MATH423 - Introduction to String Theory
Set Work: Sheet 5

1. Let A_μ be the electromagnetic vector potential. The electromagnetic field strength tensor is then defined as

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

(a) Show that Maxwell's equations (in the absence of sources) in four-vector notation can be derived from the electromagnetic Lagrangian given by

$$L_{\text{e.m.}} \;=\; -\frac{1}{4}\, F_{\alpha\beta}F^{\alpha\beta} \,.$$

(b) Show that $L_{\text{e.m.}}$ is invariant under the transformation $A_\mu \rightarrow A_\mu - \partial_\mu \Lambda$, where Λ is a scalar function.

(c) Show that by imposing a local $U(1)$ symmetry a mass term for the photon is forbidden.

2. Consider the action

$$S[x,e] = \frac{1}{2}\int e d\tau \left(\frac{1}{e^2}\left(\frac{dx^\mu}{d\tau}\right)^2 - m^2\right) \;, \tag{1}$$

where τ is an arbitrary parameter and $ed\tau$ is an invariant line element.

(a) Show that the action is invariant under reparameterisation of the world–line and under Poincare transformations of Minkowski space.

(b) Perform variations of x^μ and of e to obtain the equations of motions for x^μ and e respectively.

(c) For $m^2 > 0$, eliminate e by its equation of motion, and substitute the result back into (1). Show that you obtain the action for a free massive particle.

(d) Instead of eliminating e by its equation of motion, you can set it to a constant value by reparameterisation. Derive the equation of motion and the contraint equation for the two cases $m^2 > 0$ and $m^2 = 0$.

3. The action for the relativistic string is given by (with $c = 1$)

$$S_{\text{NG}}[X] = \int d^2\sigma \mathcal{L} = -T \int_\Sigma d^2\sigma \sqrt{|det(\partial_\alpha X^\mu \partial_\beta X_\mu)|} \;. \tag{2}$$

(a) Show that the action is invariant under reparametrisations of the world–sheet Σ:

$$\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha(\sigma^0, \sigma^1), \quad \text{where } \det\left(\frac{\partial\tilde{\sigma}^\alpha}{\partial\sigma^\beta}\right) > 0\ . \tag{3}$$

(b) Compute the momentum densities

$$P^0_\mu = \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\ , P^1_\mu = \frac{\partial\mathcal{L}}{\partial X'^\mu}\ . \tag{4}$$

(c) Show that the canonical momenta $\Pi^\mu = P^\mu_0$ are subject to the two constraints

$$\begin{aligned} \Pi^\mu X'_\mu &= 0 \\ \Pi^2 + T^2(X')^2 &= 0 \end{aligned}$$

and that the Hamiltonian vanishes

$$\mathcal{H} = \dot{X}\Pi - \mathcal{L} = 0 \tag{5}$$

(d) Compute the nonrelativistic limit of the Nambu–Goto action. Use the static gauge, which fixes the longitudinal directions $X^0 = \tau,\ X^1 = \sigma$, while leaving the transverse directions X^i free. Show that the kinetic energy contains only the transverse velocity.

4. Consider the transformations $q(t) \rightarrow q(t) + \delta q(t)$ that leave the Lagrangian invariant up to a a total time derivative, *i.e.*

$$L + \delta L = L(q(t) + \delta q(t), \dot{q}(t) + \delta\dot{q}(t), t) = L(q(t), \dot{q}(t), t) + \frac{d}{dt}(\epsilon\Lambda), \tag{6}$$

where ϵ is an infinitesimal constant and Λ is a calculable function of the coordinates velocities and possibly of time.

(a) Show that is a associated conversed charge that takes the form

$$\epsilon Q = \frac{\partial L}{\partial\dot{q}}\delta q - \epsilon\Lambda. \tag{7}$$

(b) Cconsider a Lagrangian $L(q(t), \dot{q}(t))$ that has no explicit time dependence and the transformation

$$q(t) \rightarrow q(t+\epsilon) \simeq q(t) + \epsilon\dot{q}(t) \tag{8}$$

that represents a constant inifinitesimal time translation. Show that the transformantion 8 leaves the Lagrangian invariant up to an added term which is total time derivative. Calculate Λ and the conserved charge Q. Give an interpretation of the result.