MATH423 - Introduction to String Theory

Set Work: Sheet 2

1. Consider the plane (x, y) with the identification

$$(x, y) \sim (x + 2\pi R, y + 2\pi R)$$

What is the resulting space.

2. (a) Consider the circle S^1. Presented as the real line with the identifications $x \sim x + 2$. Choose $-1 < x \leq 1$ as the fundamental domain. The circle is the space $-1 < x \leq 1$ with the points $x \pm 1$ identified. The orbifold S^1/Z_2 is defined by imposing the Z_2 identification $x \sim -x$. Describe the action of this identification on the circle. Show that there are two points on the circle that are left fixed by the Z_2 action find a fundamental domain for the two identifications. Describe the orbifold S^1/Z_2 in simple terms.

(b) Consider a torus T^2, presented as the (x, y) plane with the identifications $x \sim x + 2$ and $y \sim y + 2$. Choose $-1 < x, y \leq -1$ as the fundamental domain. The orbifold T^2/Z_2 is defined by imposing the Z_2 identification $(x, y) \sim (-x, -y)$. Prove that there are four points on the torus that are left fixed by the Z_2 transformation. Show that the orbifold $T^2/Z2$ is a two dimensional sphere, naturally presented as a square pillowcase with seamed edges.

3. The Lorentz force equation can be written relativistically as

$$\frac{dp^\mu}{ds} = \frac{q}{c} F^{\mu\nu} \frac{dx_\nu}{ds},$$

where p_μ is the four momentum. Check explicitly that that equation reproduce the Lorentz force law equation,

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right),$$

when μ is a spatial index. What does the relativistic equation give when $\mu = 0$

4. (a) Show explicitly that the source free Maxwell equations emerge from $T_{\mu\lambda\nu} = 0$.

(b) Show explicitly that the Maxwell equations with sources emerge from

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{1}{c} j^\mu.$$

5. Consider the electromagnetic field in three spacetime dimensions.

(a) Find the reduced Maxwell equations in three dimensions by starting with Maxwell's equations and the force law in four dimensions, using the ansatz $E_z = B_x = B_y = 0$, and assuming that no field can depend on the z direction.

(b) Write down the field strength tensor in three dimensions in terms of the three dimensionsal scalar and vector potentials.

(c) Write down the relativistic covariant form of the Lorentz force equation and show that the three dimensional Lorentz force equations are reproduced.

6. Assume a five-dimensional space-time $(t, \mathbf{x}, y)$, where $x = (t, \mathbf{x})$ are the usual four-dimensional space-time coordinates and y is the coordinate of an additional compact extra dimension, $-R/2 \leq y \leq R/2$.

Consider the free Klein-Gordon equation (KG) in this space-time,

$$(\partial_\sigma \partial^\sigma + m^2)\phi \;=\; 0\,,$$

where $\sigma = 0, 1, 2, 3, 4$ and $\mathbf{x} = \{x^1, x^2, x^3\}$ and $y = \{x^4\}$, *i.e.* $\partial_\sigma \partial^\sigma \equiv \partial_\mu \partial^\mu - \partial^2/\partial y^2$, with $\partial_\mu \partial^\mu = \partial_0^2 - \nabla^2$ the usual d'Alembert operator. The general solution of KG equation is obtained by a Fourier expansion and by imposing appropriate boundary conditions.

Show that

$$\phi(x, y) \;=\; \sum_{n=1}^{\infty} \phi_n(x)\, \mathrm{cs}\left(\frac{n\pi y}{R}\right)\,,$$

where $\mathrm{cs}(n\pi y/R) = \cos(n\pi y/R)$ if n is odd and $\mathrm{cs}(n\pi y/R) = \sin(n\pi y/R)$ for even n, is a solution of the KG equation, provided that the Fourier coefficients $\phi_n(x)$ are solutions of a four-dimensional Klein-Gordon equation. Determine the mass spectrum as seen by a four-dimensional observer, and show that for $m = 0$ the masses are equally spaced. What is this infinite set of massive particles called?