MATH423 - Introduction to String Theory
Set Work: Sheet 4

1. If the path of a point particle is parameterised by proper time, the equation of motion of a free particle is

$$\frac{d^2x}{ds^2} = 0. \tag{1}$$

Consider a new parameter $\tau = f(s)$. Find the most general function f for which (1) implies

$$\frac{d^2x}{d\tau^2} = 0. \tag{2}$$

2. Consider the point particle action given by

$$S = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\,d\tau \ .$$

Vary the action to find a manifestly reparameterization invariant form of the free particle equation of motion.

3. The relativistic version of Newton's second law is

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}}\right) = \vec{F}.$$

Show that this can be written in manifestly covariant form

$$\frac{dp^\mu}{ds} = f^\mu$$

and find the relation between $\vec{F}$ and the relativistic force vector f^μ.

4. The action for a non–relativistic particle of mass m and charge q coupled to an electromagentic field is obtained by replacing the first term in

$$S = -mc\int_{\mathcal{P}} ds + \frac{q}{c}\int_{\mathcal{P}} A_\mu(x)dx^\mu \ .$$

by the non–relativistic action for a free point particle:

$$S = \int \frac{1}{2}mv^2 dt + \frac{q}{c}\int A_\mu(x)\frac{dx^\mu}{dt}dt \ .$$

Time t is used to parameterize the second integral.

(a) Rewrite the action S in terms of the potentials $(\Phi \vec{A})$ and the ordinary velocity $\vec{v}$. What is the Lagrangian?

(b) Calculate the canonical momentum $\vec{p}$ conjugate to the position of the particle and show that it is given by

$$\vec{p} = m\vec{v} + \frac{q}{c}\vec{A} \ .$$

(c) Construct the Hamiltonian for the charged particle and show that it is given by

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\Phi \ .$$

5. The action for a relativistic particle of mass m and charge q coupled to an electromagentic field is given by

$$S = -mc\int_{\mathcal{P}} ds + \frac{q}{c}\int_{\mathcal{P}} d\tau A_\mu(x(\tau))\frac{dx^\mu}{d\tau}(\tau) \ .$$

Derive the equation of motion from this action by taking the variation $x^\mu(\tau) \rightarrow x^\mu(\tau) + \delta x^\mu(\tau)$.

6. Consider the invariant interval $ds^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu$ in a curved space with metric $g_{\mu\nu}(x)$. The motion of a point particle of mass m on curved space is studied using the action

$$S \ = \ -mc\int ds.$$

Show that the equation of motion obtained by variation of the world–line is

$$\frac{d}{ds}\left[g_{\mu\rho}\frac{dx^\mu}{ds}\right] = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}.$$

Show that this is equivalent to the familiar form of the geodesic equation

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0.$$