MATH423 - Introduction to String Theory

Set Work: Sheet 3

1. (a)Show that for time independent fields, the Maxwell equation $T_{0ij} = 0$ implies that $\partial_i E_j - \partial_j E_i = 0$. Show that this condition is satisfied by the ansatz $\vec{E} = -\vec{\nabla}\Phi$.

(b) Show that in d spatial dimensions, with $d > 2$, the potential due to a point charge q is given by

$$\Phi(r) = \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}} \frac{q}{r^{d-2}}.$$

2. (a) The Standard Bohr radius is $a_0 = \frac{\hbar^2}{me^2} \approx 5.29 \times 10^{-9}$cm, and arises from the electric potential $V = -\frac{e^2}{r}$. What would be the gravitational Bohr radius if the attraction force binding the electron to the proton was gravitational?

(b) In units where G, c and $\hbar$ are set equal to one, the temperature of a black hole is given by $kT = \frac{1}{8\pi M}$. Insert back the factors of G, c and $\hbar$ into this formula. Evaluate the temperature of a black hole of a million solar masses. What is the mass of a black hole whose temperature is room temperature.

3. A string with tension T_0 is stretched from $x = 0$ to $x = 2a$. The part of the string $x \in (0, a)$ has constant mass density μ_1 and the part of the string $x \in (a, 2a)$ has constant mass density μ_2. Consider the differential equation

$$\frac{d^2 y}{dx^2} + \frac{\mu(x)}{T_0}\omega^2 y(x) = 0.$$

that determines the normal oscillations

(a) What boundary conditions should be imposed on $y(x)$ and $\frac{dy}{dx}(x)$ at $x = a$?

(b) Write the conditions that determine the possible frequencies of oscillation.

(c) Calculate the lowest frequency of oscillation of this string when $\mu_1 = \mu_0$ and $\mu_2 = 2\mu_0$.

4. (a) The Planck mass $m_{\rm Planck}$ is defined as a function of Newton Gravitational constant G, the constant speed of light c, and the Planck constant $\hbar$, as

$$m_P = (G)^\alpha (c)^\beta (\hbar)^\gamma \ .$$

Using dimensional analysis determine the constants α, β and γ.

Find the numerical value of the Planck mass in grams.

(b) Similarly, find the numerical value of the Planck time in seconds.

(c) The vacuum energy associated with the current acceleration of the universe is $\rho_{\rm vac} = 7.7 \times 10^{-27}{\rm kg/m}^3$. Derive a fundamental length scale, $\ell_{\rm vac}$ associated with this vacuum energy in terms of $\rho_{\rm vac}$, the Planck constant $\hbar$, and the speed of light c. Express the numerical value for ℓ_{vac} in μm where $1\mu{\rm m} = 10^{-6}$m.

5.

(a) Consider an action for a particle moving in three space dimensions with dynamical variables $q_i(t)$ and velocities $\dot{q}_i(t)$:

$$S = \int dt L(q_i(t), \dot{q}_i(t), t).$$

Calculate the variation δS of the action under the variation $\delta q_i(t)$ of the coordinates and derive the Euler–Lagrange equations of motion.

(b) Consider the Lagrangian $L(q(t), \dot{q}(t), t)$ and variation of

$$q(t) \quad \to \quad q(t) \ + \ \delta(q(t)) = q(t) \ + \ \epsilon h(q(t), t)$$

where ϵ is an infinitesimal constant. Show that if the Lagrangian is invariant under the variation then the charge defined as

$$\epsilon Q \equiv \frac{\partial L}{\partial \dot{q}} \delta q \qquad \text{is conserved.}$$