MATH423 String Theory Solutions 4

1.

$$\frac{d^2x^\mu}{ds^2} = 0 \qquad \tau = f(s). \tag{1}$$

$$\frac{dx^\mu}{ds} = \frac{dx^\mu}{d\tau}\frac{d\tau}{ds} = \frac{dx^\mu}{d\tau}f'(s) \tag{2}$$

$$\frac{d^2x^\mu}{ds^2} = \frac{d^2x^\mu}{d\tau^2}[f'(s)]^2 + \frac{dx^\mu}{d\tau}f''(s) \tag{3}$$

$$\Rightarrow \text{ equation of motion is } \frac{d^2x^\mu}{ds^2} = 0 \text{ if and only if } f''(s) = 0$$

i.e. $f(s) = As + B$ with A, B constants.
i.e. allowed reparametrisations are a shift of origin and a rescaling.

+ +

2. We have to take the variation $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$ in the action

$$S = -mc\int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau\ .$$

The variation gives

$$\begin{aligned}
S + \delta S &= -mc\int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu}\frac{d(x^\mu + \delta x^\mu)}{d\tau}\frac{d(x^\nu + \delta x^\nu)}{d\tau}}d\tau \\
&= -mc\int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu}\left(\frac{dx^\mu}{d\tau} + \frac{d\delta x^\mu}{d\tau}\right)\left(\frac{dx^\nu}{d\tau} + \frac{d\delta x^\nu}{d\tau}\right)}d\tau \\
&= -mc\int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu}\left(\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau} + \frac{d\delta x^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}\right)}d\tau \\
&= -mc\int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu}\left(\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{d\delta x^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}\right)}d\tau
\end{aligned}$$

We have to Taylor expand the square root. Recall that Taylor expansion for an arbitrary function to first order is

$$f(x_0 + \Delta) = f(x_0) + f'(x_0)\Delta.$$

Here,

$$f(x_0) = \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \quad \text{and} \quad \Delta = -\eta_{\mu\nu}\left(2\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{d\delta x^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}\right)$$

Hence we get (dropping terms of order δ^2)

$$S + \delta S \;=\; -mc\int_{\tau_i}^{\tau_f}\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau - mc\int_{\tau_i}^{\tau_f}\frac{1}{2}\left(\frac{-2\eta_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau}}{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}}\right)d\tau$$

$$\Longrightarrow \quad \delta S = -mc\int_{\tau_i}^{\tau_f}\frac{1}{2}\left(\frac{-2\eta_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau}}{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}}\right)d\tau = mc\int_{\tau_i}^{\tau_f}\frac{\frac{d\delta x^\mu}{d\tau}\frac{dx_\mu}{d\tau}}{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}}d\tau$$

Integrating by parts ($\int V dU = \int d(VU) - UdV$) with,

$$V = \frac{\frac{dx_\mu}{d\tau}}{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}} \quad , \quad U = \delta x^\mu \quad \text{we have}$$

$$\delta S = mc\int_{\tau_i}^{\tau_f}\frac{d}{d\tau}\left(\frac{\frac{dx^\mu}{d\tau}}{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}}\delta x_\mu\right)d\tau - \int_{\tau_i}^{\tau_f}\frac{d}{d\tau}\left[\frac{mc\frac{dx_\mu}{d\tau}}{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}}\right]\delta x^\mu(\tau)d\tau.$$

The first term vanishes by the conditions $\delta x^\mu(\tau_i) = \delta x^\mu(\tau_f) = 0$. Since $\delta x^\mu(\tau)$ is arbitrary in the domain of integration, The second term vanishes iff the integrand is identicaly zero *i.e.*

$$\frac{d}{d\tau}\left[\frac{mc\frac{dx_\mu}{d\tau}}{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}}\right] = 0 \tag{4}$$

This equation is in manifestly reparameterization invariant form. Indeed, the object between the brackets is clearly reparameterization invariant:

$$\frac{mc\frac{dx_\mu}{d\tau}}{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}} = \frac{mc\frac{dx_\mu}{d\tau'}}{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau'}\frac{dx^\beta}{d\tau'}}}$$

This follows from the chain rule $\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}$. The derivative in front of the square bracket does not spoil the reparameterization invariance since $\frac{d}{d\tau}[\cdots] = \frac{d}{d\tau'}[\cdots]\frac{d\tau'}{d\tau} = 0 \to \frac{d}{d\tau'}[\cdots] = 0$. When we choose $\tau = s$

$$-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{(ds)^2}{(ds)^2} = 1$$

Equation (4) then becomes

$$\frac{d}{ds}\left[mc\frac{dx_\mu}{ds}\right] = \frac{dp_\mu}{ds} = 0.$$

+ +

3. The relativistic version of Newton's second law is

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}}\right) = \vec{F}. \tag{5}$$

or in covariant form

$$\frac{dp^\mu}{ds} = \frac{d}{ds}\left(m\frac{dx^\mu}{ds}\right) = f^\mu \tag{6}$$

with

$$f^\mu = \left(\frac{\vec{v}\cdot\vec{F}, \vec{F}}{\sqrt{1-\vec{v}^2}}\right).$$

we want to show that (6) is the same as (5). Note

$$\vec{v}\cdot\frac{d\vec{p}}{dt} = \vec{v}\cdot\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}}\right) = \frac{m\vec{v}\cdot\frac{d\vec{v}}{dt}}{(1-\vec{v}^2)^{\frac{3}{2}}} = \frac{d}{dt}\left(\frac{m}{\sqrt{1-\vec{v}^2}}\right) = \frac{dp^0}{dt}$$

$$\Longrightarrow \frac{dp^0}{dt} = \vec{v}\cdot\vec{F}$$

$$\Longrightarrow \frac{dp^0}{ds}\frac{ds}{dt} = \vec{v}\cdot\vec{F} \quad \text{or} \quad \frac{dp^0}{ds} = \frac{\vec{v}\cdot\vec{F}}{\sqrt{1-\vec{v}^2}}$$

Similarly,

$$\frac{d\vec{p}}{dt} = \frac{d\vec{p}}{ds}\frac{ds}{dt} = \vec{F}$$

or

$$\frac{d\vec{p}}{ds} = \frac{\vec{F}}{\sqrt{1-\vec{v}^2}}$$

and

$$\frac{dp^\mu}{ds} = f^\mu$$

Here s is the proper time and $ds = \sqrt{1-\vec{v}^2}dt$. Note that we fixed $c = 1$.

+ +

4.

$$S = \int \frac{1}{2} m v^2 dt + \frac{q}{c} \int A_\mu(x) \frac{dx^\mu}{dt} dt \qquad \text{where } A^\mu = (\Phi, \vec{A}).$$

a.

$$S = \int \frac{1}{2} m v^2 dt + \frac{q}{c} \int \vec{A} \cdot \vec{v} dt - q \int \Phi dt = \int L dt \ .$$

where

$$L = \frac{1}{2} m v^2 + \frac{q}{c} \vec{A} \cdot \vec{v} - q\Phi$$

b.

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + \frac{q}{c}\vec{A}$$

c.

$$H = \vec{p} \cdot \vec{v} - L = m v^2 + \frac{q}{c} \vec{A} \cdot \vec{v} - \frac{1}{2} m v^2 - \frac{q}{c} \vec{A} \cdot \vec{v} + q\Phi = \frac{1}{2} m v^2 + q\Phi$$

We have to replace $\vec{v}$ by $\vec{p}$. From part (b)

$$\vec{v} = \frac{1}{m} \left(\vec{p} - \frac{q}{c} \vec{A} \right)$$

hence

$$H = \frac{1}{2m} \left(\vec{p} - \frac{q}{c} \vec{A} \right)^2 + q\Phi$$

+ +

5.

We are interested in variation of the action

$$S = -mc \int_{\mathcal{P}} ds + \frac{q}{c} I, \qquad I = \int_{\mathcal{P}} d\tau A_\mu(x(\tau)) \frac{dx^\mu}{d\tau}(\tau) \ . \tag{7}$$

when we let $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$. We note that

$$\delta A_\mu(x(\tau)) \equiv A_\mu \left(x(\tau) + \delta x(\tau) \right) - A_\mu \left(x(\tau) \right) = \frac{\partial A_\mu}{\partial x^\nu} \delta x^\nu(\tau),$$

where $\frac{\partial A_\mu}{\partial x^\nu}$ is calculated at $x = x(\tau)$. The variation of the firstpart wss done in problem 2. The variation of the second part is obtained from

$$\begin{aligned} I + \delta I &= \int_{\mathcal{P}} d\tau A_\mu(x(\tau) + \delta x(\tau)) \frac{d}{d\tau}(x^\mu + \delta x^\mu) \\ &= \int_{\mathcal{P}} d\tau (A_\mu(x(\tau)) + \frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu)(\frac{dx^\mu}{d\tau} + \frac{d\delta x^\mu}{d\tau}) \\ &= \int_{\mathcal{P}} d\tau \left(A_\mu(x(\tau))\frac{dx^\mu}{d\tau} + \frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu \frac{dx^\mu}{d\tau} + A_\mu(x(\tau))\frac{d\delta x^\mu}{d\tau} \right) \end{aligned}$$

where we dropped the term of order δ^2 in the last line. We therefore have

$$\delta I = \int_{\mathcal{P}} d\tau \frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu \frac{dx^\mu}{d\tau} + \int_{\mathcal{P}} d\tau A_\mu(x(\tau))\frac{d\delta x^\mu}{d\tau}.$$

We exchange $\mu \leftrightarrow \nu$ in the first term and rewrite the second using a total derivative:

$$\delta I = \int_{\mathcal{P}} d\tau \delta x^\mu \frac{\partial A_\nu}{\partial x^\mu}\frac{dx^\nu}{d\tau} + \int_{\mathcal{P}} d\tau \left[\frac{d}{d\tau}(A_\mu \delta x^\mu) - \delta x^\mu \frac{dA_\mu}{d\tau} \right].$$

We assume that δx vanishes at the ends of $\mathcal{P}$, so the total derivative (first term in square brackets) vanishes. Using the chain rule for the second term in square brackets we find

$$\delta I = \int_{\mathcal{P}} d\tau \delta x^\mu \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) \frac{dx^\nu}{d\tau} = \int_{\mathcal{P}} d\tau \delta x^\mu F_{\mu\nu}\frac{dx^\nu}{d\tau}$$

This concludes the variation of I.

The variation of the first term in (7) is obtained by varying the path $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$, as was done in problem 2, or alternatively as we did in the lectures,

$$\begin{aligned} \delta S &= -mc \int \delta(ds) = mc \int \eta_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{ds} d\tau \\ &= mc \int_{\tau_i}^{\tau_f} d\tau \frac{d}{d\tau}\left(\eta_{\mu\nu}\delta(x^\mu(\tau))\frac{dx^\nu}{ds} \right) - \int_{\tau_i}^{\tau_f} d\tau \delta(x^\mu(\tau)) \left(mc\eta_{\mu\nu}\frac{d}{d\tau}\left(\frac{dx^\nu}{ds} \right) \right) \end{aligned}$$

The first term is a boundary term and vanishes by imposing

$$\delta(x^\mu(\tau_i)) = \delta(x^\mu(\tau_f)) = 0 \ .$$

$$\Rightarrow \quad \delta S = -\int_{\tau_i}^{\tau_f} d\tau \delta(x^\mu(\tau)) \left(mc\eta_{\mu\nu} \frac{d}{d\tau}\left(\frac{dx^\nu}{ds}\right)\right)$$

The momentum four vector is given by

$$p^\nu = mu^\nu = mc\frac{dx^\nu}{ds},$$

where u^ν is the velocity four vector. Hence,

$$\delta S = -\int_{\tau_i}^{\tau_f} d\tau \delta(x^\mu(\tau))\eta_{\mu\nu}\frac{dp^\nu}{d\tau} = -\int_{\tau_i}^{\tau_f} d\tau \delta(x^\mu(\tau))\frac{dp_\mu}{d\tau}$$

Combining the variation of the two terms in the action for a charged particle in an electromagnetic field we get

$$\frac{dp_\mu}{d\tau} = \frac{q}{c}F_{\mu\nu}\frac{dx^\nu}{d\tau}$$

6.

Our starting point is the variation

$$\delta S \;=\; -\, mc\int \delta(ds). \tag{8}$$

With the path parametrised by an arbitrary τ we have

$$(ds)^2 = -g_{\mu\nu}(x(\tau))\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}(d\tau)^2. \tag{9}$$

Under the variation $x(\tau) \to x(\tau) + \delta x(\tau)$ the variation of the metric is

$$\delta g_{\mu\nu}(x(\tau)) = g_{\mu\nu}(x+\delta x) - g_{\mu\nu}(x) = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\delta x^\alpha(\tau).$$

The variation of 9 gives

$$2\ ds\ \delta(ds) = -\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\delta x^\alpha \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}(d\tau)^2 - 2g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}(d\tau)^2$$

Dividing by 2 ds and cancelling one factor of $d\tau$, we have

$$\delta(ds) = -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\delta x^\alpha\frac{dx^\mu}{ds}\frac{dx^\nu}{d\tau}d\tau - g_{\mu\alpha}\frac{dx^\mu}{ds}\frac{d\delta x^\alpha}{d\tau}d\tau.$$

inserting this result into 8 we have,

$$\delta S = mc \int d\tau \left(\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \delta x^\alpha \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau} - g_{\mu\alpha} \frac{dx^\mu}{ds} \frac{d\delta x^\alpha}{d\tau} \right).$$

Integrating by parts and dropping total derivatives that vanish since $\delta x = 0$ at the boundaries of the world–line

$$\delta S = mc \int d\tau \delta x^\alpha \left(\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau} - \frac{d}{d\tau} \left[g_{\mu\alpha} \frac{dx^\mu}{ds} \right] \right).$$

Since δx^α is arbitrary and requiring that the variation vanishes fernishes the equation of motion which is given by

$$\frac{d}{d\tau} \left[g_{\mu\alpha} \frac{dx^\mu}{ds} \right] = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau}.$$

This is the geodesic equation. Since τ is an arbitrary parameter we can choose $\tau = s$. We then have

$$\frac{d}{ds} \left[g_{\mu\alpha} \frac{dx^\mu}{ds} \right] = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}. \tag{10}$$

Expanding the derivatives in 10 gives

$$g_{\mu\alpha} \frac{d^2 x^\mu}{ds^2} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} \frac{dx^\nu}{ds} \frac{dx^\mu}{ds} - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.$$

The last two terms on the left–hand side can be combined to give

$$g_{\mu\alpha} \frac{d^2 x^\mu}{ds^2} + \frac{1}{2} \left(2 \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.$$

Since the expression in parenthesis multiplies $\frac{dx^\nu}{ds} \frac{dx^\mu}{ds}$, and is symmetric in the summation indices μ and ν, it can be rewritten in a form which is also symmetric in μ and ν as

$$g_{\mu\alpha} \frac{d^2 x^\mu}{ds^2} + \frac{1}{2} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.$$

multiplying throughout by $g^{\lambda\alpha}$ and using $g^{\lambda\alpha} g^{\mu\alpha} = \delta^\lambda_\alpha$ gives

$$\frac{d^2 x^\lambda}{ds^2} + \frac{1}{2} g^{\lambda\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.$$

The equation takes the form

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0.$$

with

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right).$$