MATH423 - Introduction to String Theory

Set Work: Sheet 1

* **Reading**: Zwiebach chapters 1 and 2

1. Write down the weight lattice of the spinorial 16 representation of $SO(10)$ and how it decomposes under the $SU(5) \times U(1)$, $SO(6) \times SO(4)$ and] $SU(3) \times SU(2) \times U(1)^2$ subgroups.

Indentify how the Standard Model states fit into the spinorial 16 of $SO(10)$.

2. Consider the infinitesimal line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2.$$

(a) Write the metric $g_{\mu\nu}$ and its inverse in an explicit in matrix form.

(b) Find the set of independent transformations of the form

$$\begin{aligned} t &\to t + \epsilon A(t, x) \\ x &\to x + \epsilon B(t, x) \ , \end{aligned}$$

where ϵ is an infinitesimal constant and the functions A and B have to be determined by the requirement that ds^2 is invariant. State what each transformation represents in space time.

3. Consider two Lorentz vectors a^μ and b^μ. Write the Lorentz transformations $a^\mu \to a'^\mu$ and $b^\mu \to b'^\mu$. Verify that $a^\mu b_\mu$ is invariant under these transformations.

4. (a) Give the Lorentz transformations for the components a_μ of a vector under a boost along the x^1 axis.

(b) Show that the object $\frac{\partial}{\partial x^\mu}$ transforms under a boost along the x^1 axis as the a_μ vector considered in (a) do. This checks, in a particular case, that partial derivatives with respect to upper–index coordinates x^μ behave as a four–vector with lower indices, which is why they are written as ∂_μ.

(c) Show that, in quantum mechanics, the expression for the energy and momentum in terms of derivatives can be written compactly as $p_\mu = \frac{\hbar}{i} \frac{\partial}{\partial x^\mu}$.

5. Consider the infinitesimal line element on a two dimensional surface

$$ds^2 = g_{\mu\nu} d\theta^\mu d\theta^\nu = d\theta^2 + \sin^2\theta d\phi^2$$

(a convenient notation is $\theta^\mu \equiv (\theta, \phi)$ $(\mu = 1, 2)$)

(a) Write the metric $g_{\mu\nu}$ in an explicit matrix form.

Write $g^{\mu\nu}$ in matrix form.

(b) Consider the set of infinitesimal transformations of the form

$$\theta^\mu \to \theta^\mu + \epsilon \zeta^\mu(\theta, \phi) \ ,$$

for which the line element ds^2 is invariant and where ϵ is an infinitesimal constant. Derive the conditions that the functions ζ^1 and ζ^2 must satisfy for ds^2 to remain invariant.

6. A matrix L that satisfies

$$L^T \eta L = \eta,$$

where η is the Minkowski metric, is a Lorentz transformation (LT).

(a) Show that the Lorentz transformations form a group (*hint: to show that the Lorentz transformations form a group you a have to show that the product of two Lorentz transformations is a LT; that the inverse transformation is a LT; and that the idenity is a LT*).

(b) Show that if L is a LT so is the transpose matrix L^T.

(c) A special class of LT are those that are contiously connected to the identity. State the conditions on this class of LT.

(d) Consider LT in five spacetime dimensions, with one time, and four spatial, directions. How many LT connected to the identity are there in this space? What are they?

(e) Consider the extention of the Lorentz group in the five dimensional spacetime to the corresponding Poincare group. What is the dimensionality of the Poincare group in this space?

(f) Consider the infinitesimal line element in $d+1$ spacetime dimensions

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu,$$

where $\eta_{\mu\nu}$ is the Minkowski metric in $d+1$ dimensions. Enumerate the set of continuous transformations that leave this line element invariant and state what they correspond to. What is the dimensionality of the Poincare group in $d+1$ dimensions?