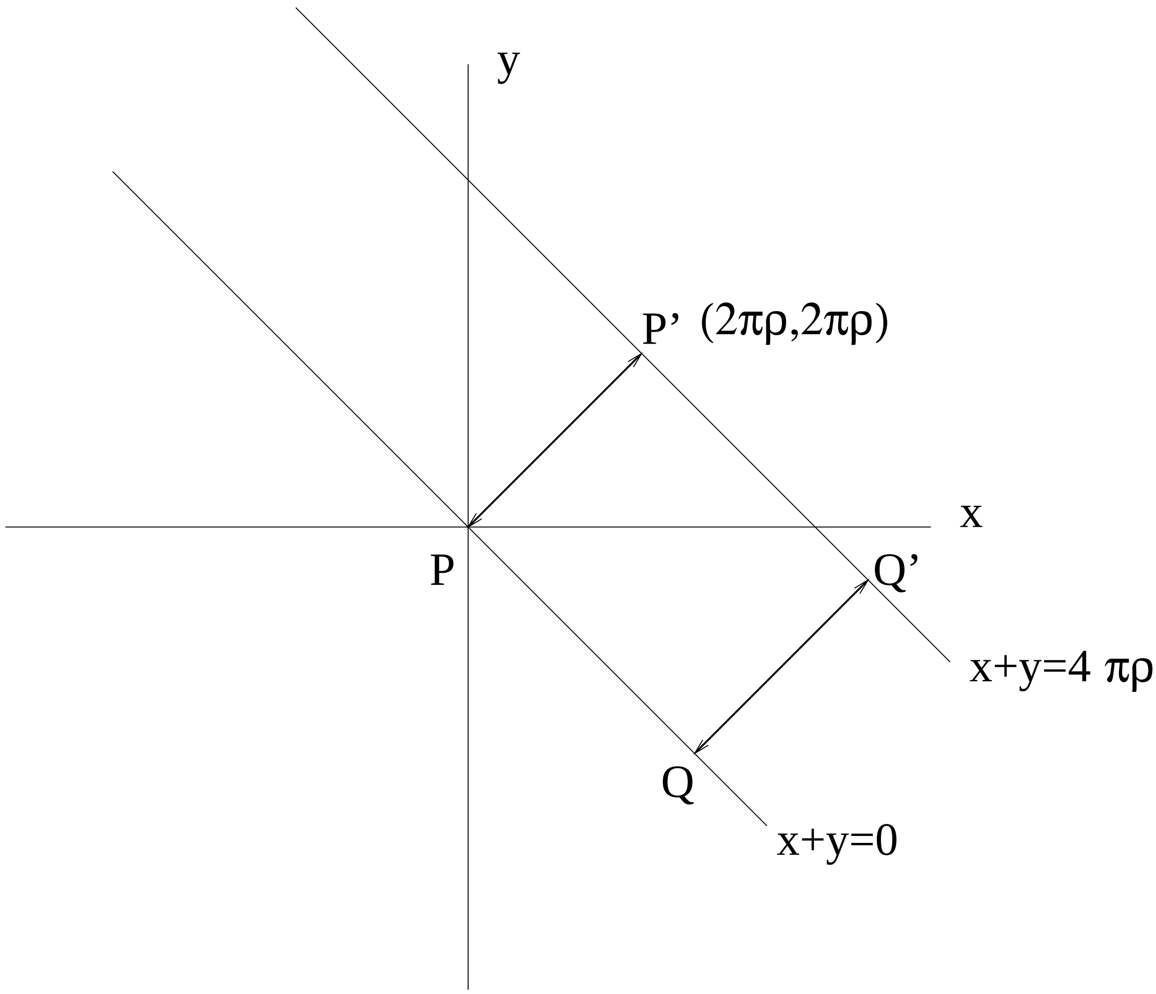
MATH423 String Theory Solutions 2

1.

Plane (x, y) with identification $(x, y) \sim (x + 2\pi R, y + 2\pi R)$

There is just one identification. Note $P \sim P'$ and more generally, $Q \sim Q'$. The lines $x + y = 0$ and $x + y = 4\pi R$ bound the fundamental domain $\{\mathcal{F} : (x, y) \text{ with } 0 \leq x + y < 4\pi R\}$. Identifying the two lines we get a cylinder.

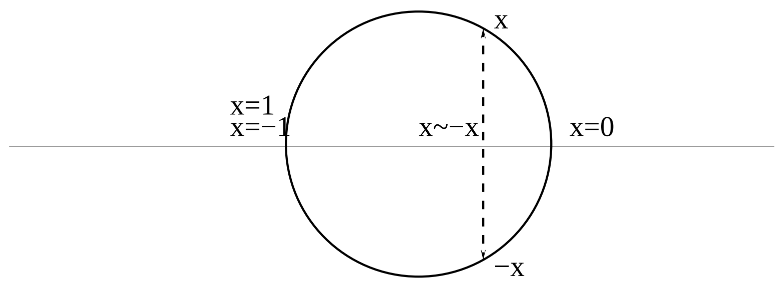


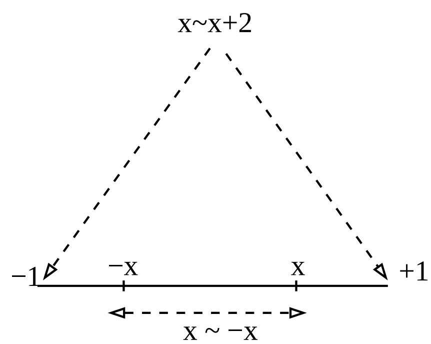
2.

a.

The Z_2 identification $x \sim -x$ sends points in $-1 \leq x \leq 1$ into points in $-1 \leq x \leq 1$. Two ways of picturing this are shown in the figures. If we picture S^1 as $-1 \leq x \leq 1$ with $x = \pm 1$ identified, then $x \to -x$ is reflection in the midpoint $x = 0$. Clearly, $x = 0$ is invariant under $x \to -x$, and so is $x = 1$ which goes into $x = -1$ already identified with $x = 1$ under $x \sim x + 2$.

If we picture S^1 as a circle in the two dimensional plane, $x \to -x$ is reflexion in a diameter and the end of the diameter are left invariant. The interval $0 \leq x \leq 1$ is a fundamental domain since no two points in it are identified by $x \to -x$ and the remaining points on the circle $-1 < x < 0$, are obtained from $0 < x < 1$ by $x \to -x$. This means that S^1/Z_2 is just the closed interval $0 \leq x \leq 1$.





b.

T^2 is $-1 \leq x, y, \leq 1$ with opposite edges identified, *i.e.* $(-1, y) \sim (1, y)$ and $(x, -1) \sim (x, 1)$.

(x, y) is invariant under $(x, y) \to (-x, -y) \Leftrightarrow$

we want to solve the equation

$$(x, y) \sim (-x, -y) + m(2, 0) + n(0, 2)$$

where m and n take the values 0,1. This equation has the four solutions in the fundamental domain

$$(x, y) = \{(0, 0), (1, 0), (0, 1), (1, 1)\} \tag{1}$$

These are the four fixed points of the orbifold T^2/Z_2.

As a fundamental domain we can choose the rectangle $0 < x < 1$, $-1 < y < 1$ together with part of its boundary.

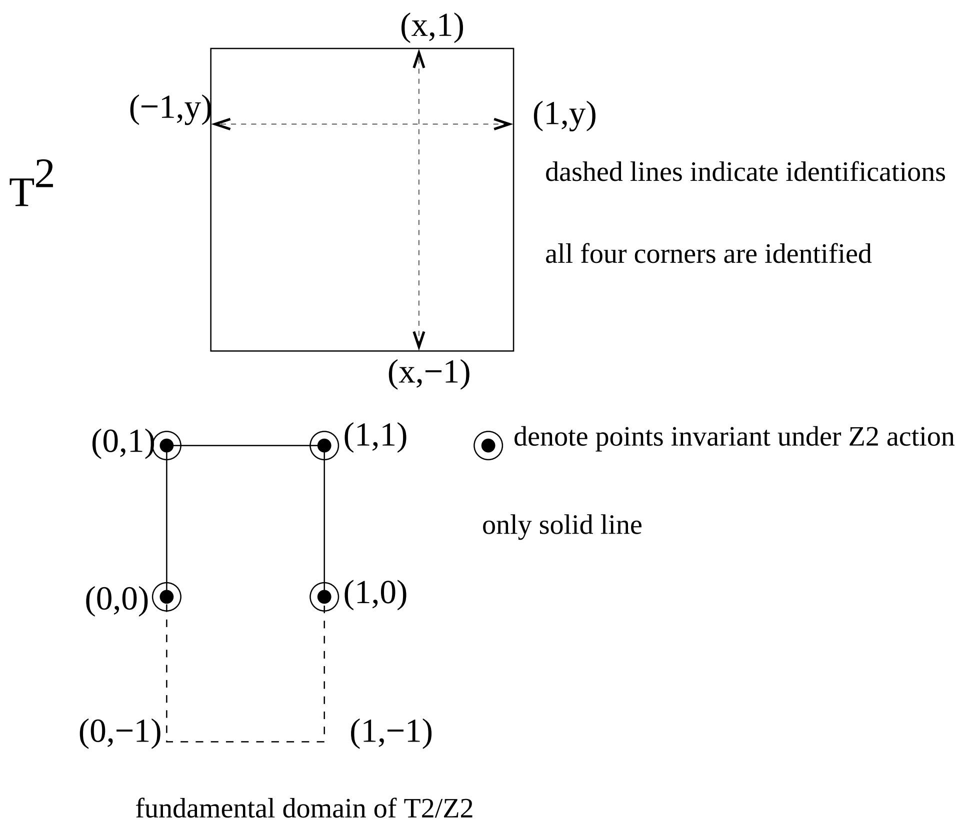
Since $(0, y) \sim (0, -y)$ we need only keep $0 \leq y \leq 1$ on $x = 0$

Since $(1, y) \sim (-1, -y) \sim (1, -y)$ we need only keep $0 \leq y \leq 1$ on $x = 1$

Since $(x, 1) \sim (-x, -1) \sim (-x, 1)$ we need only keep $0 \leq x \leq 1$ on $y = 1$

This leaves the fundamental domain shown in the figure below.

To present T^2/Z_2, take the fundamental domain and its boundary, *i.e.* $-1 \leq y \leq 1$ and on the boudary identify $(x, -1) \sim (x, 1)$; $(0, y) \sim (0, -y)$; $(1, y) \sim (1, -y)$, *i.e.* take the rectangle $0 \leq x \leq 1$, $-1 \leq y \leq 1$, fold it over along $y = 0$ and sew together the edges. Now puff up the pillowcase and we a 2–sphere, S^2.


(x,1)
(−1,y)
(1,y)
T2
dashed lines indicate identifications
all four corners are identified
(x,−1)
(0,1)
(1,1)
denote points invariant under Z2 action
only solid line
(0,0)
(1,0)
(0,−1)
(1,−1)
fundamental domain of T2/Z2

3.

We rearrange the indices by raising the μ in equation of the Lorentz force

$$\frac{dp^\mu}{ds} = \frac{q}{c} F^{\mu\nu} \frac{dx_\nu}{ds}.$$

Multiplying both sides of the equation by ds/dt we find

$$\frac{dp^\mu}{dt} = \frac{q}{c} F^{\mu\nu} \frac{dx_\nu}{dt}.$$

we test this equation using $F^{\mu\nu}$, as given in the lectures, and $\frac{dx_\nu}{dt} = (-c, v_x, v_y, v_z)$:

$$\begin{aligned}
\frac{dp^1}{dt} &= \frac{q}{c}(F^{10}(-c) + F^{12} v_y + F^{13} v_z) = qE_x + \frac{q}{c}(v_y B_z - v_z B_y) \\
&= q(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B})_x \qquad \text{good!}
\end{aligned}$$

$$\begin{aligned}
\frac{dp^2}{dt} &= \frac{q}{c}(F^{20}(-c) + F^{21} v_x + F^{23} v_z) = qE_y + \frac{q}{c}(v_z B_x - v_x B_z) \\
&= q(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B})_y \qquad \text{good!}
\end{aligned}$$

$$\begin{aligned}
\frac{dp^3}{dt} &= \frac{q}{c}(F^{30}(-c) + F^{31} v_x + F^{32} v_y) = qE_z + \frac{q}{c}(v_x B_y - v_y B_x) \\
&= q(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B})_z \qquad \text{good!}
\end{aligned}$$

The last equation is

$$\frac{dp^0}{dt} = \frac{q}{c} F^{0i} \frac{dx_i}{dt} = \frac{q}{c} \vec{E} \cdot \vec{v}.$$

With $p^0 = E/c$, it becomes

$$\frac{dE}{dt} = (q\vec{E}) \cdot \vec{v} = \text{force} \times \text{velocity}.$$

where on the left E stands for energy and on the right $\vec{E}$ stands for the electric field. The rate of change of the particle energy equals the rate at which the fields do the work on the particle. The magnetic force is perpendicular to the velocity and does not do work.

4a. T is totally antisymmetric, so T vanishes unless all the indices are different. This yields four equations $T_{012}=0$, $T_{013}=0$, $T_{023}=0$, $T_{123}=0$. The first three of them give

$$
\begin{aligned}
T_{012}=0 \quad &\rightarrow \quad \partial_0 F_{12}+\partial_1 F_{20}+\partial_2 F_{01}=0 \\
&\quad\quad \frac{1}{c}\frac{\partial}{\partial t}B_z+\frac{\partial}{\partial x}E_y-\frac{\partial}{\partial y}E_x=0 \\
&\rightarrow \quad \frac{\partial}{\partial x}E_y-\frac{\partial}{\partial y}E_x=-\frac{1}{c}\frac{\partial}{\partial t}B_z \\
T_{013}=0 \quad &\rightarrow \quad \partial_0 F_{13}+\partial_1 F_{30}+\partial_3 F_{01}=0 \\
&\quad\quad \frac{1}{c}\frac{\partial}{\partial t}(-B_y)+\frac{\partial}{\partial x}E_z-\frac{\partial}{\partial z}E_x=0 \\
&\rightarrow \quad \frac{\partial}{\partial z}E_x-\frac{\partial}{\partial x}E_z=-\frac{1}{c}\frac{\partial}{\partial t}B_y \\
T_{023}=0 \quad &\rightarrow \quad \partial_0 F_{23}+\partial_2 F_{30}+\partial_3 F_{02}=0 \\
&\quad\quad \frac{1}{c}\frac{\partial}{\partial t}B_x+\frac{\partial}{\partial y}E_z-\frac{\partial}{\partial z}E_y=0 \\
&\rightarrow \quad \frac{\partial}{\partial y}E_z-\frac{\partial}{\partial z}E_y=-\frac{1}{c}\frac{\partial}{\partial t}B_x
\end{aligned}
$$

They are the three components of $\nabla\times\vec{E}=\frac{-1}{c}\frac{\partial\vec{B}}{\partial t}$. Finally

$$T_{123}=\partial_1 F_{23}+\partial_2 F_{31}+\partial_3 F_{12}=0,$$

gives

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0 \quad \rightarrow \quad \nabla\cdot\vec{B}=0. \tag{2}$$

b.

Test values for μ in the equation $\partial F^{\mu\nu}/\partial x^\nu=j^\mu/c$. For $\mu=0$

$$\frac{\partial F^{0i}}{\partial x^i}=\frac{j^0}{c}=\rho \;\text{ which is }\; \nabla\cdot\vec{E}=\rho.$$

$\mu=i$: Note that we can write $F^{ij}=\epsilon^{ijk}B^k$ where ϵ is totally antisymmetric and $\epsilon^{123}=1$. For example $F^{12}=\epsilon^{12k}B^k=\epsilon^{123}B^3=B^3$. Using this, we have

$$\frac{1}{c}\frac{\partial F^{i0}}{\partial t}+\frac{\partial F^{ij}}{\partial x^j}=\frac{j^i}{c} \quad \rightarrow \quad -\frac{1}{c}\frac{\partial E^i}{\partial t}+\frac{\partial}{\partial x^j}(\epsilon^{ijk}B^k)=\frac{j^i}{c}$$

A little rearrangement gives

$$\epsilon^{ijk}\frac{\partial}{\partial x^j}B^k = \frac{j^i}{c} + \frac{1}{c}\frac{\partial E^i}{\partial t}, \quad \text{which is} \quad (\vec{\nabla}\times\vec{B})_i = \left(\frac{\vec{j}}{c} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right)_i .$$

5a.

To construct a consistent three dimensional theory, we must ensure that the dynamics do not depend on the z–direction. The motion of the particle must be confined in the (x, y) plane. Since the Lorents force is perpendicular to the magnetic field, it follows that the components of the magnetic fields in the (x, y) plane has to vanish. Similarly, the component of the electric field in the z–direction is zero. Hence, we take

$$E_z = B_x = B_y = 0.$$

The remaining components E_x, E_y and B_z can only depend on x and y. Similarly, the velocity and current components in the z–directions are zero, *i.e.* $v_z = j_z = 0$. Maxwell's equations then become

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \rho \quad \text{from} \quad \vec{\nabla}\cdot\vec{E} = \rho \tag{3}$$

$$\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} = -\frac{1}{c}\frac{\partial B_z}{\partial t} \quad \text{from} \quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{4}$$

$$\begin{aligned} \frac{\partial B_z}{\partial y} &= \frac{j_x}{c} + \frac{1}{c}\frac{\partial E_x}{\partial t} \\ -\frac{\partial B_z}{\partial x} &= \frac{j_y}{c} + \frac{1}{c}\frac{\partial E_y}{\partial t} \end{aligned}$$

$$\text{from} \quad \vec{\nabla}\times\vec{B} = \frac{\vec{j}}{c} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

The remaining Maxwell equation $\vec{\nabla}\cdot\vec{B} = 0$ is trivial because $B_x = B_y = 0$ and $B_z = B_z(x, y)$. The Lorentz force law gives nontrivial equations only for the x and y components:

$$\begin{aligned} \frac{dp_x}{dt} &= q\left(E_x + \frac{v_y}{c}B_x\right), \\ \frac{dp_y}{dt} &= q\left(E_y - \frac{v_x}{c}B_y\right). \end{aligned} \tag{5}$$

5b. In three dimensions we have $A^\mu = (\Phi, A^1, A^2)$, $A_\mu = (-\Phi, A_1, A_2)$, and $j^\mu = (c\rho, j^1, j^2)$. Moreover, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, so

$$\begin{aligned} F_{0i} &= \frac{1}{c}\frac{\partial A_i}{\partial t} + \frac{\partial \Phi}{\partial x^i} \equiv -E_i \\ F_{0i} &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \equiv B_z \end{aligned}$$

Thus, the field strength tensor takes the form

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & B_z \\ E_y & -B_z & 0 \end{pmatrix} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B_z \\ -E_y & -B_z & 0 \end{pmatrix} \tag{6}$$

The above $D = 3$ field strength F can be viewed as the $D = 4$ one with $E_z = B_x = B_y = 0$. The $D = 3$ current j can be viewed as the $D = 4$ current with $j_z = 0$. The three dimensional Maxwell equations are therefore the truncation to $E_z = B_x = B_y = 0$ and $j_z = 0$ of the original four dimensional ones

5c. The force law

$$\frac{dp^\mu}{dt} = \frac{q}{c} F^{\mu\nu} \frac{dx_\nu}{dt} \tag{7}$$

gives

$$\begin{aligned} \frac{dp^x}{dt} &= q\frac{F^{10}}{c}(-c) + q\frac{F^{12}}{c} v_y = q\left(E_x + \frac{v_y}{c} B_z\right), \\ \frac{dp^y}{dt} &= q\frac{F^{20}}{c}(-c) + q\frac{F^{21}}{c} v_x = q\left(E_y - \frac{v_x}{c} B_z\right), \end{aligned}$$

which is the expected result.

6. Inserting the given solution

$$\phi(x,y) = \sum_{n=1}^{\infty} \phi_n(x) \operatorname{cs}\left(\frac{n\pi y}{R}\right)$$

into the five-dimensional Klein-Gordon equation and carrying out $\partial^2/\partial y^2$, one obtains a four-dimensional Klein-Gordon equation for each of the Fourier coefficients $\phi_n(x)$:

$$\left(\partial_0^2 - \nabla^2 + m^2 + \left(\frac{n\pi}{R}\right)^2\right) \phi_n(x) = 0\,.$$

Therefore the given $\phi(x, y)$ is a solution of the five-dimensional Klein-Gordon equation *if* the $\phi_n(x)$ are solutions of the four-dimensional equations. The masses m_n of the fields ϕ_n are given by

$$m_n^2 = m^2 + \left(\frac{n\pi}{R}\right)^2 .$$

If the five-dimensional mass m is zero, we get the equally spaced mass spectrum

$$m_n = \frac{n\pi}{R} .$$

The infinite set of particles are called Kaluza-Klein tower.