ and $U(1)_{h5}$. One possibility is:

$$U_7 = U(1)_{\bar{\phi}^1} - U(1)_{\bar{\phi}^8} \tag{108}$$

$$U_8 = U(1)_{\bar{\phi}^1} + 4U(1)_{\bar{\phi}^2} - 2U(1)_{\bar{\phi}^{567}} + U(1)_{\bar{\phi}^8}. \tag{109}$$

The fermion mass spectrum

In this model, the following sectors give rise to fermionic states:

$$b_1, \quad b_2, \quad b_3, \quad b_1 + 2\gamma, \quad b_2 + 2\gamma, \quad b_3 + 2\gamma, \quad 1 + \alpha + 2\gamma, \tag{110}$$

$$1 + b_1 + b_2 + 2\gamma, \quad 1 + b_2 + b_3 + 2\gamma, \quad 1 + b_1 + b_3 + 2\gamma. \tag{111}$$

The sectors that transform under the hidden gauge group are merged according to:

$$\xi \oplus \xi + (1 + b_1 + b_2 + b_3), \tag{112}$$

since the hidden symmetry is enhanced by $1+b_1+b_2+b_3$. Thus the following pairs of sectors are merged:

$$b_1 + 2\gamma \oplus 1 + b_2 + b_3 + 2\gamma, \tag{113}$$

$$b_2 + 2\gamma \oplus 1 + b_1 + b_3 + 2\gamma, \tag{114}$$

$$b_3 + 2\gamma \oplus 1 + b_1 + b_2 + 2\gamma. \tag{115}$$

All states of these three pairs transform as singlets under the observable gauge symmetry.

Of all the sectors, only b_1, b_2 and b_3 give rise to spinorial 16 in the observable gauge group. The sector γ contains 8 states. They transform as 3, $\bar{3}$, and two singlets under $SU(3)_C$.

All $U(1)$ currents carry charges. For convenience the charges are redefined in appendix E. If we further define:

$$Q_y = \frac{1}{2}\Big(Q_{B-L} + Q_{T_3 Y}\Big), \tag{116}$$

we find that Q_y's of the spinorial 16 of the observable gauge group matches exactly to those in the standard model.

The full fermion mass spectrum is given in appendix F.

Anomalous $U(1)$ symmetries

The model contains three anomalous $U(1)$ symmetries. This means that the sum of charges over all sectors are found to be non-zero for the $U(1)$ currents of $\bar{\eta}^1$, $\bar{\eta}^2$ and $\bar{\eta}^3$. In particular the values are found to be:

$$\sum_{sectors} Q(\bar{\eta}^1) = \sum_{sectors} Q(\bar{\eta}^2) = \sum_{sectors} Q(\bar{\eta}^3) = 24. \tag{117}$$

Two of the three anomalous $U(1)$'s may be rotated away, but not all three simultaneously.

The custodial $SU(2)_c$ symmetries

In this model, three spinorial 16 arise from the sectors b_1, b_2 and b_3. In a class of models, however, the observable gauge group is enhanced to $SU(3)_C \times SU(2)_L \times SU(2)_c$. This is due to the sector $1 + S + \alpha + 2\gamma$. It produces two additional spacetime vector bosons, which are singlets of the nonabelian group but carry $U(1)$ charges. It was shown in [22] that only the lepton supermultiplets, $\{L, e^c_L, N^c_L\}$ transform as doublets under the custodial $SU(2)_c$, while the quarks are singlets. In our model these two vector bosons are projected out. The remaining states in this sector are scalars.

The GSO coefficient for $\xi = 1 + S + \alpha + 2\gamma$ is given by:

$$\delta_\xi C\begin{pmatrix}\xi \\ j\end{pmatrix}^* = e^{-i\xi\cdot j\frac{\pi}{2}}(\delta_j)^4 C\begin{pmatrix}j \\ 1\end{pmatrix} C\begin{pmatrix}j \\ S\end{pmatrix} C\begin{pmatrix}j \\ \alpha\end{pmatrix} C\begin{pmatrix}j \\ \gamma\end{pmatrix}^2, \tag{118}$$

where j represents all the basis vectors. A general method to calculate GSO coefficients is given in appendix C.

In the current model, the GSO coefficient in this sector for $j = \beta$ is $+1$. I show that if the initial phase choice is altered in such a way to change this particular coefficient from $+1$ to -1, the sector produces two spacetime vector bosons in addition to six scalars. These bosons carry charges of the currents $U(1)_{\bar{y}^1\bar{w}^5}$, $U(1)_{\bar{w}^2\bar{w}^4}$, $U(1)_{\bar{y}^3\bar{y}^6}$, $U(1)_{B-L}$, $U(1)_{\bar{\phi}^1}$ and $U(1)_{\bar{\phi}^8}$. The spinorial 16 of b_1, b_2 and b_3 are not affected by this particular phase change. The $U(1)$ symmetry in the observable gauge group is enhanced to $SU(2)_c$ as a result of these additional spacetime vector bosons.

6 Conclusion

In this thesis a realistic string model based on the free fermionic formulation was investigated. It was free of the anomalous custodial $SU(2)_c$ symmetry in which only the leptons transform. We trace the origin of this extra symmetry as group enhancement of the $U(1)_y$ in the observable gauge group.

In the class of model that contains this $SU(2)_c$ symmetry, the NS sector gives rise to gauge bosons of the $SU(3)_C \times SU(2)_L \times U(1)_y$ group, while the sector $1 + S + \alpha + 2\gamma$ produces two additional spacetime vector bosons which are singlets but carry $U(1)$ charges. These bosons enhance the $U(1)$ symmetries to $SU(2)_c$. In our model, these additional vector bosons are projected out by GSO projection, thus leaving the standard model gauge group intact.

It was shown that these extra spacetime bosons may be recovered by a suitable change of phase in the sector $1 + S + \alpha + 2\gamma$. While this is not a direction one wish to proceed, there is currently no mechanism to forbid this. In other words, different models are equivalent in the sense that there is no way to select particular models over the others, apart from appealing to phenomenology.

In this study, a way was found to construct a realisitc string model that possesses the group structure of the standard model. It could serve as a guide to future research in devising a mechanism which selects particular models. It would be a triumph of string theory if the standard model gauge symmetry comes out inevitably in the low energy limit.

In this thesis the stated results were verified as much as possible. It was tried to be clear whenever results were adopted from other sources. All the results in chapter 2 and the section on bosonic strings in chapter 3 were verified. For the latter, verifications follow the line of [10]. The ideas in chapter 4 follows [11] closely. All the results related to the realistic string model presented were verified by the author.

A Basis Vector

The basis vectors are stated below. The first five vectors $\{1, S, b_1, b_2, b_3\}$ constitute the NAHE set and is the basis of the realistic string model. The three additional basis vectors are responsible for symmetry breaking and reduction to 3 generations of spinorial 16 in the observable gauge group. The notations of the fermions are described in section 5.2.

	ψ^μ	χ^{12},	χ^{34},	χ^{56}	$\bar{\psi}^1$,	$\bar{\psi}^2$,	$\bar{\psi}^3$,	$\bar{\psi}^4$,	$\bar{\psi}^5$,	$\bar{\eta}^1$,	$\bar{\eta}^2$,	$\bar{\eta}^3$
1	1	1,	1,	1	1,	1,	1,	1,	1,	1,	1,	1
S	1	1,	1,	1	0,	0,	0,	0,	0,	0,	0,	0
b_1	1	1,	0,	0	1,	1,	1,	1,	1,	1,	0,	0
b_2	1	0,	1,	0	1,	1,	1,	1,	1,	0,	1,	0
b_3	1	0,	0,	1	1,	1,	1,	1,	1,	0,	0,	1
α	0	0,	0,	0	1,	1,	1,	0,	0,	0,	0,	0
β	0	0,	0,	0	1,	1,	1,	0,	0,	0,	0,	0
γ	0	0,	0,	0	$\frac{1}{2}$,	$\frac{1}{2}$,	$\frac{1}{2}$,	$\frac{1}{2}$,	$\frac{1}{2}$,	$\frac{1}{2}$,	$\frac{1}{2}$,	$\frac{1}{2}$

	$\bar{\phi}^1$,	$\bar{\phi}^2$,	$\bar{\phi}^3$,	$\bar{\phi}^4$,	$\bar{\phi}^5$,	$\bar{\phi}^6$,	$\bar{\phi}^7$,	$\bar{\phi}^8$
1	1,	1,	1,	1,	1,	1,	1,	1
S	0,	0,	0,	0,	0,	0,	0,	0
b_1	0,	0,	0,	0,	0,	0,	0,	0
b_2	0,	0,	0,	0,	0,	0,	0,	0
b_3	0,	0,	0,	0,	0,	0,	0,	0
α	1,	1,	1,	1,	0,	0,	0,	0
β	1,	1,	1,	1,	0,	0,	0,	0
γ	$\frac{1}{2}$,	0,	1,	1,	$\frac{1}{2}$,	$\frac{1}{2}$,	$\frac{1}{2}$,	0

	$y^3y^6, y^4\bar{y}^4, y^5\bar{y}^5, \bar{y}^3\bar{y}^6$	$y^1w^5, y^2\bar{y}^2, w^6\bar{w}^6, \bar{y}^1\bar{w}^5$	$w^2w^4, w^1\bar{w}^1, w^3\bar{w}^3, \bar{w}^2\bar{w}^4$
1	1, 1, 1, 1	1, 1, 1, 1	1, 1, 1, 1
S	0, 0, 0, 0	0, 0, 0, 0	0, 0, 0, 0
b_1	1, 1, 1, 1	0, 0, 0, 0	0, 0, 0, 0
b_2	0, 0, 0, 0	1, 1, 1, 1	0, 0, 0, 0
b_1	0, 0, 0, 0	0, 0, 0, 0	1, 1, 1, 1
α	1, 1, 1, 0	1, 1, 1, 0	1, 1, 1, 0
β	0, 1, 0, 1	0, 1, 0, 1	0, 1, 0, 0
γ	0, 0, 1, 1	1, 0, 0, 0	0, 1, 0, 1

B Phase Tables

The following tables show the values of phase $C\binom{a}{b}$.

General Structure

	$b=1$	S	b_1	b_2	b_3	α	β	γ
$a=1$	±1	±1	±1	±1	±1	±1	±1	$\pm i$
S	$C\binom{1}{S}$	$C\binom{1}{S}$	±1	±1	±1	±1	±1	±1
b_1	$C\binom{1}{b_1}$	$-C\binom{S}{b_1}$	$C\binom{1}{b_1}$	±1	±1	±1	±1	±1
b_2	$C\binom{1}{b_2}$	$-C\binom{S}{b_2}$	$C\binom{b_1}{b_2}$	$C\binom{1}{b_2}$	±1	±1	±1	±1
b_3	$C\binom{1}{b_3}$	$-C\binom{S}{b_3}$	$C\binom{b_1}{b_3}$	$C\binom{b_2}{b_3}$	$C\binom{1}{b_3}$	±1	±1	±1
α	$C\binom{1}{\alpha}$	$C\binom{S}{\alpha}$	$-C\binom{b_1}{\alpha}$	$-C\binom{b_2}{\alpha}$	$-C\binom{b_3}{\alpha}$	$C\binom{1}{\alpha}$	±1	$\pm i$
β	$C\binom{1}{\beta}$	$C\binom{S}{\beta}$	$C\binom{b_1}{\beta}$	$C\binom{b_2}{\beta}$	$-C\binom{b_3}{\beta}$	$-C\binom{\alpha}{\beta}$	$-C\binom{1}{\beta}$	$\pm i$
γ	$-iC\binom{1}{\gamma}^*$	$C\binom{S}{\gamma}^*$	$C\binom{b_1}{\gamma}^*$	$-C\binom{b_2}{\gamma}^*$	$C\binom{b_3}{\gamma}^*$	$iC\binom{\alpha}{\gamma}^*$	$-iC\binom{\beta}{\gamma}^*$	$-C\binom{1}{\gamma}^*$

The Phase of the $SU(3)_C \times SU(2)_L \times U(1)_y$ String Model

	$b=1$	S	b_1	b_2	b_3	α	β	γ
$a=1$	1	1	1	-1	-1	1	1	$-i$
S	1	1	1	1	1	-1	-1	-1
b_1	1	-1	1	-1	-1	-1	-1	-1
b_2	-1	-1	-1	-1	-1	-1	-1	-1
b_3	-1	-1	-1	-1	-1	-1	-1	-1
α	1	-1	1	1	1	1	1	$-i$
β	1	-1	-1	-1	1	-1	-1	i
γ	1	-1	-1	1	-1	-1	-1	$-i$

C Calculation of Phase

In general, a vector $\zeta \in \Xi$ may be expressed as $\zeta = \sum_j m_j b_j$, where $b_j \in B$. Similarly we may also have $\varrho = \sum_i m_i b_i$, where m_j, n_i are all integers. The phase of spin structure $Z\begin{bmatrix}\zeta\\ \varrho\end{bmatrix}$ can be determined using the ABK rules:

$$C\begin{pmatrix}\zeta\\ \varrho\end{pmatrix} = C\begin{pmatrix}\zeta\\ \sum_i n_i b_i\end{pmatrix} = \delta_\zeta^{(\sum_i n_i)-1}\prod_i^n C\begin{pmatrix}\zeta\\ b_i\end{pmatrix}^{n_i}.$$

It is possible to express $C\binom{\zeta}{b_i}$ as:

$$C\begin{pmatrix}\zeta\\ b_i\end{pmatrix} = e^{i\pi\zeta\cdot b_i/2} C\begin{pmatrix}b_i\\ \zeta\end{pmatrix}^*$$

$$= e^{i\pi\zeta\cdot b_i/2}\delta_{b_i}^{(\sum_j m_j)-1}\prod_j^m C\binom{b_i}{b_j}^{*m_j}. \quad (119)$$

Hence we obtain:

$$C\binom{\zeta}{\varrho} = C\binom{\sum_j m_j b_j}{\sum_i n_i b_i}$$
$$= \delta_\zeta^{(\sum_i n_i)-1}\prod_i^n e^{in_i\pi\zeta\cdot b_i/2}\delta_{b_i}^{n_i[(\sum_j m_j)-1]}\prod_j^m C\binom{b_i}{b_j}^{*m_j n_i}. \quad (120)$$

D The States of the b_1 Sector

In this appendix, the GSO projection of the sector b_1 is shown. This should serve as an example of how the states in different sectors are obtained in general.

The GSO coefficients, defined in (98), in this sector are calculated to be:

b_j	1	S	b_1	b_2	b_3	α	β	γ
$\delta_{b_1}C\binom{b_1}{b_j}^*$	-1	1	-1	1	1	1	1	1

In this sector, $\alpha_L\cdot\alpha_L = 4$ and $\alpha_R\cdot\alpha_R = 8$, so from (96) there are no oscillators. The 12 periodic complex fermions need to be specified by either $|+\rangle$ or $|-\rangle$, as they live in the R vacua. The following representation is used:

$\binom{m}{n}$	denotes n among the m complex fermions are in states $\vert-\rangle$, while the rest are in states $\vert+\rangle$.

Recall (98), the coefficient on the two sides of the equation can be matched by choosing suitable combinations of $|-\rangle$, since the fermion number associated with this state is -1, from (93).

The 12 complex fermions in the R vacua are:

$$\psi^{12},\ \xi^{12},\ y^{36},\ y^4\bar{y}^4,\ y^5\bar{y}^5,\ \bar{y}^{36},\ \bar{\psi}^{1\cdots5},\ \bar{\eta}^1.$$

In the presentation below the above order of the 12 fermions will be followed.

b_j	Possible Combinations
1	$\binom{12}{\text{odd}}$
S	$\binom{2}{0}\binom{10}{\text{odd}}$
b_1	$\binom{2}{0}\binom{10}{\text{odd}}$
b_2	$\binom{2}{0}\binom{4}{\text{odd}}\binom{5}{\text{even}}\binom{1}{0}\oplus\binom{2}{0}\binom{4}{\text{even}}\binom{5}{\text{even}}\binom{1}{1}$
b_3	$\binom{2}{0}\binom{4}{\text{odd}}\binom{5}{\text{even}}\binom{1}{0}\oplus\binom{2}{0}\binom{4}{\text{even}}\binom{5}{\text{even}}\binom{1}{1}$

$$
\begin{array}{lll}
\alpha & & \binom{2}{0}\left[\binom{3}{1}\binom{1}{0}+\binom{3}{3}\binom{1}{0}\right]\left[\binom{3}{1}\binom{2}{1}+\binom{3}{3}\binom{2}{1}\right]\binom{1}{0} \\
 & \oplus & \binom{2}{0}\left[\binom{3}{1}\binom{1}{1}+\binom{3}{3}\binom{1}{1}\right]\left[\binom{3}{1}\binom{2}{1}+\binom{3}{3}\binom{2}{1}\right]\binom{1}{1} \\
 & \oplus & \binom{2}{0}\left[\binom{3}{0}\binom{1}{0}+\binom{3}{2}\binom{1}{0}\right]\left[\binom{3}{0}\binom{2}{0}+\binom{3}{0}\binom{2}{2}+\binom{3}{2}\binom{2}{0}+\binom{3}{2}\binom{2}{2}\right]\binom{1}{1} \\
 & \oplus & \binom{2}{0}\left[\binom{3}{0}\binom{1}{1}+\binom{3}{2}\binom{1}{1}\right]\left[\binom{3}{0}\binom{2}{0}+\binom{3}{0}\binom{2}{2}+\binom{3}{2}\binom{2}{0}+\binom{3}{2}\binom{2}{2}\right]\binom{1}{0} \\
\beta & & \binom{2}{0}\left[\binom{1}{0}\binom{1}{1}\binom{1}{0}\binom{1}{0}+\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{0}\right]\left[\binom{3}{1}\binom{2}{1}+\binom{3}{3}\binom{2}{1}\right]\binom{1}{0} \\
 & \oplus & \binom{2}{0}\left[\binom{1}{1}\binom{1}{0}\binom{1}{0}\binom{1}{1}+\binom{1}{0}\binom{1}{0}\binom{1}{1}\binom{1}{1}\right]\left[\binom{3}{1}\binom{2}{1}+\binom{3}{3}\binom{2}{1}\right]\binom{1}{1} \\
 & \oplus & \binom{2}{0}\left[\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}+\binom{1}{1}\binom{1}{0}\binom{1}{1}\binom{1}{0}\right]\left[\binom{3}{0}\binom{2}{0}+\binom{3}{0}\binom{2}{2}+\binom{3}{2}\binom{2}{0}+\binom{3}{2}\binom{2}{2}\right]\binom{1}{1} \\
 & \oplus & \binom{2}{0}\left[\binom{1}{0}\binom{1}{1}\binom{1}{1}\binom{1}{1}+\binom{1}{1}\binom{1}{1}\binom{1}{0}\binom{1}{1}\right]\left[\binom{3}{0}\binom{2}{0}+\binom{3}{0}\binom{2}{2}+\binom{3}{2}\binom{2}{0}+\binom{3}{2}\binom{2}{2}\right]\binom{1}{0} \\
\gamma & & \binom{2}{0}\left[\binom{1}{0}\binom{1}{1}\binom{1}{0}\binom{1}{0}\right]\left[\binom{3}{3}\binom{2}{1}\right]\binom{1}{0}\oplus\binom{2}{0}\left[\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{0}\right]\left[\binom{3}{1}\binom{2}{1}\right]\binom{1}{0} \\
 & \oplus & \binom{2}{0}\left[\binom{1}{0}\binom{1}{1}\binom{1}{1}\binom{1}{1}\right]\left[\binom{3}{0}\binom{2}{0}+\binom{3}{2}\binom{2}{2}\right]\binom{1}{0} \\
 & \oplus & \binom{2}{0}\left[\binom{1}{1}\binom{1}{1}\binom{1}{0}\binom{1}{1}\right]\left[\binom{3}{0}\binom{2}{2}+\binom{3}{2}\binom{2}{0}\right]\binom{1}{0}
\end{array}
$$

By the end of the projection, 16 states remain. These states are naturally grouped into six batches, correspond to one generation of particles in the standard model. For clarity, the CTP partners of the states are not shown.

E Definition of Charges

The charges of the fermion mass specturm are defined as follows. Certain charges are re-scaled to match the $U(1)$ charges appear in the standard model.

$$
\begin{aligned}
Q_{B-L} &= \frac{2}{3}\sum_{i=1}^{3} Q(\bar{\psi}^i). \\
Q_{T_3Y} &= \sum_{i=4}^{5} Q(\bar{\psi}^i). \\
Q_i &= Q(\bar{\eta}^i), \qquad \text{where } i = 1, 2, 3. \\
Q_4 &= Q(\bar{y}^3\bar{y}^6). \\
Q_5 &= Q(\bar{y}^1\bar{w}^5). \\
Q_6 &= Q(\bar{w}^2\bar{w}^4). \\
Q_{h3} &= Q(\bar{\phi}^1) + Q(\bar{\phi}^2) + Q(\bar{\phi}^{567}) + Q(\bar{\phi}^8). \\
Q_{h5} &= -3Q(\bar{\phi}^1) + 3Q(\bar{\phi}^2) + Q(\bar{\phi}^{567}) - 3Q(\bar{\phi}^8). \\
Q_7 &= Q(\bar{\phi}^1) - Q(\bar{\phi}^8). \\
Q_8 &= Q(\bar{\phi}^1) + 4Q(\bar{\phi}^2) - 2Q(\bar{\phi}^{567}) + Q(\bar{\phi}^8),
\end{aligned}
$$

where $Q(\bar{\phi}^{567}) = \sum_{i=5}^{7} Q(\bar{\phi}^i)$.

F The Fermion Mass Spectrum

The 3 generations of spinorial 16 derived from $SO(10)$.

F	SEC	$SU(3)_C \times SU(2)_L$	Q_y	Q_C	Q_L	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
Q_1	b_1	$(3,2)$	$\frac{1}{6}$	$\frac{1}{3}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
$\bar{u}_1$		$(\bar{3},1)$	$-\frac{2}{3}$	$-\frac{1}{3}$	-1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
$\bar{d}_1$		$(\bar{3},1)$	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
L_1		$(1,2)$	$-\frac{1}{2}$	-1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
e_1		$(1,1)$	1	1	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
n_1		$(1,1)$	0	1	-1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
Q_2	b_2	$(3,2)$	$\frac{1}{6}$	$\frac{1}{3}$	0	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0
$\bar{u}_2$		$(\bar{3},1)$	$-\frac{2}{3}$	$-\frac{1}{3}$	-1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0
$\bar{d}_2$		$(\bar{3},1)$	$\frac{1}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0
L_2		$(1,2)$	$-\frac{1}{2}$	-1	0	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0
e_2		$(1,1)$	1	1	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0
n_2		$(1,1)$	0	1	-1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0
Q_3	b_3	$(3,2)$	$\frac{1}{6}$	$\frac{1}{3}$	0	0	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
$\bar{u}_3$		$(\bar{3},1)$	$-\frac{2}{3}$	$-\frac{1}{3}$	-1	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
$\bar{d}_3$		$(\bar{3},1)$	$\frac{1}{3}$	$-\frac{1}{3}$	1	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
L_3		$(1,2)$	$-\frac{1}{2}$	-1	0	0	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
e_3		$(1,1)$	1	1	1	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
n_3		$(1,1)$	0	1	-1	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$

F	SEC	$SU(5) \times SU(3)$	Q_7	Q_8
Q_1	b_1	$(1,1)$	0	0
$\bar{u}_1$		$(1,1)$	0	0
$\bar{d}_1$		$(1,1)$	0	0
L_1		$(1,1)$	0	0
e_1		$(1,1)$	0	0
n_1		$(1,1)$	0	0
Q_2	b_2	$(1,1)$	0	0
$\bar{u}_2$		$(1,1)$	0	0
$\bar{d}_2$		$(1,1)$	0	0
L_2		$(1,1)$	0	0
e_2		$(1,1)$	0	0
n_2		$(1,1)$	0	0
Q_3	b_3	$(1,1)$	0	0
$\bar{u}_3$		$(1,1)$	0	0
$\bar{d}_3$		$(1,1)$	0	0
L_3		$(1,1)$	0	0
e_3		$(1,1)$	0	0
n_3		$(1,1)$	0	0

The additional fermionic states.

F	SEC	$SU(3)_C \times SU(2)_L$	Q_y	Q_C	Q_L	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
V_1	$b_1 + 2\gamma\ \oplus$	$(1,1)$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\bar{V}_1$	$1 + b_2 + b_3 + 2\gamma$	$(1,1)$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
T_1		$(1,1)$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\bar{T}_1$		$(1,1)$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
V_2	$b_2 + 2\gamma\ \oplus$	$(1,1)$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$\bar{V}_2$	$1 + b_1 + b_3 + 2\gamma$	$(1,1)$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
T_2		$(1,1)$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$\bar{T}_2$		$(1,1)$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
V_3	$b_3 + 2\gamma\ \oplus$	$(1,1)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
$\bar{V}_3$	$1 + b_1 + b_2 + 2\gamma$	$(1,1)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$
T_3		$(1,1)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
$\bar{T}_3$		$(1,1)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$
K	$1 + \alpha + 2\gamma$	$(1,1)$	$\frac{1}{2}$	1	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
K'		$(1,1)$	$-\frac{1}{2}$	-1	0	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
M		$(3,1)$	$\frac{1}{6}$	$\frac{1}{3}$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{M}$		$(\bar{3},1)$	$-\frac{1}{6}$	$-\frac{1}{3}$	0	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

F	SEC	$SU(5) \times SU(3)$	Q_7	Q_8
V_1	$b_1 + 2\gamma\ \oplus$	$(1,3)$	$-\frac{1}{2}$	$\frac{5}{2}$
$\bar{V}_1$	$1 + b_2 + b_3 + 2\gamma$	$(1,\bar{3})$	$\frac{1}{2}$	$-\frac{5}{2}$
T_1		$(5,1)$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\bar{T}_1$		$(\bar{5},1)$	$\frac{1}{2}$	$\frac{3}{2}$
V_2	$b_2 + 2\gamma\ \oplus$	$(1,3)$	$-\frac{1}{2}$	$\frac{5}{2}$
$\bar{V}_2$	$1 + b_1 + b_3 + 2\gamma$	$(1,\bar{3})$	$\frac{1}{2}$	$-\frac{5}{2}$
T_2		$(5,1)$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\bar{T}_2$		$(\bar{5},1)$	$\frac{1}{2}$	$\frac{3}{2}$
V_3	$b_3 + 2\gamma\ \oplus$	$(1,3)$	$-\frac{1}{2}$	$\frac{5}{2}$
$\bar{V}_3$	$1 + b_1 + b_2 + 2\gamma$	$(1,\bar{3})$	$\frac{1}{2}$	$-\frac{5}{2}$
T_3		$(5,1)$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\bar{T}_3$		$(\bar{5},1)$	$\frac{1}{2}$	$\frac{3}{2}$
K	$1 + \alpha + 2\gamma$	$(1,1)$	1	0
K'		$(1,1)$	-1	0
M		$(1,1)$	1	0
$\bar{M}$		$(1,1)$	-1	0

In the tables above, Q_C and Q_L are equivalent to Q_{B-L} and Q_{T_3Y}, defined in appendix E, respectively.

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