MATH423 String Theory Solutions 3

1a.

$$T_{\lambda\mu\nu} = \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}.$$

where

$$F_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$T_{0ij} = \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i}.$$

For time independent fields $\partial_0 F_{ij} = 0$, so $T_{0ij} = 0$ implies

$$\partial_i F_{0j} - \partial_j F_{0i} = 0 \to \partial_i E_j - \partial_j E_i = 0 \ . \tag{1}$$

$\vec{E} = -\vec{\nabla}\Phi$ says $E_i = -\partial_i \Phi$, so condition (1) is satisfied since

$$\partial_i \partial_j \Phi = \partial_j \partial_i \Phi.$$

b.

With d spatial dimensions and a point charge q at $\vec{x} = 0$, the magnitude $E(r)$ of the electric field is

$$E(r) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{q}{r^{d-1}} \quad \text{(derived in lectures)} \tag{2}$$

Since $E(r) = -\frac{d\Phi}{dr}(r)$.

$$\Phi(r) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{1}{d-2} \frac{q}{r^{d-2}} \tag{3}$$

Setting $\Phi = 0$ at $r = \infty$ for $d > 2$. Using $\Gamma(x+1) = x\Gamma(x)$,

$$\Gamma\left(\frac{d}{2}\right) = \left(\frac{d}{2} - 1\right)\Gamma\left(\frac{d}{2} - 1\right) = \frac{1}{2}(d-2)\Gamma\left(\frac{d}{2} - 1\right) \tag{4}$$

so we get

$$\Phi(r) = \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{\frac{d}{2}}} \frac{q}{r^{d-2}} \qquad d > 2. \tag{5}$$

2a.

The standard Bohr radius

$$a_0 = \frac{\hbar^2}{m_e e^2} = 5.29 \times 10^{-9}\text{cm}, \quad \text{arises from the potential} \quad V = -\frac{e^2}{r}. \quad (6)$$

In the gravitational case, the potential is

$$V = -Gm_e m_p/r,$$

so the gravitational Bohr radius a_0^g is obtained by replacing e^2 with $Gm_e m_p$:

$$a_0^g = \frac{\hbar^2}{m_e(Gm_e m_p)} = \frac{\hbar^2}{m_e e^2}\left(\frac{e^2}{Gm_e m_p}\right) . \quad (7)$$

The ratio between the "gravitational" and the conventional Bohr radius is therefore

$$\frac{a_0^g}{a_0} = \frac{e^2}{Gm_e m_p} = \frac{(4.8 \times 10^{-10})^2}{(6.67 \times 10^{-8})(0.911 \times 10^{-27})(1.672 \times 10^{-24})} \approx 2.27 \times 10^{39}. \quad (8)$$

This gives

$$a_0^g = 1.2 \times 10^{31}\text{cm} \quad (9)$$

Since 1 light year $\approx$ 9.5×10^{17}cm, $a_0^g = 1.3 \times 10^{13}$ light years!! The universe is about 10^{10} years old, so this distance is quite large.

b.

We introduce factors of G, c and $\hbar$ into the formula, with exponents α, β and γ to be determined:

$$kT = \frac{1}{8\pi M} \to \frac{G^\alpha c^\beta \hbar^\gamma}{8\pi M}. \quad (10)$$

Since $[kT] = [E] = ML^2/T^2$, using $[G] = L^3M^{-1}T^{-2}$, $[c] = LT^{-1}$, and $[\hbar] = ML^2T^{-1}$, we find

$$\frac{ML^2}{T^2} = L^{3\alpha+\beta+2\gamma}M^{-\alpha+\gamma-1}T^{-2\alpha-\beta-\gamma} \quad (11)$$

This gives three equations for three unknowns

$$3\alpha + \beta + 2\gamma = 2, \quad -\alpha + \gamma - 1 = 1, \quad -2\alpha - \beta - \gamma = -2 \quad . \quad (12)$$

The solution is $\alpha = -1$, $\beta = 3$ and $\gamma = 1$. As a result

$$kT = \frac{\hbar c^3}{8\pi GM} = \left(\frac{\hbar c}{G}\right)\frac{c^2}{8\pi M} = \frac{m_P^2 c^2}{8\pi M}, \tag{13}$$

where $m_P = 2.17 \times 10^{-5}$gr is the Planck mass. For a mass with million solar masses $M = 10^6 M_{sun} \approx 2 \times 10^{39}$gr,

$$kT = \frac{(2.17 \times 10^{-5})^2 (3 \times 10^{10})^2}{8\pi 2 \times 10^{39}} = 8.43 \times 10^{-30}\text{eV} \quad \Rightarrow \quad T \approx 6.1 \times 10^{-14}\text{K}. \tag{14}$$

For a room temperature black hole, $T \approx 300$K, and we find $M = \frac{6.1\times10^{-14}\text{K}}{300\text{K}} 2 \times 10^{39}$gr $\approx 4.1 \times 10^{20}$kg, which is approximately $6.8 \times 10^{-5} M_{\text{earth}}$.

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3.

$$\frac{d^2 y_1}{dx^2} + \frac{\mu_1}{T_0} w^2 y_1(x) = 0 \quad \text{for} \quad 0 \le x \le a,$$
$$\frac{d^2 y_2}{dx^2} + \frac{\mu_2}{T_0} w^2 y_2(x) = 0 \quad \text{for} \quad a \le x \le 2a.$$

Here $y_1(x) = y(x)$ for $x \in [0, a]$ and $y_2(x) = y(x)$ for $x \in [a, 2a]$. We let

$$\lambda_1^2 = \frac{\mu_1}{T_0} w^2 \quad \text{and} \quad \lambda_2^2 = \frac{\mu_2}{T_0} w^2. \tag{15}$$

with this notation, the two equations take similar form

$$\frac{d^2 y_i}{dx^2} + \lambda_i^2 y_i(x) = 0, \quad \lambda_i = \sqrt{\frac{\mu_i}{T_0}} w, \; i = 1, 2. \tag{16}$$

Since $y(x = 0) = y(x = 2a) = 0$ we write

$$y_1(x) = A \sin(\lambda_1 x), \quad \text{and} \quad y_2(x) = B \sin(\lambda_2 (2a - x)), \tag{17}$$

which satisfy the differential equation and the boundary conditions at $x = 0$ and $x = 2a$.

a.

Since the string does not break at $x = a$, we must have $y_1(a) = y_2(a)$. Since there is no mass at the joining point the slopes must agree too: $y_1'(a) = y_2'(a)$.

b.

Applying the boundary conditions in (a) we find

$$\begin{aligned} A \sin \lambda_1 a &= B \sin \lambda_2 a \\ A\lambda_1 \cos \lambda_1 a &= -B\lambda_2 \cos \lambda_2 a \end{aligned}$$

These two equations can be written as

$$\begin{pmatrix} \sin \lambda_1 a & -\sin \lambda_2 a \\ \lambda_1 \cos \lambda_1 a & \lambda_2 \cos \lambda_2 a \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \ . \tag{18}$$

Since we need solutions where both A and B are not zero, the determinant of the above matrix must vanish:

$$\lambda_2 \sin \lambda_1 a \cos \lambda_2 a + \lambda_1 \cos \lambda_1 a \sin \lambda_2 a = 0 \ . \tag{19}$$

This equation can be used to find the oscillations frequencies. As a check assume $\mu_1 = \mu_2 = \mu_0$, in which case $\lambda_1 = \lambda_2 \equiv \lambda$. The condition becomes $\sin(2\lambda a) = 0$, which gives $\lambda(2a) = n\pi$. This gives the familiar frequencies

c.

With $\mu_1 = \mu_0$ and $\mu_2 = 2\mu_0$ we write

$$\lambda_1 = \sqrt{\frac{\mu_0}{T_0}} w \equiv \lambda \quad \text{and} \quad \lambda_2 = \sqrt{\frac{2\mu_0}{T_0}} w = \sqrt{2}\lambda \ .$$

Eq (19) then gives

$$\sqrt{2} \sin \lambda a \cos \sqrt{2}\lambda a + \cos \lambda a \sin \sqrt{2}\lambda a = 0 \ .$$

Solving this equation numerically we find that to the lowest root is $\lambda a \approx 1.26567$. This gives the lowest frequency

$$w_L = \sqrt{\frac{T_0}{\mu_0}} \frac{1.26567}{a} = \sqrt{\frac{T_0}{\mu_0}} \frac{2.53134}{2a} .$$

Note that w_L is bounded as

$$\sqrt{\frac{T_0}{\mu_0}} \frac{\pi/\sqrt{2}}{(2a)} < w_L < \sqrt{\frac{T_0}{\mu_0}} \frac{\pi}{(2a)} \ .$$

The lower and upper bounds are the lowest frequencies of strings of contant mass density $2\mu_0$ and μ_0 respectively ($\pi/\sqrt{2} \approx 2.22144$).

4.

a.

The Planck mass is given by

$$m_P = G^\alpha c^\beta \hbar^\gamma \tag{20}$$

where G is Newton's gravitational constant; c is the speed of light; and $\hbar$ is Planck constant. The dimensions of G, c and $\hbar$ are

$$\begin{aligned} [G] &= \frac{[L]^3}{[M]\,[T]^2} \\ [c] &= \frac{[L]}{[T]} \\ [\hbar] &= \frac{[M]\,[L]^2}{[T]} \end{aligned}$$

where $[L]$, $[M]$ and $[T]$ are the dimensions of length, time and mass, respectively. Since m_P has the dimension of mass we have

$$[m_P] = [M]^1 = \left(\frac{[L]^3}{[M]\,[T]^2}\right)^\alpha \left(\frac{[L]}{[T]}\right)^\beta \left(\frac{[M]\,[L]^2}{[T]}\right)^\gamma \tag{21}$$

Hence we obtain the equations

$$\begin{aligned} 3\alpha + \beta + 2\gamma &= 0 \\ -\alpha + \gamma &= 1 \\ -2\alpha - \beta - \gamma &= 0 \end{aligned}$$

which are solved with $\beta = \gamma = 1/2$ and $\alpha = -1/2$. Hence

$$m_P = \sqrt{\frac{c\hbar}{G}} \tag{22}$$

The numerical values of G, c and $\hbar$ are

$$\begin{aligned} G &= 6.674 \times 10^{-8} \frac{cm^3}{g \cdot s^2} \\ c &= 3 \times 10^{10} \frac{\text{cm}}{s} \\ \hbar &= 1.054 \times 10^{-27} erg \cdot s \end{aligned}$$

hence we get

$$m_P = 2.17 \times 10^{-5} g. \tag{23}$$

b.

Since t_P has the dimension of time we have

$$[t_P] = [T]^1 = \left(\frac{[L]^3}{[M]\,[T]^2}\right)^\alpha \left(\frac{[L]}{[T]}\right)^\beta \left(\frac{[M]\,[L]^2}{[T]}\right)^\gamma \tag{24}$$

Hence we obtain the equations

$$\begin{aligned} 3\alpha + \beta + 2\gamma &= 0 \\ -2\alpha - \beta - \gamma &= 1 \\ -\alpha + \gamma &= 0 \end{aligned}$$

which are solved with $\alpha = \gamma = 1/2$ and $\beta = -5/2$. Hence

$$t_P = \sqrt{\frac{G\hbar}{c^5}} \tag{25}$$

Inserting the numerical values of G, c and $\hbar$ we get

$$t_P = 5.38 \times 10^{-44} s. \tag{26}$$

c.

In the formula

$$\ell_{\text{vac}} = \rho_{\text{vac}}^\alpha \hbar^\beta c^\gamma \tag{27}$$

The units on both sides must agree:

$$L = \left(\frac{M}{L^3}\right)^\alpha \left(\frac{ML^2}{T}\right)^\beta \left(\frac{L}{T}\right)^\gamma \tag{28}$$

This gives three equations for matching powers of M, L and T:

$$\alpha + \beta = 0, \quad -3\alpha + 2\beta + \gamma = 1, \quad -\beta - \gamma = 0 \quad . \tag{29}$$

The solution is $\alpha = -1/4$, $\beta = 1/4$ and $\gamma = -1/4$. This means that (27) gives

$$\ell_{\text{vac}} = \left(\frac{\hbar}{c\rho_{\text{vac}}}\right)^{\frac{1}{4}} \tag{30}$$

Taking $\rho_{\text{vac}} = 7.7 \times 10^{-27} kg/m^3$ gives

$$\ell_{\text{vac}} = 8.22 \times 10^{-5} m = 82.2 \mu\text{m}$$

5a.

Under a variation $\delta q_i(t)$ of the coordinate, the variation of the velocity is $\frac{d\delta q_i}{dt}$. Since the Lagrangian L depends on q_i and $\dot{q}_i$ the full variation is

$$\delta S = \int dt \left\{ \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d\delta q_i}{dt} \right\}. \tag{31}$$

The second term in the brackets is rewritten in terms of a total derivative and a term proportional to δq_i

$$\delta S = \int dt \left\{ \frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right\}. \tag{32}$$

The total time derivative vanishes when we set the variations to vanish at the initial and final times. The variation δS is therefore

$$\delta S = \int dt \delta q_i(t) \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right\}. \tag{33}$$

If this variation is to vanish for all $\delta q_i(t)$ we must have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0. \tag{34}$$

These are the Euler–Lagrange equations of motion for the coordinates q_i.

5b.

Considering the Lagrangian

$$L(q(t), \dot{q}(t), t)$$

and the variation

$$\begin{aligned} q(t) \quad &\rightarrow \quad q(t) \; + \; \delta(q(t)) = q(t) \; + \; \epsilon h(q(t), t) \\ \dot{q}(t) \quad &\rightarrow \quad \dot{q}(t) \; + \; \frac{d}{dt} \left(\delta(q(t)) \right) \end{aligned}$$

Variation of the action gives rise to the Euler-Lagrange equation of motion

$$S(t) = \int dt L(q(t), \dot{q}(t), t) \quad \Rightarrow \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

since L is invariant under the variation

$$L(q + \delta q, \dot{q} + \frac{d}{dt}\delta q, t) = L + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q + \cdots = L$$

$$\implies \quad \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q = 0$$

Now take

$$\begin{aligned} \epsilon\frac{dQ}{dt} &= \tfrac{d}{dt}\left(\tfrac{\partial L}{\partial \dot{q}}\delta q\right) = \tfrac{d}{dt}\left(\tfrac{\partial L}{\partial \dot{q}}\right)\delta q + \tfrac{\partial L}{\partial \dot{q}}\tfrac{d\delta q}{dt} = \tfrac{\partial L}{\partial q}\delta q + \tfrac{\partial L}{\partial \dot{q}}\tfrac{d}{dt}\delta q = 0 \\ \implies \frac{dQ}{dt} &= 0, \end{aligned}$$

where we have used the Euler–Lagrange equation to go from the second to third equality. Therefore, we have that

$$Q(t) = \frac{\partial L}{\partial \dot{q}}h(q(t), t)$$

is conserved.