MATH423 String Theory Solutions 1

1.

+	+	+	+	+
+	+	+	-	-
-	+	+	-	+
+	-	+	-	+
+	+	-	-	+
-	+	+	+	-
+	-	+	+	-
+	+	-	+	-
-	-	+	+	+
-	+	-	+	+
+	-	-	+	+
-	-	+	-	-
-	+	-	-	-
+	-	-	-	-
-	-	-	-	+
-	-	-	+	-

Table 1: The weight lattice of the spinorial 16 representation of $SO(10)$. Each entry should be multiplied by the $\frac{1}{2}$. The $\pm\frac{1}{2}$ entries are the charges with respect to the five $U(1)$ generators of the Cartan subalgebra. The product of the five charges should be either positive or negative. If we choose to be positive then we can have either, zero, two or four negative charges (or spins) of the total five. Alternatively, if we choose the product to be negative we can have either one, three or five of the charges (or spins) to have a minus sign. In either case the total number of possibilities is 16. These are the two spinorial representations of $SO(10)$ being the chiral 16 and the anti–chiral $\overline{16}$. The table above shows the chiral 16 representation.

From the table above we see that there is one state with zero minus signs. This is a singlet of $SU(5)$. There are five states with four minus signs. This is a $\overline{5}$ representation of $SU(5)$. Finally there are 10 states with two minus sighs. This is the 10 representation of $SU(5)$. Hence, under $SU(5)\times U(1)_X$ the 16 representation of $SO(10)$ decomposes as:

$$16 = (1, \frac{5}{2}) + (\bar{5}, -\frac{3}{2}) + (10, \frac{1}{2})$$

where the $U(1)_X$ charges are obtained by taking the trace $Q_X = Q_1 + Q_2 + Q_3 + Q_4 + Q_5$, and the Q_i are the $\pm\frac{1}{2}$ charges with respect to the five generators of the Cartan subalgebra.

in our combinatorial notation we can write

$$16 = \binom{5}{0} + \binom{5}{2} + \binom{5}{4}$$

where the combinatorial factor counts the number of $-$ in a given state.

To find the decomposition under $SO(6) \times SO(4) \equiv SU(4) \times SU(2)_L \times SU(2)_R$ we split the five slots into the first three which correspond to $SO(6)$ and the last two which correspond to $SO(4)$. We now split the 16 again by counting how many minus signs there are under the first three times how many there are under the last two. Note that here we have to take the product of the signs with respect to the first three slots and last two separately. So, for example, states with zero or two minus signs under the first three slots belong to the same $SO(6)$ representation. Hence, under $SO(6) \times SO(4) \equiv SU(4) \times SU(2)_L \times SU(2)_R$ it decomposes as:

$$16 = (4, 2, 1) + (\bar{4}, 1, 2)$$

In the combinatorial notation this decomposes as:

$$16 = \left[\binom{3}{0} + \binom{3}{2}\right]\left[\binom{2}{0} + \binom{2}{2}\right] + \left[\binom{3}{1} + \binom{3}{3}\right]\left[\binom{2}{1}\right]$$

To find the decomposition under $SU(3) \times U(1)_C \times SU(2) \times U(1)_L$ we again split the five slots into the first three, which correspond to $SU(3) \times U(1)_C$ and the last two that correspond to $SU(2) \times U(1)_L$. We again count how many minus signs there are in each state under the first three and last two slots to find the multiplicity. The charges under the $U(1)$s are given by the sums under $Q_1 + Q_2 + Q_3$ and $Q_4 + Q_5$ for $U(1)_C$ and $U(1)_L$, respectively. Hence, under $SU(3) \times U(1)_C \times SU(2) \times U(1)_L$ the 16 decomposes as:

$$16 = (1, \frac{3}{2}, 1, +1) + (3, \frac{1}{2}, 2, 0) + (\bar{3}, -\frac{1}{2}, 1, +1) + (1, \frac{3}{2}, 1, -1) + (\bar{3}, -\frac{1}{2}, 1, -1) + (1, -\frac{3}{2}, 1, 0)$$

where Q_C and Q_L are defined as $Q_C = Q_1 + Q_2 + Q_3$ and $Q_L = Q_4 + Q_5$. In combinatorial notation these are:

$$16 = \left[\binom{3}{0}\binom{2}{0}\right] + \left[\binom{3}{2}\binom{2}{0}\right] + \left[\binom{3}{0}\binom{2}{2}\right] + \left[\binom{3}{1}\binom{2}{1}\right] + \left[\binom{3}{2}\binom{2}{2}\right] + \left[\binom{3}{3}\binom{2}{1}\right]$$

From the last line we can read off the Standard Model states. These are in order respectively

$$\mathrm{e}^c_L \ , \mathrm{d}^c_L \ , \mathrm{N}^c_L \ , Q \ , \mathrm{u}^c_L \ , L$$

where the subscript L indicates that these are all left–handed fields and the upperscript c denotes charge conjugation (corresponding to antiparticles). Therefore the states are: 1. the positron which is an $SU(2)$ and $SU(3)$ singlet; 2. down-type quark $SU(2)$ singlet; 3. standard model singlet corresponding to the right–handed neutrino; 4. quark $SU(2)$ doublet; 5. up-type quark $SU(2)$ singlet; 6. lepton $SU(2)$ doublet.

The weak hypercharge is given by

$$U(1)_Y = \frac{1}{3} U_C + \frac{1}{2} U_L$$

and the electric charge by

$$U(1)_{e.m.} = T_{3_L} + U(1)_Y$$

where T_{3_L} is the diagonal generator of the $SU(2)$ subgroup.

2.

a.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

b.

$$\begin{aligned} t &\rightarrow t + \epsilon A(t, x) \\ x &\rightarrow x + \epsilon B(t, x) \end{aligned}$$

$$\begin{aligned} dt &\rightarrow dt + \epsilon(\tfrac{\partial A}{\partial t} dt + \tfrac{\partial A}{\partial x} dx) \\ dx &\rightarrow dx + \epsilon(\tfrac{\partial B}{\partial t} dt + \tfrac{\partial B}{\partial x} dx) \end{aligned}$$

$$ds^2 \rightarrow [(1+\epsilon\frac{\partial A}{\partial t})dt + \epsilon\frac{\partial A}{\partial x}dx]^2 - [(1+\epsilon\frac{\partial B}{\partial x})dx + \epsilon\frac{\partial B}{\partial t}dt]^2$$

we require invariance of ds^2. Expanding to first order in ϵ we impose that the coefficients of the addtional terms vanish. These yield the constraints on the functions A and B.

$$\begin{aligned}
dt^2 &: \frac{\partial A}{\partial t} = 0 \Rightarrow A = A(x) \\
dx^2 &: \frac{\partial B}{\partial x} = 0 \Rightarrow B = B(t) \\
dxdt &: \frac{\partial A}{\partial x} - \frac{\partial B}{\partial t} = 0 \Rightarrow \frac{dA}{dx} = \frac{dB}{dt} = \text{constant} = c
\end{aligned}$$

$$\begin{aligned}
\Rightarrow A(x) &= cx + a \\
B(t) &= ct + b
\end{aligned}$$

we obtained three constants of integration a, b and c. These correspond to a shift in time a, a shift in space b, and a boost c.

3.

$$\begin{aligned}
a'^0 &= \gamma(a^0 - \beta a^1) \\
a'^1 &= \gamma(-\beta a^0 + a^1) \\
a'^2 &= a^2 \\
a'^3 &= a^3
\end{aligned}$$

$$\begin{aligned}
b'^0 &= \gamma(b^0 - \beta b^1) \\
b'^1 &= \gamma(-\beta b^0 + b^1) \\
b'^2 &= b^2 \\
b'^3 &= b^3
\end{aligned}$$

$$\begin{aligned}
b'_0 &= -\gamma(b^0 - \beta b^1) \\
b'_1 &= \gamma(-\beta b^0 + b^1) \\
b'_2 &= b^2 \\
b'_3 &= b^3
\end{aligned}$$

Then

$$
\begin{aligned}
a'^{\mu} b'_{\mu} &= -\gamma(a^0 - \beta a^1)\gamma(b^0 - \beta b^1) + \gamma(-\beta a^0 + a^1)\gamma(-\beta b^0 + b^1) + a^2 b_2 + a^3 b_3 \\
&= \gamma^2 \left[a^0 b^0(-1+\beta^2) + a^1 b^1 (1-\beta^2)\right] + a^2 b_2 + a^3 b_3 \\
&= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3 \\
&= a^{\mu} b_{\mu}
\end{aligned}
$$

4.

Lorentz transformations, derivatives and quantum operators

a.

$$
\begin{aligned}
a_0 &= -a^0,\ a_1 = a^1,\ a_2 = a^2,\ a_3 = a^3 \\
a'_0 &= -a'^0 = -\gamma(a^0 - \beta a^1) = \gamma(a_0 + \beta a_1) \\
a'_1 &= a'^1 = \gamma(-\beta a^0 + a^1) = \gamma(\beta a_0 + a_1) \qquad (1) \\
\text{and}\ a'_2 &= a_2,\ a'_3 = a_3
\end{aligned}
$$

b.

Suppose we have a function $f(x^0, x^1, x^2, x^3)$ which we express as a function of $x'^0,\ x'^1,\ x'^2,\ x'^3$ by expressing x^{μ} as a function of x'^{μ}. The standard chain rule for partial differentiation says that

$$
\frac{\partial f}{\partial x'^{\mu}} = \sum_{\nu=0}^{3} \frac{\partial f}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}} \quad \text{for} \quad \mu = 0,1,2,3 \tag{2}
$$

Using the summation convention and writing as an operator equation we get

$$
\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} \tag{3}
$$

We need x as a function of x', the inverse of the Lorentz transformation that gives x' as a function of x. For a boost along the x' axis, the inverse is a boost with the opposite speed, so

$$
x^0 = \gamma(x'^0 + \beta x'^1),\ x^1 = \gamma(\beta x'^0 + x'^1) \tag{4}
$$

Hence

$$
\frac{\partial}{\partial x'^0} = \gamma(\frac{\partial}{\partial x^0} + \beta \frac{\partial}{\partial x^1}), \quad \frac{\partial}{\partial x'^1} = \gamma(\beta \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}) \tag{5}
$$

which is the same as as (1) with $a_\mu = \frac{\partial}{\partial x^\mu}$

c.

The operator for momentum $\vec{p}$ is $\frac{\hbar}{i}\vec{\nabla}$, *i.e.*

$$p_1 = \frac{\hbar}{i}\frac{\partial}{\partial x^1},\ p_2 = \frac{\hbar}{i}\frac{\partial}{\partial x^2},\ p_3 = \frac{\hbar}{i}\frac{\partial}{\partial x^3} \tag{6}$$

The Schrödinger equation says

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

where H is the energy operator. Since $p^0 = \frac{E}{c}$ and $x^0 = ct$, this can be written as

$$p^0 = i\hbar\frac{\partial}{\partial x_0} = -\frac{\hbar}{i}\frac{\partial}{\partial x_0} \tag{7}$$

If we write (6) and (7) in terms of p_μ, we remove the sign difference between the 0 component and the others.

$$p_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x^\mu}. \tag{8}$$

This is Lorentz invariant since we proved in (b) that $\frac{\partial}{\partial x^\mu}$ tranforms like p_μ.

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5.

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

a.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad , \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

b.

$$\theta^\mu \longrightarrow \theta^\mu + \epsilon\zeta^\mu(\theta,\phi)$$

We want to find functions

$$\begin{aligned} A(\theta,\phi) &= \zeta^1(\theta,\phi) \\ \text{and } B(\theta,\phi) &= \zeta^2(\theta,\phi) \end{aligned} \tag{9}$$

such that ds^2 remains invariant under the transformations.

$$\begin{aligned}
\theta &\to \theta + \epsilon A(\theta,\phi) \\
\phi &\to \phi + \epsilon B(\theta,\phi) \\
\sin\theta &\to \sin(\theta+\epsilon A) \approx \sin\theta + \epsilon A\cos\theta \\
\sin^2\theta &\to \sin^2\theta + 2\epsilon A\sin\theta\cos\theta + O(\epsilon^2)
\end{aligned} \tag{10}$$

Keeping terms to first order in ϵ

$$\begin{aligned}
d\theta &\to (1+\epsilon\tfrac{\partial A}{\partial\theta})d\theta + \epsilon\tfrac{\partial A}{\partial\phi}d\phi \\
d\phi &\to \epsilon\tfrac{\partial B}{\partial\theta}d\theta + (1+\epsilon\tfrac{\partial B}{\partial\phi})d\phi \\
d\theta^2 &\to (1+2\epsilon\tfrac{\partial A}{\partial\theta})d\theta^2 + 2\epsilon\tfrac{\partial A}{\partial\phi}d\theta d\phi \\
d\phi^2 &\to (1+2\epsilon\tfrac{\partial B}{\partial\phi})d\phi^2 + 2\epsilon\tfrac{\partial B}{\partial\theta}d\theta d\phi \\
\sin^2\theta d\phi^2 &\to \sin^2\theta(1+2\epsilon\tfrac{\partial B}{\partial\phi})d\phi^2 + \sin^2\theta 2\epsilon\tfrac{\partial B}{\partial\theta}d\theta d\phi + 2\epsilon A\sin\theta\cos\theta d\phi^2
\end{aligned} \tag{11}$$

we demand that ds^2 remains invariant.

$$d\theta^2 + \sin\theta d\phi^2 \to d\theta^2 + \sin\theta d\phi^2 + \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{terms that vanish}}$$

demanding that the additional terms vanish we obtain the following constraints

$$\begin{aligned}
d\theta^2 &: \tfrac{\partial A}{\partial\theta} = 0 \Rightarrow A = A(\phi) = f'(\phi) = \tfrac{df}{d\phi} \\
d\phi^2 &: \sin^2\theta\tfrac{\partial B}{\partial\phi} + A\sin\theta\cos\theta = 0 \Rightarrow \tfrac{\partial B}{\partial\phi} = -\tfrac{df}{d\phi}\tfrac{\cos\theta}{\sin\theta} \Rightarrow B = -f(\phi)\tfrac{\cos\theta}{\sin\theta} + g(\theta) \\
d\theta d\phi &: \tfrac{\partial A}{\partial\phi} + \sin^2\theta\tfrac{\partial B}{\partial\theta} = 0 \Rightarrow f'' + \sin^2\theta\left(\tfrac{f(\phi)}{\sin^2\theta} + g'(\theta)\right) = 0
\end{aligned} \tag{12}$$

+ +

6a. To show that Lorentz transformations form a group we have to show that they satify three properties:

- a product of two Lorentz transformations is a Lorentz transformation (LT).
- the inverse of a Lorentz transformation is a Lorentz transformation.
- the identity transformation is a Lorentz transformations.

In matrix form the Lorentz transformations satisfy the property that

$$L^T \eta L = \eta,$$

where L is a general LT in matrix form and η is the Minkowski metric. Given two LTs we have

$$L_1^T \eta Ł_1 = \eta \qquad , \qquad L_2^T \eta Ł_2 = \eta$$

then for the product of the two LT we have

$$(L_1 L_2)^T \eta (L_1 L_2) = L_2^T L_1^T \eta L_1 L_2 = L_2^T \eta L_2 = \eta$$

Hence, the product of LT satisfies the defining property of LTs and is therefore a LT.

Given a LT L its inverse satifies $L^{-1} L = I$ where I is the identity matrix. Hence we can check that

$$(L^{-1})^T \eta L^{-1} = (L^{-1})^T L^T \eta L L^{-1} = (L L^{-1})^T \eta L L^{-1} = \eta$$

Hence, the inverse transformation satisfies the defining property of LT.

Finally, it is obvious that the identity transformation is a LT, *i.e.* $I^T \eta I = \eta$, and the identity matrix satisfies the definig property of LT.

b. In this case we have to show that L^T satifies the defining property of LT, *i.e.*

$$(L^T)^T \eta L^T = L \eta Ł_T = \eta$$

Starting with the defining property of LT $L^T \eta L = \eta$ and multiplying from the left both side of the equation by $L\eta$ we get

$$L \eta L^T \eta L = L \eta \eta = L$$

where we used the fact that $\eta^2 = 1$. Multiplying both sides of the equation from the right by $L^{-1}\eta$ we get

$$(L \eta L^T) \eta L L^{-1} \eta = L (L^{-1} \eta) \quad \Rightarrow \quad L \eta L^T = \eta$$

Hence the transpose matrix is a LT.

c. These are the proper orthochronos Lorentz transformations that satisfy the considitions that $\mathrm{Det}(L) = +1$ and $L_{00} \geq +1$.

d. in five space time dimension there are 10 proper orthochronos LT corresponding to 4 boosts $(dtdx, dtdy, dtdz, dtdw)$ and 6 rotations $(dxdy, dxdz, dxdw, dydz, dydw, dzdw)$.

e. The dimensionality of the Poincare group in 5 space time dimensions if 15 corresponding to 10 boosts and rotations transformations and 5 translations.

f. In $d+1$ dimensions the infinitesimal line element is invariant under $d+1$ translations. In addition the set of LT in $d+1$ dimensions correspond to independent parameters in a $d+1$ asymmetric matrix, *i.e.*

$$\frac{(d+1-1)(d+1)}{2} = \frac{d(d+1)}{2}$$

Of these d correspond to boosts and

$$\frac{d(d+1)}{2} - d = \frac{d(d-1)}{2}$$

correspond to rotations. The dimensionality of the Poincare group in $d+1$ dimensions is therefore

$$d+1 \quad + \quad d \quad + \quad \frac{d(d-1)}{2} = \frac{(d+2)(d+1)}{2},$$

corresponding to translations, boosts and rotations in $d+1$ spacetime dimensions.